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CHANGES OF SIGN OF SUMS OF RANDOM VARIABLES

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1. Introduction. Let $x_1, x_2, \cdots$ be independent random variables all having the same continuous symmetric distribution, and let

$$s_k = x_1 + \cdots + x_k .$$

Our purpose is to prove statements concerning the changes of sign in the sequence of partial sums $s_1, s_2, \cdots$ which do not depend on the particular distribution the x_k may have.

The first theorem estimates the expectation of N_n, the number of changes of sign in the finite sequence $s_1, \cdots, s_{n+1}$. Here and later we write $\phi(k)$ for

$$\frac{2([k/2]+1)}{k+1} \binom{k}{[k/2]} 2^{-k} \approx (2\pi k)^{-1/2}.$$

THEOREM 1.

$$\sum_{k=1}^{n} \frac{1}{2(k+1)} \le \mathrm{E}\{N_n\} \le \frac{1}{2} \sum_{k=1}^{n} \phi(k).$$

It is known (see [**1**]) that, with probability one,

$$\limsup_{n \to \infty} \frac{N_n}{(n \log \log n)^{1/2}} = 1 \tag{1}$$

when the x_k are the Rademacher functions. We conjecture, but have not been able to prove, that (1) remains true, provided the equality sign be changed to $\le$, for all sequences of identically distributed independent symmetric random variables. We have had more success with lower limits:

THEOREM 2. *With probability one,*

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Pacific J. Math. 3 (1953), 673-687

$$\liminf_{n \to \infty} \frac{N_n}{\log n} \geq \frac{1}{2}.$$

By considering certain subsequences of the partial sums we obtain an exact limit theorem which is still independent of the distribution of the x_k: Let α be a positive number and a the first integer such that $(1+\alpha)^a \geq 2$; let $1', 2', \cdots$ be any sequence of natural numbers satisfying $(k+1)' \geq (1+\alpha)k'$; and let N_n' be the number of changes of sign in the sequence $s_1', \cdots, s_{n+1}'$, where s_k' stands for $s_{k'}$.

THEOREM 3. $\mathrm{E}\{N_n'\} \geq [n/a]/8$, *and, with probability one,*

$$\lim_{n \to \infty} \frac{N_n'}{\mathrm{E}\{N_n'\}} = 1.$$

For $k' = 2^k$, it is easy to see that $\mathrm{E}\{N_n'\} = n/4$; so with probability one the number of changes of sign in the first n terms of the sequence $s_1, s_2, \cdots, s_{2^k}, \cdots$ is asymptotic to $n/4$.

The basis of our proofs is the combinational Lemma 2 of the next section. When translated into the language of probability, this gives an immediate proof of Theorem 1. We prove Theorem 3 in § 3 and then use it to prove Theorem 2. A sequence of random variables for which $N_n/\log n \longrightarrow 1/2$ is exhibited in § 4; thus the statement of Theorem 2 is in a way the best possible. Finally we sketch the proof of the following theorem, which was discovered by Paul Lévy [2] when the x_k are the Rademacher functions.

THEOREM 4. *With probability one,*

$$\sum_{k=1}^{n} \frac{\operatorname{sgn} s_k}{k} = o(\log n).$$

Our results are stated only for random variables with continuous distributions. Lemma 3, slightly altered to take into account cases of equality, remains true however for discontinuous distributions; the altered version is strong enough to prove the last three theorems as they stand and the first theorem with the extreme members slightly changed. The symmetry of the x_k is of course essential in all our arguments.

2. Combinatorial lemmas. Let $a_1, \cdots, a_n$ be positive numbers which are free in the sense that no two of the sums $\pm a_1 \pm \cdots \pm a_n$ have the same value.

These sums, arranged in decreasing order, we denote by $S_1, \cdots, S_{2^n}$; q_i is the excess of plus signs over minus signs in S_i; and $Q_i = q_1 + \cdots + q_i$. It is clear that $Q_{2^n} = 0$ and that $Q_i = Q_{2^n - i}$ for $1 \le i < 2^n$.

LEMMA 1. *For* $1 \le i \le 2^{n-1}$,

$$0 \le Q_i - i \le ([n/2]+1)\binom{n}{[n/2]} - 2^{n-1}.$$

The proof of the first inequality, which is evident for $n = 1$, goes by induction. Suppose $n > 1$ and $i \le 2^{n-1}$. Define S_j' and Q_j' for $1 \le j \le 2^{n-1}$ just as S_j and Q_j were defined above, but using only $a_1, \cdots, a_{n-1}$. Let k and l be the greatest integers such $S_k' - a_n \ge S_i$ and $S_l' + a_n \ge S_i$. It may happen that no such k exists; then $i = l$ and the proof is relatively easy. Otherwise $k \le l$, $k \le 2^{n-2}$, and $i = k + l$. If $l \le 2^{n-2}$ then

$$Q_i = Q_k' - k + Q_l' + l = (Q_k' - k) + (Q_l' - l) + 2l \ge i.$$

If $2^{n-2} < l < 2^{n-1}$ then

$$Q_i = Q_k' - k + Q_l' + l = Q_k' - k + Q'_{2^{n-1}-l} + l$$

$$= (Q_k' - k) + (Q_{2^{n-1}-l} - 2^{n-1} + l) + 2^{n-1} - l + l \ge 2^{n-1} \ge i.$$

Finally, if $l = 2^{n-1}$ then, recalling $Q'_{2^{n-1}} = 0$, we get

$$Q_i = Q_k' - k + Q'_{2^{n-1}} + 2^{n-1} \ge 2^{n-1} \ge i.$$

In order to prove the second inequality we note that for each i the maximum of Q_i is attained if the a_i are given such values that $S_j > S_k$ implies $q_j \ge q_k$, —this happens if the a_j are nearly equal. Assume this situation. Then if n is odd q_i is positive for $i \le i_0 = 2^{n-1}$ and $Q_i - i$ is maximum for $i = i_0$. We have

$$Q_{i_0} - i_0 = \sum_{k=0}^{[n/2]} (n-2k)\binom{n}{k} - 2^{n-1} = ([n/2]+1)\binom{n}{[n/2]} - 2^{n-1}.$$

A similar computation for n even gives

$$2^{n-1} - \binom{n}{n/2}$$

for the index i_0 of the maximum and the same expression for $Q_{i_0} - i_0$. This completes the proof.

If $c_1, \cdots, c_{n+1}$ are real numbers let $m(c_1, \cdots, c_{n+1})$ be the number of indices j for which

$$|c_j| > |\sum_{i \neq j} c_i|.$$

We now consider $n+1$ positive numbers $a_1, \cdots, a_{n+1}$ which are 'free' in the sense explained above, and define

$$M = M(a_1, \cdots, a_{n+1}) = \sum m(\pm a_1, \cdots, \pm a_{n+1}),$$

the summation being taken over all combinations of plus signs and minus signs.

LEMMA 2.

$$2^{n+1} \leq M \leq 4([n/2]+1)\binom{n}{[n/2]}.$$

It is clear that $M = 2^{n+1}$ if

$$a_{n+1} > a_1 + \cdots + a_n,$$

and we reduce the other cases to this one by computing the change in M as a_{n+1} is increased to $a_1 + \cdots + a_n + 1$. Using the notation of Lemma 1, we suppose that $S_{i+1} < a_{n+1} < S_i$, where i of course is not greater than 2^{n-1}, and that a'_{n+1} is a number slightly greater than S_i. We now compare $M(a_1, \cdots, a_n, a_{n+1})$ with $M(a_1, \cdots, a_n, a'_{n+1})$. The inequality $a_{n+1} < S_i$ becomes $a'_{n+1} > S_i$ if a_{n+1} is replaced by a'_{n+1}, and we see that there is a contribution $+4$ to M coming from the terms $\pm a'_{n+1}$ in the four sums $\pm S_i \pm a'_{n+1}$. In like manner, each $+a_j$ occurring in S_i contributes -4 to M, and each $-a_j$ in S_i contributes $+4$ if j is less than $n+1$. So

$$M(a_1, \cdots, a_n, a_{n+1}) - M(a_1, \cdots, a_n, a'_{n+1}) = 4(q_i - 1),$$

where q_i has the meaning explained at the beginning of this section. Thus increasing a_{n+1} to $a_1 + \cdots + a_n + 1$ decreases M by

$$4(Q_i - i) = 4 \sum_{j \leq i} (q_j - 1),$$

and Lemma 2 follows from Lemma 1.

There is another more direct way of establishing the first inequality of Lemma 2. Since the inequality is trivial for $n = 1$, we proceed by induction. Considering the numbers $(a_1 + a_2)$, $a_3, \cdots, a_{n+1}$ we assume that there are at least 2^{n-2} inequalities of the form

$$a_j > U \qquad (j > 2) \tag{2}$$

or

$$(a_1 + a_2) > V, \tag{3}$$

where the right members are positive, and U is a sum over $(a_1 + a_2)$, $a_3, \cdots, a_{j-1}, a_{j+1}, \cdots, a_{n+1}$ with appropriate signs, and V is a sum over $a_3, \cdots, a_{n+1}$. From (2) we obtain an inequality (2′) by dropping the parentheses from $(a_1 + a_2)$ in U; from (3) we obtain an inequality (3′): $a_1 > a_2 - V$ or $a_1 > V - a_2$ according as a_2 is greater or less than V (we assume without loss of generality that $a_1 > a_2$). We consider also the numbers $(a_1 - a_2)$, $a_3, \cdots, a_{n+1}$ and inequalities

$$a_j > \overline{U} \qquad (j > 2) \tag{4}$$

$$(a_1 - a_2) > \overline{V}, \tag{5}$$

of which we assume there are at least 2^{n-2}. From (4) we derive an inequality (4′) by dropping the parentheses from $(a_1 - a_2)$ in $\overline{U}$, and from (5) we derive an inequality (5′): $a_1 > a_2 + \overline{V}$. It is easy to see that no two of the primed inequalities are the same. Hence there must be at least $2 \cdot 2^{n-2} = 2^{n-1}$ inequalities

$$a_i > \sum_{j \neq i} \pm a_j \qquad (1 \leq i \leq n + 1)$$

in which the right member is positive. Taking into account the four possibilities of attributing signs to the members of each inequality we get the first statement of the lemma.

We now translate our result into terms of probability.

LEMMA 3.

$$\frac{1}{n+1} \leq \Pr\{|x_{n+1}| > |x_1 + \cdots + x_n|\} \leq \phi(n).$$

Here of course the random variables satisfy the conditions imposed at the beginning of § 1, and $\phi(n)$ is the function defined there. Since the joint distribution of the x_i is unchanged by permuting the x_i or by multiplying an x_i by -1, we have

$$\Pr\left\{|x_{n+1}| > \left|\sum_1^n x_i\right|\right\} = \frac{1}{n+1} \sum_{i=1}^{n+1} \Pr\left\{|x_i| > \left|\sum_{j \neq i} x_j\right|\right\}$$

$$= \frac{1}{n+1} \mathrm{E}\{m(x_1, \cdots, x_{n+1})\}$$

$$= \frac{1}{n+1} \mathrm{E}\left\{\frac{1}{2^{n+1}} \sum_{+,-} m(\pm|x_1|, \cdots, \pm|x_{n+1}|)\right\}$$

$$= \frac{1}{(n+1)2^{n+1}} \mathrm{E}\{M(|x_1|, \cdots, |x_{n+1}|)\},$$

where m and M are the functions defined above. Since $|x_1|, \cdots, |x_{n+1}|$ are 'free' with probability one (because the distribution of the x_i is continuous), Lemma 3 follows at once from Lemma 2.

Our later proofs could be made somewhat simpler than they stand if we could use the inequality

$$\frac{m}{m+n} \leq P_{m,n} \equiv \Pr\left\{\left|\sum_1^n x_i\right| < \left|\sum_{n+1}^{n+m} x_i\right|\right\} \leq \phi([n/m])$$

for $m \leq n$. This generalization of Lemma 3 we have been unable to prove; and indeed a corresponding generalization of Lemma 2 is false. However, we shall use

$$(6) \qquad P_{m,n} \leq 6\phi([n/m]) < 3[n/m]^{-1/2},$$

and establish it in the following manner:

Let $a = [n/m]$, and write

$$u = x_1 + \cdots + x_{am},$$

$$v = x_{am+1} + \cdots + x_n,$$

$$w = x_{n+1} + \cdots + x_{n+m},$$

$$z = y_{n+1} + \cdots + y_{am+m},$$

where the y_k have the same distribution as the x_j, and the x_j and y_k taken together form an independent set of random variables. Let E be the set on which the four inequalities

$$|w| < |u \pm v \pm z|$$

hold; by Lemma 3 the probability of any one of these inequalities is at least $1 - \phi(a+1)$; hence E has probability at least $1 - 4\phi(a+1)$. Similarly the probability of the set F on which the two inequalities $|v \pm z| < |u|$ hold in at least $1 - 2\phi(a)$. Now clearly $|u + v| > |w|$ on EF and also

$$\Pr\{EF\} \geq 1 - 2\phi(a) - 4\phi(a+1) \geq 1 - 6\phi(a).$$

3. Proofs of Theorems 1, 2, 3. It is easy to see that the probability of s_k and s_{k+1} differing in sign is one-half the probability of s_{k+1} being larger in absolute value than s_k. Thus

$$\mathrm{E}\{N_n\} = \sum_1^n \Pr\{s_k\, s_{k+1} < 0\} = \frac{1}{2} \sum_1^n \Pr\{|x_{k+1}| > |s_k|\},$$

and Lemma 3 implies Theorem 1.

Let us turn to Theorem 3. Clearly the probability of s_k' and

$$s_{2k'} = \sum_1^{2k'} x_j$$

differing in sign is 1/4. Also, $s_{k+a} - s_{2k'}$ is independent of both s_k' and $s_{2k'}$, for

$$(k+a)' \geq (1+\alpha)^a k' \geq 2k'.$$

Thus $s_{k+a}' - s_{2k'}$ has an even chance of taking on the same sign as $s_{2k'}$; so we must have

$$\Pr\{s_k' s_{k+a}' < 0\} \geq \frac{1}{2} \Pr\{s_k' s_{2k'}' < 0\} = 1/8.$$

Now, if $s_k' s_{k+a}' < 0$ then must be at least one change of sign in the sequence $s_k', s_{k+1}', \cdots, s_{k+a}'$. Hence, if p_k is the probability of s_k' and s_{k+1}' differing in sign, we have

$$p_k + \cdots + p_{k+a-1} \geq \frac{1}{8},$$

and consequently

$$(7) \qquad E\{N_n'\} = \sum_1^n p_k \geq \frac{1}{8}[n/a].$$

This proves the first half of the theorem.

As a preliminary to proving the second half of the theorem we show that the variance of N_n' is $O(n)$ by estimating the probabilities

$$p_{i,j} = \Pr\{s_i' s_{i+1}' < 0 \ \& \ s_j' s_{j+1}' < 0\}.$$

Suppose that $i < j$; set

$$u = s_i', \quad v = s_{i+1}' - s_i', \quad w = s_j' - s_{i+1}', \quad z = s_{j+1}' - s_j';$$

and define the events

$$A : uv < 0,$$
$$B : |u| < |v|,$$
$$C : (u+v+w)z < 0,$$
$$D : |u+v+w| < |z|,$$
$$D' : |w| < |z|,$$
$$E : |z-w| > |u+v|.$$

Then

$$p_i = \Pr\{AB\},\ p_j = \Pr\{CD\},\ \text{and}\ p_{i,j} = \Pr\{ABCD\}.$$

One sees immediately that A, B, C, D' are independent, and that $ED = ED'$. Writing $\widetilde{E}$ for the complement of E, we have

$$ABCD = \widetilde{E}ABCD + EABCD' \subset \widetilde{E} + ABCD',$$

and

$$D' \subset \widetilde{E} + D.$$

Hence

$$\begin{aligned}\Pr\{ABCD\} &\le \Pr\{\widetilde{E}\} + \Pr\{ABC\}\Pr\{D'\}\\ &\le \Pr\{\widetilde{E}\} + \Pr\{ABC\}(\Pr\{\widetilde{E}\} + \Pr\{D\})\\ &\le \Pr\{AB\}\Pr\{C\}\Pr\{D\} + 2\Pr\{\widetilde{E}\} = p_i\, p_j + 2\Pr\{\widetilde{E}\}.\end{aligned}$$

Note now that $z - w$ is the sum of $(j+1)' - (i+1)'$ of the x's, and $u + v$ is the sum of $(i+1)'$, of the x's, and that moreover

$$(j+1)' - (i+1)' \ge [(1+\alpha)^{j-i} - 1]\,(i+1)'.$$

We may thus apply the inequality (6) following Lemma 3 to obtain

$$\Pr\{\widetilde{E}\} < 3\,[(1+\alpha)^{j-i} - 2]^{-1/2}$$

provided $j - i \ge a$. This yields an upper bound for $p_{i,j}$; a similar argument yields a corresponding lower bound. We have finally

$$p_{i,j} = p_i\, p_j + O\{|1+\alpha|^{-|i-j|/2}\}$$

for all i and j. This estimate shows that

$$(8)\qquad \begin{aligned}\mathrm{E}\{N_n'^{\,2}\} &\equiv \sum_{1\le i,j\le r} p_{ij}\\ &= \sum p_i\, p_j + \sum O\{(1+\alpha)^{-|i-j|/2}\} = \mathrm{E}\{N_n'\}^2 + O(n).\end{aligned}$$

Let us denote $\mathrm{E}\{N_k'\}$ by b_k. It follows from (7), (8), and Tchebycheff's inequality that

$$\Pr\left\{\left|\frac{N'_k}{b_k} - 1\right| > \epsilon\right\} < \frac{c}{\epsilon^2 k}$$

for an appropriate constant c and for all positive ϵ. Thus

$$\Pr\left\{\left|\frac{N'_{k^2}}{b_{k^2}} - 1\right| > \epsilon\right\}$$

is the kth term of a convergent series, so that according to the lemma of Borel and Cantelli

$$\frac{N'_{k^2}}{b_{k^2}} \longrightarrow 1$$

with probability one. Note also that

$$\frac{b_{k^2}}{b_{(k+1)^2}} \longrightarrow 1.$$

Now for every natural number n we have

$$\frac{N'_{k^2}}{b_{(k+1)^2}} \leq \frac{N'_n}{b_n} \leq \frac{N'_{(k+1)^2}}{b_{k^2}},$$

with k so chosen that $k^2 \leq n < (k+1)^2$. Since the extreme members tend to one as n increases, the proof of the second half of Theorem 3 is complete.

Theorem 2 is obtained from Theorem 3 in the following way. Let r be a large integer and let $1', 2', \cdots$ be the sequence

$$\begin{array}{l} r, (r+1), \\ r^2, r(r+1), (r+1), (r+1)^2, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ r^l, r^{l-1}(r+1), \cdots, (r+1)^l, \\ r^m, r^{m-1}(r+1), \cdots, (r+1)^m, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \end{array}$$

where m is defined by

$$r^{m+1} \geq (r+1)^{l+1} > r^m.$$

Let us call j 'favorable' if $(j+1)' = (1+1/r)j'$. Then it is easy to see that:

a) $(1+1/r)j' \leq (j+1)' \leq (1+r)j'$ for all j;

b) there are $k + o(k)$ favorable j less than k (as $k \longrightarrow \infty$);

c) $\log k' = k \log (1+1/r) + o(k)$.

Now, if j is favorable then

$$j' = r\{(j+1)' - j'\}$$

and we may apply Lemma 3 to s_j' and $s_{j+1}' - s_j$. Thus

$$\Pr\{s_j' s_{j+1}' < 0\} = \frac{1}{2}\Pr\{|s_{j+1}' - s_j'| > |s_j'|\} \geq \frac{1}{2(1+r)} .$$

Hence

$$\mathrm{E}\{N_k'\} = \sum_{j=1}^{k} \Pr\{s_j' s_{j+1}' < 0\}$$

$$\geq \sum_{j \text{ favorable}} \Pr\{s_j' s_{j+1}' < 0\} \geq \frac{k}{2(r+1)} + o(k).$$

Note that for every natural number n

$$\frac{N_n}{\log n} \geq \frac{N_k'}{\log (k+1)'} ,$$

where k is chosen so that $k' \leq n < (k+1)'$. Consequently

$$\liminf_{n\to\infty} \frac{2N_n}{\log n} \geq \liminf_{k\to\infty} \frac{2N_k'}{\log (k+1)'} = \liminf \frac{2N_k'}{(k+1)\log(1+1/r)}$$

$$\geq \liminf \frac{N_k'}{\mathrm{E}\{N_k'\}(r+1)\log(1+1/r)} = \frac{1}{(r+1)\log(1+1/r)}.$$

Letting $r \longrightarrow \infty$ we have Theorem 2.

4. An example. Our construction of a sequence $x_1, x_2, \cdots$ for which $N_n/\log n \longrightarrow 1/2$ with probability one depends on the following observations. For given k define the random index $i = i(k)$ by the condition

$$|x_i| = \max_{1 \leq j \leq k+1} |x_j|,$$

and let A_k be the event $|x_i| > \sum |x_j|$, where the summation is over $j \neq i$, $1 \leq j \leq k+1$. Let f_k be the characteristic function of the event '$s_k\, s_{k+1} < 0$,' and g_k is the characteristic function of the event '$i(k) = k+1$ and further $(x_1 + \cdots + x_k)x_{k+1} < 0$'. It is clear that $g_1, g_2, \cdots$ are independent random variables, that

$$2\Pr\{g_k = 1\} = \frac{1}{(k+1)},$$

and that the strong law of large numbers applies to the sequence $g_1, g_2, \cdots$ also $f_k = g_k$ on A_k; if moreover $\sum \Pr\{\widetilde{A}_k\} < \infty$ (here $\widetilde{A}_k$ is the complement of A_k) then, with probability one, $f_k = g_k$ for all but a finite number of indices. In this case we have, with probability one,

$$N_n = \sum_{k=1}^{n} f_k = \sum_{k=1}^{n} g_k + O(1) = \sum_{k=1}^{n} \frac{1}{2(k+1)} + o(\log n),$$

the last step being the strong law of large numbers applied to $g_1, g_2, \cdots$. Thus, in order to produce the example, we have only to choose the x_j so that, say,

$$\Pr\{\widetilde{A}_k\} = O(k^{-2}).$$

To do this we take $x_j = \pm \exp(\exp 1/u_j)$, where $u_1, u_2, \cdots$ is a sequence of independent random variables each of which is uniformly distributed on the interval $(0, 1)$ and the $\pm$ stands for multiplication by the jth Rademacher function. For a given k let y and z be the least and the next to least of $u_1, \cdots, u_{k+1}$. The joint density function of y and z is

$$(k+1)\,k(1-z)^{k-1} \qquad (0 < y < z < 1).$$

Consequently the event

$$D_k : \frac{1}{y} > \frac{1}{z} + \frac{1}{k^2}$$

has probability

$$k(k+1)\int_0^{k^2/(k^2+1)} dy \int_{k^2y/(k^2-y)}^{1} (1-z)^{k-1}\,dz = 1 + O(k^{-2}),$$

and the event $E_k : 1/z > 3 \log k$ also has probability $1 + O(k^{-2})$. It is easy to verify that the event A_k defined above contains $D_k\,E_k$; thus

$$\Pr\{\widetilde{A}_k\} = O(k^{-2}),$$

and our example is completed.

5. Proof of Theorem 4. We prove Theorem 4 in the form

$$T_n \equiv \sum_{\substack{1 \le k \le n \\ s_k > 0}} \frac{1}{k} = \frac{1}{2} \log n + o(\log n)$$

by much the same method as we proved Theorem 2. First,

$$\mathrm{E}\{T_n\} = \frac{1}{2} \sum_{1}^{n} \frac{1}{k} = \frac{1}{2} \log n + o(1).$$

Next, the inequality following Lemma 3 yields

$$\Pr\{|s_l - s_k| < |s_k|\} \le 3\left[\frac{k}{l-k}\right]^{1/2} \qquad (l \ge 2k),$$

so that

$$\Pr\{|s_l - s_k| < |s_k|\} = O\left(\frac{k}{l}\right)^{1/2}$$

for $l > k$. Consequently

$$\Pr\{s_k > 0 \ \&\ s_l > 0\} = \frac{1}{4} + O\left(\frac{k}{l}\right)^{1/2} \qquad (l > k).$$

This implies that

$$E\{T_n^2\} \equiv \sum_{1\le k,\, l\le n} \frac{1}{kl} \Pr\{s_k > 0 \ \&\ s_l > 0\}$$

$$= \sum_{1\le k\le n} \frac{1}{k^2} \Pr\{s_k > 0\} + 2 \sum_{1\le k< l\le n} \frac{1}{kl} \Pr\{s_k > 0 \ \&\ s_l > 0\}$$

$$= \frac{1}{2} \sum_{1}^{n} \frac{1}{k^2} + 2 \sum_{1\le k\le l\le n} \frac{1}{kl} \left\{ \frac{1}{4} + O\left(\frac{k}{l}\right)^{1/2} \right\}$$

$$= \frac{1}{4} (\log n)^2 + O(\log n).$$

Thus the variance of T_n is of the order of log n. Setting $n(k) = 2^{k^2}$, we have, according to Tchebycheff's inequality,

$$\Pr\left\{ \left| \frac{T_{n(k)}}{\log n(k)} - 1 \right| > \epsilon \right\} \le \frac{c}{\epsilon^2 k^2}$$

for an appropriate constant c and all positive ϵ. Since the right member is the kth term of a convergent series, the lemma of Borel and Cantelli implies that

$$\frac{T_{n(k)}}{\log n(k)} \longrightarrow 1$$

with probability one. Note also that

$$\frac{\log n(k+1)}{\log n(k)} \longrightarrow 1.$$

Now, for any n,

$$\frac{T_{n(k)}}{\log n(k+1)} \le \frac{T_n}{\log n} \le \frac{T_{n(k+1)}}{\log n(k)},$$

where k is so chosen that $n(k) \le n \le n(k+1)$. Here the extreme members almost certainly tend to one as n increases. This proves Theorem 4.

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ON LINEAR INDEPENDENCE OF SEQUENCES IN A BANACH SPACE

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1. A. Dvoretzky has raised the following problem:

Let $x_1, x_2, \cdots, x_n, \cdots$ be an infinite sequence of unit vectors in a Banach space which are linearly independent in the algebraic sense; that is,

$$\sum_{i=1}^{k} c_i x_{n_i} = 0 \implies c_i = 0 \qquad (i = 1, \cdots, k).$$

Does there exist an infinite subsequence $\{x_{n_i}\}$ which is linearly independent in a stronger sense?

We may consider three types of linear independence of a sequence of unit vectors in a normed linear space:

I. $$\sum_{n=1}^{\infty} c_n x_n = 0 \implies c_n = 0 \qquad (n = 1, 2, \cdots).$$

II. If $\phi(k) > 0$ is any function defined for $k = 1, 2, \cdots$, then

$$|c_n^{(k)}| < \phi(k) \qquad (n, k = 1, 2, \cdots)$$

and

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} c_n^{(k)} x_n = 0$$

imply

$$\lim_{k\to\infty} c_n^{(k)} = 0 \qquad (n = 1, 2, \cdots).$$

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III. $$\lim_{k\to\infty}\sum_{k=1}^{\infty} c_n^{(k)} x_n = 0 \implies \lim_{k\to\infty} c_n^{(k)} = 0 \qquad (n = 1, 2, \cdots).$$

It is obvious that III implies both II and I; and if

$$\liminf_{k\to\infty} \phi(k) > 0$$

then II implies I. It is easy to show that the converse implications do not hold.

In this note we give an affirmative answer to Dvoretzky's question if independence is defined in the sense I or even II for arbitrary $\phi(k)$. However the answer is in the negative if indepence is defined in the sense III.

2. The negative part is proved by the following example due to G. Szegö [**1**; I, p. 86]:

THEOREM. *If* $\{\lambda_n\}$ *is a sequence of positive number with* $\lambda_n \longrightarrow \infty$, *then the functions* $\{1/(x+\lambda_n)\}$ *are complete in every finite positive interval.*

Obviously every infinite subsequence of $\{1/(x+\lambda_n)\}$ satisfies the condition of the theorem and is therefore complete.

3. For the affirmative part of our result we prove the following:

THEOREM. *Let* $\{x_{n_i}\}$ *be an infinite sequence of algebraically linearly independent unit vectors in a Banach space and let* $\phi(k) > 0$ *be any function defined for* $k = 1, 2, \cdots$. *Then there exists an infinite subsequence* $\{x_{n_i}\}$ *such that* $|c_i^{(m)}| < \phi(i)$ $(i, m = 1, 2, \cdots)$ *and*

$$\lim_{m\to\infty}\sum_{i=1}^{\infty} c_i^{(m)} x_{n_i} = 0$$

imply

$$\lim_{m\to\infty} c_i^{(m)} = 0 \qquad (i = 1, 2, \cdots).$$

It was pointed out to us by the referee that it suffices to prove the theorem for a separable Hilbert space. The separability may be assumed since we may restrict our attention to the subspace spanned by $\{x_n\}$. Now every separable Banach space can be imbedded isometrically in the space $C(0, 1)$ of continuous functions over the interval $(0, 1)$; and $C(0, 1) \subset L_2(0, 1)$, where linear independence, in any of the above defined senses, in L_2 implies the same independence in C. Let $\{z_n\}$ be the orthonormal sequence obtained from $\{x_n\}$ by the Gram-Schmidt process; then

$$x_n = \sum_{m=1}^{n} a_{nm} z_m,$$

with $|a_{nm}| \leq 1$ and $a_{nn} \neq 0$.

Since $\{a_{nm}\}$ is bounded for fixed m, we can select a subsequence $\{x_{n_i}\}$ such that

$$\lim_{i \to \infty} a_{n_i m} = b_m$$

exists for every m.

If we prove the theorem for $\psi(k) \geq \phi(k)$, then it is proved *a fortiori* for $\phi(k)$. Hence we may set

$$\psi(n) = \max\{1, \phi(1), \cdots, \phi(n)\},$$

so that $\psi(n) \geq 1$ and $\psi(n)$ is nondecreasing.

If the theorem we false then for every infinite subsequence $\{y_k\}$ of $\{x_{n_i}\}$ there would exist a sequence of sequences $\{c_k^{(m)}\}$ with

$$|c_k^{(m)}| < \psi(k) \qquad (k, m = 1, 2, \cdots)$$

and

$$\lim_{m \to \infty} \sum_{k=1}^{\infty} c_k^{(m)} y_k = Q$$

while

$$\limsup_{m \to \infty} |c_{k_0}^{(m)}| \neq 0 \quad \textit{for some fixed } k_0.$$

We can then select a subsequence of sequences $\{c_k^{(m_i)}\}$ such that

$$\lim_{i \to \infty} c_k^{(m_i)} = c_k$$

exists for every k, and $c_{k_0} \neq 0$. For convenience of notation we assume

$$\lim_{n \to \infty} c_k^{(m)} = c_k.$$

Since $c_{k_0} \neq 0$, there would exist a least $k_1 \geq k_0$ such that

$$|c_k| < 2^{k-k_0}\, \psi(k)\, |c_{k_0}| \qquad \text{for all } k > k_1.$$

This implies

$$(1) \qquad |c_k^{(m)}| \leq 2^{k-k_1}\, \psi(k)\, |c_{k_1}^{(m)}| \qquad \text{for all } k \geq k_1;\ m > m_0.$$

Case A: $b_{n_{i_j}} = 0$ *for* $j = 1, 2, \cdots$.

In order to simplify notation we assume $b_{n_i} = 0$ for all $i = 1, 2, \cdots$ by omitting all terms with $n_i \neq n_{i_j}$ from our subsequence. We select the subsequence $\{y_k\}$ as follows:

$$y_1 = x_{n_1},\ y_{k+1} = x_{n_{i_{k+1}}}$$

where

$$|a_{n_{i_{k+1}}, n_j}| < \frac{|a_{n_j, n_j}|}{4^{k+1}\, \psi(k+1)} \qquad \text{for } j = 1, 2, \cdots, i_k.$$

We write $y_k = x_{l_k}$.

If the theorem were false then there would exist a sequence of sequences $\{c_k^{(m)}\}$ with the above properties such that

$$\left\| \sum_{k=1}^{\infty} c_k^{(m)} y_k \right\| = \epsilon_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

If we take the k_1 defined in (1), then

$$(2) \qquad \left| \sum_{k=k_1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right| \leq \epsilon_m;$$

but for all $m > m_0$ we have

$$|c_k^{(m)}| \leq 2^{k-k_1} \psi(k) |c_{k_1}| \qquad (k = k_1, k_1+1, \cdots),$$

and hence

$$\left| \sum_{k=k_1+1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right|$$

$$< \sum_{k=k_1+1}^{\infty} \frac{2^{k-k_1} \psi(k) |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|}{4^k \psi(k)} = 2^{-2k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|.$$

We can now choose m so large that

$$|c_{k_1}^{(m)} - c_{k_1}| < 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}| \text{ and } \in_m < 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|.$$

Then for the left side of (2) we obtain

$$\left| \sum_{k=k_1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right| \geq |c_{k_1}^{(m)}| |a_{l_{k_1}, l_{k_1}}| - \left| \sum_{k=k_1+1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right|$$

$$\geq |c_{k_1}| |a_{l_{k_1}, l_{k_1}}| - 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}| - 2^{-2k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|$$

$$> 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|,$$

while for the right side of (2) we have

$$\in_m < 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|,$$

a contradiction.

Case B: $b_{n_i} \neq 0$ except for a finite number of i.

Without loss of generality we may assume $b_{n_i} \neq 0$ for all i by omitting a finite number of elements from $\{x_{n_i}\}$. We select the subsequence $\{y_k\}$ as follows:

$$y_1 = x_{n_1}, \quad y_{k+1} = x_{n_{i_{k+1}}},$$

where

$$|a_{n_{i_{k+1}}, n_j} - b_{n_j}| < \frac{|b_{n_{i_{k+1}}}|}{4^{k+1}\,\psi(k+1)} \qquad \text{for } j = 1, 2, \cdots, i_k .$$

For simplicity we again write $y_k = x_{l_k}$.

If the theorem were false then there would exist sequences $\{c_k^{(m)}\}$ with the foregoing properties such that

$$\|\sum c_k^{(m)} y_k\| = \epsilon_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty .$$

If we let k_1 be defined as in (1), then on the one hand we have

$$Q = \left| \frac{1}{b_{l_{k_1}}} \sum_{k=k_1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} - \frac{1}{b_{l_{k_1+1}}} \sum_{k=k_1+1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1+1}} \right|$$

$$\geq |c_{k_1}^{(m)}| - \frac{|c_{k_1}^{(m)}|}{4^{k_1}} - \sum_{k=k_1+1}^{\infty} \frac{2|c_k^{(m)}|}{4^k \psi(k)}$$

$$\geq |c_{k_1}^{(m)}| \left(1 - \frac{1}{4^{k_1}} - \sum_{k=k_1+1}^{\infty} \frac{2 \cdot 2^{k-k_1} \psi(k)}{4^k \psi(k)}\right) > \frac{1}{2} |c_{k_1}^{(m)}| > \frac{1}{4} |c_{k_1}| > 0$$

for all $m > m_0$; on the other hand, we have

$$Q \leq \frac{1}{b_{l_{k_1}}} \left\| \sum_{k=1}^{\infty} c_k^{(m)} y_k \right\| + \frac{1}{b_{l_{k_1+1}}} \left\| \sum_{k=1}^{\infty} c_k^{(m)} y_k \right\|$$

$$\leq \left(\frac{1}{b_{l_{k_1}}} + \frac{1}{b_{l_{k_2}}} \right) \epsilon_m < \frac{1}{4} |c_{k_1}|$$

for all sufficiently large m, a contradiction.

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ON SUMS OF SERIES OF COMPLEX NUMBERS

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1. Introduction. We recall certain facts about the convergence of series.

1.1. Let $\sum_{i=1}^{\infty} a_i$ be a series of real numbers, $a_i \longrightarrow 0$. Then it is obvious that a sequence of signs $\epsilon_i = \pm 1$ $(i = 1, 2, \cdots)$ may be chosen so that $\sum_{i=1}^{\infty} \epsilon_i a_i$ is convergent. It is, furthermore, well known that all the possible sums so obtained form a perfect set, and if $\sum_{i=1}^{\infty} |a_i| = \infty$ then any preassigned sum may be obtained.

1.2. The first statement remains true also for complex numbers. Aryeh Dvoretzky and the author [2] proved that if $\sum_{i=1}^{\infty} c_i$ is a series of complex numbers with $c_i \longrightarrow 0$, then a sequence of signs $\epsilon_i = \pm 1$ $(i = 1, 2, \cdots)$ may be chosen so that $\sum_{i=1}^{\infty} \epsilon_i c_i$ converges and

$$\left| \sum_{i=1}^{n} \epsilon_i c_i \right| \leq \sqrt{3} \cdot \max |c_i| \qquad (n = 1, 2, \cdots).$$

1.3. The object of the present paper is to determine the sets of points which may be sums of the series $\sum_{i=1}^{\infty} \epsilon_i c_i$ when suitable sequences ϵ_i are chosen.

2. Notation and definitions. In this paper the following notations and definitions will be used.

2.1. NOTATION.

$c = a + ib$ denotes a term of a (finite or infinite) series of complex numbers, a being its real and ib its imaginary part;

$C = A + iB$ also denotes a complex number;

$\gamma = \alpha + i\beta$ denotes a direction in the plane of complex numbers, and also a unit vector in the same direction;

(C, C') is the scalar product of the vectors C and C'; that is $(C, C') = AA' + BB'$;

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γ' denotes a direction perpendicular to γ; that is, $(\gamma, \gamma') = 0$;

ϵ denotes ± 1;

Σ without summation limits denotes summation by the summation index from 1 to ∞. In any other cases the summation limits will be indicated.

2.2. DEFINITION. C will be called an *attainable point* of the series $\sum c_i$ if a sequence ϵ_i $(i = 1, 2, \cdots)$ exists, such that $\sum \epsilon_i c_i = C$.

2.3. DEFINITION. Let $\sum c_i$ be a series of complex numbers with $c_i \longrightarrow 0$ and $\sum |c_i| = \infty$. We say that γ is a *direction of divergence* of the series $\sum c_i$ if a subseries $\sum c_{i*}$ of $\sum c_i$ exists such that

$$\sum |c_{i*}| = \infty$$

and

$$\frac{(c_{i*}, \gamma')}{(c_{i*}, \gamma)} \longrightarrow 0.$$

If γ is a direction of divergence, then clearly also the inverse direction $-\gamma$ is such. The directions γ and $-\gamma$ form an *axis of divergence*. It can easily be seen [3, p. 93] that if $\sum |c_i| = \infty$, then $\sum c_i$ has at least one axis of divergence.

2.4. DEFINITION. Let $\sum c_i$ be a series of complex numbers with $c_i \longrightarrow 0$ and $\sum |c_i| = \infty$. We define the *convergence strip* of $\sum c_i$ as follows:

If $\sum c_i$ has at least two axes of divergence, the convergence strip is the whole plane.

If $\sum c_i$ has exactly one axis of divergence, then the convergence strip is composed of all the lines parallel to this axis which contain attainable points of the series $\sum [\gamma'(c_i, \gamma')]$, where γ' is a unit vector perpendicular to the axis of divergence.

According to 1.1, the convergence strip is either i) a cartesian product of a perfect set by a straight line or ii) the whole plane. It is obvious that every attainable point of $\sum c_i$ is a point of the convergence strip.

3. Theorem. We shall establish the following result.

3.1. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to* *zero, and let* $\sum |c_i| = \infty$; *then the attainable points of* $\sum c_i$ *form a set which is dense in the convergence strip of* $\sum c_i$, *and within this strip is dense on every straight line not parallel to the axis of divergence of* $\sum c_i$.

Proof. We may, without restricting the generality of the theorem, suppose the axis of divergence to be the real axis.

The following statement is clearly equivalent to our theorem: Let $C = A + iB$ be any point of the convergence strip, δ any real number, and η any positive number however small; then there exists an attainable point $C' = A' + iB'$ of $\sum c_i$ such that $|C - C'| < \eta$ and $A - A' = \delta(B - B')$. This will now be proved.

Put

$$\eta' = \frac{1}{4\sqrt{1+\delta^2}}\,\eta.$$

Let N_1 be such that $|c_i| < \eta'$ for every $i > N_1$. According to 1.1 , there exist $N_2 \geq N_1$ and a sequence $\epsilon_{i'}$ $(i' = 1, 2, \cdots, N_2)$ such that

$$\left| B - \sum_{i'=1}^{N_2} \epsilon_{i'} b_{i'} \right| < \eta'.$$

We put

$$C_1 = \sum_{i'=1}^{N_2} \epsilon_{i'} c_{i'}.$$

It is evident [3] that the series $\sum_{i=N_2+1}^{\infty} c_i$ can be separated into two subseries $\sum c_{i_k''}$ and $\sum c_{i_k'''}$, so that for $\sum c_{i_k''}$ we have

$$\sum |a_{i_k''}| = \infty \text{ and } \sum |b_{i_k''}| < \eta'.$$

According to 1.2, there exists a sequence $\epsilon_{i_k'''}$ $(k = 1, 2, \cdots)$ such that the series $\sum \epsilon_{i_k'''} c_{i_k'''}$ converges and

$$\left|\sum \epsilon_{i_k'''} c_{i_k'''}\right| < \sqrt{3}\,\eta'.$$

Let us put $C_2 = C_1 + \sum \epsilon_{i_k'''} c_{i_k'''}$. Now, according to 1.1 there exists a sequence $\epsilon_{i_k''}$ $(k = 1, 2, \cdots)$ such that $\sum \epsilon_{i_k''}(a_{i_k''} - \delta b_{i_k''})$ converges and

$$\sum \epsilon_{i_k''}(a_{i_k''} - \delta b_{i_k''}) = (A - A_2) - \delta(B - B_2).$$

Putting $C' = C_2 + \quad \epsilon_{i_k''} c_{i_k''}$, we get $A - A' = \delta(B - B')$ and

$$|B - B'| \leq |B - B_1| + |B_1 - B_2| + |B_2 - B'| < \eta' + \sqrt{3}\,\eta' + \eta' = \eta'(2 + \sqrt{3})$$

whence

$$|C - C'| < \eta' \sqrt{1 + \delta^2}\,(2 + \sqrt{3}) < \eta.$$

The series $\sum \epsilon_i c_i$ is composed of a finite subseries $\sum_{i'=1}^{N_2} \epsilon_{i'} c_{i'}$ and two interwoven subseries $\sum \epsilon_{i_k''} c_{i_k''}$ and $\sum \epsilon_{i_k'''} c_{i_k'''}$ which are evidently convergent and in which the order of terms remains unchanged. Consequently $\sum \epsilon_i c_i = C'$.

3.2. In special cases every point of the convergence strip can be an attainable point of $\sum c_i$, but generally this is not true. A few examples are given showing the possibility that the attainable points do not cover the convergence strip and even are not dense on every straight line parallel to the axis of divergence:

a) For

$$c_n = \frac{1}{n} + \frac{1}{3^n}\, i,$$

on every line parallel to the axis of divergence (real axis) there is at most one attainable point.

b) When the convergence strip is connected, a similar example may serve, namely:

$$c_n = \frac{1}{n} + \frac{1}{2^n}\, i.$$

Here on every line parallel to the real axis there are at most two attainable points.

c) The case when the convergence strip covers the whole plane is more complicated. The following example may suit:

$$c_k = \frac{1}{n} + \frac{1}{10^{n^2}} i, \quad 1 + \sum_{j=0}^{n-1} 10^{j^2} \leq k \leq \sum_{j=0}^{n} 10^{j^2}, \; c_1 = 0.$$

No attainable point is, for example, on the line through $(0, i/9)$ parallel to the real axis. For let us suppose that $C^* = A^* + i/9$ is such a point; then

$$C^* = \sum t_n \left(\frac{1}{n} + \frac{1}{10^{n^2}} i \right), \text{ where } |t_n| \leq 10^{n^2}.$$

On the other hand, there exists N^* such that for

$$k_i > \sum_{j=0}^{N^*} 10^{j^2} \qquad (i = 1, 2)$$

we have $|c_{k_1} - c_{k_2}| < 1$. Consequently, for $n > N^*$, we have $|t_n| < n$. It follows that

$$\frac{1}{9} = \sum \frac{t_n}{10^{n^2}},$$

where $|t_n| < n$ for $n > N^*$, which clearly is impossible.

4. Plane of attainable points. We now turn to the special cases in which every point of the complex plane is an attainable point.

4.1. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to zero, and let $\sum |c_i| = \infty$. If $\sum c_i$ has at least two axes of divergence, then every complex number C is an attainable point of $\sum c_i$.*

Proof. By an affine transformation the two axes of divergence may be identified with the real and imaginary axes respectively.

The definition of axes of divergence implies the existence of two disjoint subseries $\sum c_{i'_k}$ and $\sum c_{i''_k}$ of $\sum c_i$ such that:

$$\frac{b_{i'_k}}{a_{i'_k}} \longrightarrow 0, \sum |a_{i'_k}| = \infty \text{ and } a_{i'_k} \neq 0 \qquad (k = 1, 2, \cdots),$$

$$\frac{a_{i''_k}}{b_{i''_k}} \longrightarrow 0, \sum |b_{i''_k}| = \infty \text{ and } b_{i''_k} \neq 0 \qquad (k = 1, 2, \cdots).$$

We shall now fix finite subseries

$$\sum_{l=1}^{k'_n} c_{i'_l(n)} \quad \text{and} \quad \sum_{l=1}^{k''_n} c_{i''_l(n)} \qquad (n = 1, 2, \cdots)$$

of $\sum c_{i'_k}$ and $\sum c_{i''_k}$, respectively, and N_n $(n = 1, 2, \cdots)$ as follows:

a) for every $i > N_n$, $|c_i| < 2^{-n}$;

b) for every $i'_k > N_n$, $|b_{i'_k}/a_{i'_k}| < 2^{-n}$, and

for every $i''_k > N_n$, $|a_{i''_k}/b_{i''_k}| < 2^{-n}$;

c) $i'^{(n-1)}_{k'_{n-1}} \le N_n < i'^{(n)}_1$ and $i''^{(n-1)}_{k''_{n-1}} \le N_n < i''^{(n)}_1$;

d) $1 \le \sum_{l=1}^{k'_n} |a_{i'_l(n)}| < 1 + 2^{-n}$ and $1 \le \sum_{l=1}^{k''_n} |b_{i''_l(n)}| < 1 + 2^{-n}$.

From b) and d) we obtain

$$\sum_{l=1}^{k'_n} |b_{i'_l(n)}| < 2^{-n+1} \quad \text{and} \quad \sum_{l=1}^{k''_n} |a_{i''_l(n)}| < 2^{-n+1}.$$

We denote by $\sum c_{i'''_k}$ what remains of the series $\sum c_i$ after the subseries

$$\sum_n \sum_{l=1}^{k'_n} c_{i'_l(n)} \quad \text{and} \quad \sum_n \sum_{l=1}^{k''_n} c_{i''_l(n)}$$

are removed.

According to 1.2, there exists a sequence $\epsilon_{i'''_k}$ $(k = 1, 2, \cdots)$ such that $\sum \epsilon_{i'''_k} c_{i'''_k}$ converges. We put $C_1 = \sum \epsilon_{i'''_k} c_{i'''_k}$. We construct by induction a sequence of points C_n $(n = 1, 2, \cdots)$. Suppose that we have already fixed $C_1, C_2, \cdots, C_n$; we proceed to construct C_{n+1}. We fix signs $\epsilon_{i'_l(n)}$ $(l = 1, 2, \cdots k'_n)$ so that, by addition of $-\epsilon_{i'_l(n)} a_{i'_l(n)}$ to

$$A - \left(A_n + \sum_{q=1}^{l-1} \epsilon_{i'_q(n)} a_{i'_q(n)} \right),$$

this expression either diminishes in absolute value or changes sign*. We put then

$$C_n' = C_n + \sum_{l=1}^{k_n'} \epsilon_{i_l'(n)} c_{i_l'(n)}.$$

Similarly we put

$$C_{n+1} = C_n' + \sum_{l=1}^{k_n''} \epsilon_{i_l''(n)} c_{i_l''(n)},$$

where $\epsilon_{i_l''(n)}$ ($l = 1, 2, \cdots, k_n''$) are fixed so that, by adding $-\epsilon_{i_l''(n)} b_{i_l''(n)}$ to

$$B - \left(B_n' + \sum_{q=1}^{l-1} \epsilon_{i_q''(n)} b_{i_q''(n)}\right),$$

this expression either diminishes in absolute value or changes sign*. The series $\sum \epsilon_i c_i$ is composed of three interwoven subseries

$$\sum_n \sum_{l=1}^{k_n'} \epsilon_{i_l'(n)} c_{i_l'(n)}, \quad \sum_n \sum_{l=1}^{k_n''} \epsilon_{i_l''(n)} c_{i_l''(n)}, \quad \text{and} \quad \sum_k \epsilon_{i_k'''} c_{i_k'''},$$

which evidently are convergent and in which the order of terms remains unchanged. Consequently $\sum \epsilon_i c_i$ converges; and, as $C_n \longrightarrow C$, also $\sum \epsilon_i c_i = C$.

4.2. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to zero, having exactly one axis of divergence. If $\sum c_i$ can be separated into two subseries $\sum c_{i_k}^{-}$ and $\sum c_{i_k}^{=}$, such that the convergence strip of $\sum c_{i_k}^{-}$ is the whole plane, and the attainable points of $\sum c_{i_k}^{=}$ cover a segment not parallel to the axis of divergence, then every complex number C is an attainable point of $\sum c_i$.*

This theorem is a direct outcome of Theorem 3.1.

4.3. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to zero, having exactly one axis of divergence, and let γ' be a direction perpendicular to this axis. If $\sum c_i$ can be separated into two subseries $\sum c_{i_k'}$ and*

* Whenever this expression equals zero we put the next ϵ equal to + 1.

$\sum c_{i_k''}$ *such that the convergence strip of* $\sum c_{i_k'}$ *is the whole plane,* $\sum |c_{i_k''}|$ *converges, and*

$$(1) \qquad 0 < |(c_{i_k''}, \gamma')| \leq \sum_{l=k+1}^{\infty} |(c_{i_l''}, \gamma')| \qquad (k = 1, 2, \cdots),$$

then every complex number C *is an attainable point of* $\sum c_i$.

Proof. As usual, we assume that the axis of divergence is the real axis. Let $\sum c_{\bar{i}_k''(1)}$ be a tail of the series $\sum c_{i_k''}$ such that

$$\sum |a_{\bar{i}_k''(1)}| < \frac{1}{2},$$

and let η be any real number satisfying

$$(2) \qquad 0 < \eta < |b_{\bar{i}_1''(1)}|.$$

We form finite subseries

$$\sum_{l=1}^{k_n'} c_{i_l'(n)} \qquad (n = 1, 2, \cdots)$$

of $\sum c_{i_k'}$ so that the following conditions are satisfied:

$$i_{k_{n-1}'}'^{(n-1)} < i_1'^{(n)};$$

for every term $c_{i_l'(n)}$ $(l = 1, 2, \cdots, k_n';\ n = 1, 2, \cdots)$, we have

$$(3) \qquad |a_{i_l'(n)}| < 2^{-n-3}$$

and

$$0 < \left| \frac{b_{i_l'(n)}}{a_{i_l'(n)}} \right| < \eta \cdot 2^{-n-2};$$

$$1 \leq \sum_{l=1}^{k_n'} |a_{i_l'(n)}| < 1 + 2^{-n-3}. \tag{4}$$

Consequently we have also

$$\sum_{l=1}^{k_n'} |b_{i_l'(n)}| < \eta \cdot 2^{-n-1}. \tag{5}$$

We denote by $\sum c_{i_k'''}$ what remains from the series $\sum c_i$ after the series

$$\sum_n \sum_{l=1}^{k_n'} c_{i_l'(n)} \quad \text{and} \quad \sum c_{\bar{i}_k''(1)}$$

are removed. In consideration of

$$\sum |b_{i_k'}| = \infty \quad \text{and} \quad \sum_n \sum_{l=1}^{k_n'} |b_{i_l'(n)}| < \frac{1}{2}\eta,$$

we get

$$\sum |b_{i_k'''}| = \infty,$$

and consequently also

$$\sum |a_{i_k'''}| = \infty.$$

The convergence strip of $\sum c_{i_k'''}$ is therefore the whole plane. By Theorem 3.1, there exists a sequence $\epsilon_{i_k'''}$ $(k = 1, 2, \cdots)$ such that $\sum \epsilon_{i_k'''} c_{i_k'''} = C_1$, with

$$A_1 = A, \ |B - B_1| < \eta. \tag{6}$$

We denote by $\sum_{l=1}^{k_n''} c_{i_l''(n)}$ some head and by $\sum_k c_{\bar{i}_k''(n+1)}$ the corresponding tail of $\sum c_{\bar{i}_k''(n)}$, and we construct by induction a sequence of points C_p, a increasing sequence of integers n_p and a sequence of integers k_p'' $(p = 1, 2, \cdots)$ having the following properties:

$$(7) \qquad |A - A_p| < \sum_{l=1}^{k''_{p-1}} |a_{i''_l(p-1)}| + 2^{-n_{p-2}-1} \qquad (n_{-1} = n_0 = 0),$$

$$(8) \qquad |B - B_p| < \frac{1}{2} \sum |b_{\bar{i}''_k(p)}|,$$

$$(9) \qquad \eta \cdot 2^{-n_{p-1}} < \frac{1}{2} \sum |b_{\bar{i}''_k(p)}|.$$

It can easily be verified with the use of (6), (2), and (1) that (7), (8), and (9) hold for $p = 1$.

Let us now suppose that we have already

$$n_q \text{ and } \sum_{l=1}^{k''_q} c_{i''_l(q)} \quad (q = 1, 2, \cdots, p-1) \text{ and } C_q \quad (q = 1, 2, \cdots, p),$$

and we proceed to construct

$$n_p, \ \sum_{l=1}^{k''_p} c_{i''_l(p)}, \text{ and } C_{p+1}.$$

We fix $\epsilon_{i'_l(n_{p-1}+1)}$ $(l = 1, 2, \cdots, k'_{n_{p-1}+1} - 1)$ so that by addition of

$$-\epsilon_{i'_l(n_{p-1}+1)}\, a_{i'_l(n_{p-1}+1)} \text{ to } \left[A - \left(A_p + \sum_{q=1}^{l-1} \epsilon_{i'_q(n_{p-1}+1)}\, a_{i'_q(n_{p-1}+1)}\right)\right],$$

this expression either diminishes in absolute value or changes sign. Now $\epsilon_{i'_Q(n_{p-1}+1)}$, where $Q = k'_{n_{p-1}+1}$, is fixed so that

$$B - \left(B_p + \sum_{l=1}^{a} \epsilon_{i'_l(n_{p-1}+1)}\, b_{i'_l(n_{p-1}+1)}\right) \neq 0.$$

We put then

$$C'_p = C_p + \sum_{l=1}^{a} \epsilon_{i'_l(n_{p-1}+1)}\, c_{i'_l(n_{p-1}+1)},$$

and fix $n_p > n_{p-1}$ so that

$$(10) \qquad \eta \cdot 2^{-n_p} < \frac{1}{6} |B - B_p'|.$$

We proceed as before and fix $\bar{\epsilon}_{i_l'(q)}$ $(l = 1, 2, \cdots, k_q'; q = n_{p-1}+2, n_{p-1}+3, \cdots, n_p)$ so that by addition of

$$-\bar{\epsilon}_{i_l'(q)} a_{i_l'(q)} \text{ to } \left[A - \left(A_p' + \sum_{r=n_{p-1}+2}^{q-1} \sum_{s=1}^{k_r'} \bar{\epsilon}_{i_s'(r)} a_{i_s'(r)} + \sum_{s=1}^{l-1} \bar{\epsilon}_{i_s'(q)} a_{i_s'(q)}\right)\right]$$

this expression either diminishes in absolute value or changes sign. If

$$\left|B - \left(B_p' + \sum_{q=n_{p-1}+2}^{n_p} \sum_{l=1}^{k_q'} \bar{\epsilon}_{i_l'(q)} b_{i_l'(q)}\right)\right| \geq |B - B_p'|,$$

we leave $\epsilon_{i_l'(q)} = \bar{\epsilon}_{i_l'(q)}$; otherwise we put $\epsilon_{i_l'(q)} = -\bar{\epsilon}_{i_l'(q)}$. In either case, we denote

$$C_p'' = C_p' + \sum_{q=n_{p-1}+2}^{n_p} \sum_{l=1}^{k_q'} \epsilon_{i_l'(q)} c_{i_l'(q)}.$$

By (8), (5), and (9), we have

$$|B - B_p''| \leq |B - B_p| + |B_p - B_p''|$$

$$< \frac{1}{2} \sum |b_{\bar{i}_k''(p)}| + \eta \cdot 2^{-n_{p-1}-1} < \frac{3}{4} \sum |b_{\bar{i}_k''(p)}|.$$

On the other hand we have, by (10), $|B - B_p''| \geq |B - B_p'| > 6\eta \cdot 2^{-n_p}$. Consequently,

$$(11) \qquad 6\eta \cdot 2^{-n_p} < |B - B_p''| < \frac{3}{4} \sum |b_{\bar{i}_k''(p)}|.$$

We now fix $\epsilon_{i_1''(p)}$ so that

$$|B - (B_p'' + \epsilon_{i_1''(p)}\, b_{i_1''(p)})| < |B - B_p''|,$$

or

$$[B - (B_p'' + \epsilon_{i_1''(p)}\, b_{i_1''(p)})] \cdot (B - B_p'') < 0,$$

and $\epsilon_{i_l''(p)}$ $(l = 2, 3, \cdots, k_p'')$ so that by addition of

$$-\epsilon_{i_l''(p)}\, b_{i_l''(p)} \text{ to } \left[B - \left(B_p'' + \sum_{q=1}^{l-1} \epsilon_{i_q''(p)}\, b_{i_q''(p)}\right)\right],$$

this expression diminishes in absolute value without changing sign, $b_{i_{k_p''}''(p)}$ being the last term of $\sum b_{\bar{i}_l''(p)}$ for which such operation is possible.

Such $b_{i_{k_p''}''(p)}$ exists in view of (11) and (1).

We put

$$C_{p+1} = C_p'' + \sum_{l=1}^{k_p''} \epsilon_{i_l''(p)}\, c_{i_l''(p)}.$$

The construction of n_p, $\sum_{l=1}^{k_p''} c_{i_l''(p)}$, and C_{p+1} is thus completed. It remains to show that conditions (7)-(9) are fulfilled for these indices.

We have

$$|A - A_{p+1}| \le |A - A_p'| + |A_p' - A_p''| + |A_p'' - A_{p+1}|;$$

but in view of (7), (4), and (3),

$$|A - A_p'| \le 2 \max_{l=1,2,\cdots,k'_{n_{p-1}+1}} |a_{i_l'(n_{p-1}+1)}| < 2^{-n_{p-1}-3},$$

$$|A_p' - A_p''| \le |A - A_p'| + \max_{l=1,2,\cdots,k'_{n_p}} |a_{i_l'(n_p)}| < 3.2^{-n_{p-1}-4},$$

and

$$|A_p'' - A_{p+1}| \le \sum_{l=1}^{k_p''} |a_{i_l''(p)}|.$$

Consequently,

$$|A - A_{p+1}| < \sum_{l=1}^{k_p''} |a_{i_l''(p)}| + 2^{-n_{p+1}-1},$$

so that (7) holds for this index.

For (8), we note that clearly

$$|B - B_{p+1}| < |b_{\bar{i}_1''(p+1)}|,$$

and therefore, in view of (1),

$$|B - B_{p+1}| < \frac{1}{2} \sum |b_{\bar{i}_k''(p+1)}|.$$

Finally, if

$$[B - (B_p'' + \epsilon_{i_1''(p)} b_{i_1''(p)})] \cdot (B - B_p'') \geq 0,$$

then in view of (11) we have

$$\sum |b_{\bar{i}_k''(p+1)}| = \sum |b_{\bar{i}_k''(p)}| - \sum_{l=1}^{k_p''} |b_{i_l''(p)}| \geq \sum |b_{\bar{i}_k''(p)}| - |B - B_p''|$$

$$> \frac{1}{4} \sum |b_{\bar{i}_k''(p)}| > 2\eta \cdot 2^{-n_p}.$$

If, on the other hand

$$[B - (B_p'' + \epsilon_{i_1''(p)} b_{i_1''(p)})] \cdot (B - B_p'') < 0,$$

then by (1) and (11) we have

$$\sum |b_{\bar{i}_k''(p+1)}| \geq |B - B_p''| > 6\eta \cdot 2^{-n_p}.$$

Thus (9) holds in either case.

In order to prove that $\sum \epsilon_i c_i$ converges it is sufficient to point out that this series is composed of three interwoven subseries

$$\sum \epsilon_{i_k'''} c_{i_k'''}, \quad \sum_n \sum_{l=1}^{k_n'} \epsilon_{i_l'(n)} c_{i_l'(n)}, \quad \text{and} \quad \sum \epsilon_{\bar{i}_k''(1)} c_{\bar{i}_k''(1)}$$

which are evidently convergent and in which the order of the terms remains unchanged.

As, according to (7) and (8) above, we have $C_p \longrightarrow C$, it follows that $\sum \epsilon_i c_i = C$.

4.4. The following examples illustrate the way in which the above result may be applied:

a) Let

$$\sum_n \left(\frac{1}{\sqrt{n}} + \frac{1}{n} i \right)$$

be the series in question.

If we put

$$\sum_k c_{n_k''} = \sum_k \left(\frac{1}{\sqrt{2^k}} + \frac{1}{2^k} i \right),$$

that is, the subseries of those terms for which n is a power of 2, and $\sum_l c_{n_l'}$, the remaining subseries, then the assumptions of Theorem 4.3 are fulfilled, and therefore every complex number C is an attainable point of our series.

b) The terms of the subseries $\sum_k c_{i_k''}$ may be composed of two or more terms of the series $\sum c_i$, as the following example shows:

$$\sum c_n = \sum_n \left(a_n + \frac{1}{n} i \right),$$

where

$$0 < a_{n+1} \leq a_n \quad (n = 1, 2, \cdots), \; a_n \longrightarrow 0, \; \text{and} \; n a_n \longrightarrow \infty.$$

If we put

$$\sum_k c_{n_k''} = \sum_{k=2}^{\infty} \left[(a_{3k} - a_{3k+1}) + \left(\frac{1}{3k} - \frac{1}{3k+1} \right) i \right],$$

and $\sum_l c_{n_l}'$ the remaining subseries, the assumptions of Theorem 4.3 are fulfilled, and in this case too every complex number C is an attainable point of c_i.

5. Further considerations. We make the following observations.

5.1. For an absolutely convergent series $\sum c_i$ of complex numbers, the attainable points form a perfect set. The proof does not vary from the proof of a well-known similar theorem for series of real numbers (see 1.1).

5.2. Instead of $\epsilon = \pm 1$, more general convergence- and sum-factors have been introduced by E. Calabi and A. Dvoretzky [1]. They call a set Z of complex numbers a sum-factor set if, given any series $\sum c_i$ ($\sum |c_i| = \infty$, $c_i \longrightarrow 0$), and any number C, there exists a sequence $\zeta_n \in Z$ ($n = 1, 2, \cdots$) for which $\sum_n \zeta_n c_n = C$. It was shown by them that a bounded set Z is a sum-factor set if and only if 0 is an interior point of its convex hull.

5.3. All the theorems proved in this paper may reasonably be extended to results concerning vectors in n-dimensional Euclidean spaces.

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ON THE PRIME IDEALS OF THE RING OF ENTIRE FUNCTIONS

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1. Introduction. Let R be the ring of entire functions, and let K be the complex field. In an earlier paper [6], the author investigated the ideal structure of R, particular attention being paid to the maximal ideals. In 1946, Schilling [9, Lemma 5] stated that every prime ideal of R is maximal. Recently, I. Kaplansky pointed out to the author (in conversation) that this statement is false, and constructed a nonmaximal prime ideal of R (see Theorem 1(a), below). The purpose of the present paper is to investigate these nonmaximal prime ideals and their residue class fields. The author is indebted to Prof. Kaplansky for making this investigation possible.

The nonmaximal prime ideals are characterized within the class of prime ideals, and it is shown that each prime ideal is contained in a unique maximal ideal. The intersection P^* of all powers of a maximal free ideal M is the largest nonmaximal prime ideal contained in M. The set P_M of all prime ideals contained in M is linearly ordered under set inclusion, and distinct elements P of P_M correspond in a natural way to distinct rates of growth of the multiplicities of the zeros of functions f in P.

It is shown that the residue class ring R/P of a nonmaximal prime ideal P of R is a valuation ring whose unique maximal ideal is principal; R/P is Noetherian if and only if $P = P^*$. The residue class ring R/P^* is isomorphic to the ring $K\{z\}$ of all formal power series over K. The structure theory of Cohen [2] of complete local rings is used.

2. Notation and preliminaries. A familiarity with the contents of [6] is assumed, but some of it will be reproduced below for the sake of completeness.

DEFINITION 1. If $f \in R$, and I is any nonvoid subset of R, let:

(a) $A(f) = [z \in K \mid f(z) = 0]$ (Note that multiple zeros are repeated. Unions and intersections are taken in the same sense.);

(b) $A(I) = [A(f) \mid f \in I]$;

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(c) $A^*(f)$ be the sequence of distinct zeros of f, arranged in order of increasing modulus.

In 1940, Helmer showed [**5**, Theorem 9] that if $A(f) \cap A(g)$ is empty, there exist s, t in R such that

$$sf + tg = 1. \tag{2.1}$$

More generally, if d is any element of R such that

$$A(d) = A(f) \cap A(g),$$

then d is a greatest common divisor of f and g, unique to within a unit factor, and the ideal (f, g) generated by f and g is the principal ideal (d). It easily follows that every finitely generated ideal of R is principal.

He proved this by showing that if $\{a_n\}$ is any sequence of complex numbers such that

$$\lim_{n\to\infty} a_n = \infty,$$

and $w_{n,k}$ is any set of complex numbers, then there is an s in R such that

$$s^{(k)}(a_n) = w_{n,k}, \quad (n = 1, 2, \cdots; k = 0, \cdots, 1_n). \tag{2.2}$$

The latter was shown independently by Germay [3].

REMARK. In [**4**], Germay extended (2.2) to the ring of functions analytic in $|z| < r$, where $\lim_{n\to\infty} a_n$ lies on $|z| = r$. Hence (2.1) follows for this ring, as will most of the results in [**6**] and the present paper, with minor modification.

It follows that if I is an ideal of R, then $A(I)$ has the finite intersection property. So we make the following definition.

DEFINITION 2. If $\bigcap_{f\in I} A(f)$ is nonempty, then I is called a *fixed* ideal. Otherwise, I is called a *free* ideal.

DEFINITION 3. (a) If $A^*(f) = \{a_n\}$, let $0_n(f)$ be the multiplicity of a_n as a zero of f.

(b) If A is a nonvoid subset of $A^*(f)$, let $0_n(f:A)$ be the function $0_n(f)$ with domain restricted to A.

(c) Let $m(f) = \sup_{n\geq 1} 0_n(f)$, if $f \neq 0$. Let $m(0) = \infty$.

3. Prime ideals of R. Kaplansky's construction of nonmaximal, prime ideals of R is given in Theorem 1(a), below. The only fallacy in Schilling's demonstration (referred to in the Introduction) is the false assumption that a prime ideal necessarily contains an f such that $m(f) = 1$. Hence a characterization of these nonmaximal prime ideals may be given.

THEOREM 1. (a) *There exist nonmaximal prime ideals of* R.

(b) *A necessary and sufficient condition that a prime ideal* P *of* R *be nonmaximal is that* $m(f) = \infty$, *for all* $f \in P$.

Proof. (a) Let

$$S = [f \in R \mid m(f) < \infty].$$

Clearly, S is closed under multiplication and does not contain 0. If $g \neq 0$ is in $R - S$, g is contained in a prime ideal P not intersecting S (see [**8**, p. 105]). Since, as noted in [**6**, p. 183], any maximal ideal contains an f such that $m(f) = 1$, P cannot be maximal.

(b) The sufficiency is clear from the above. If $f \in P$ with $m(f) < \infty$, the primality of P ensures that there is a $g \in P$ with $m(g) = 1$. Suppose the maximal ideal M contains P, and let $h \in M$. By (2.1), there is a $d \in M$ such that

$$A(d) = A(g) \cap A(h).$$

Now $g = g_1 d$, where $A(g_1) \cap A(d)$ is empty, since $m(g) = 1$. Since P is prime, it follows that either $g_1 \in P$ or $d \in P$. But $M \neq R$, so g_1 is not in P. It follows that d, and hence h, is in P, whence $P = M$.

COROLLARY. *Any prime, fixed ideal of* R *is maximal.*

THEOREM 2. *Every prime ideal* P *of* R *is contained in a unique maximal (free) ideal* M.

Proof. By Theorem 1(b) and [**6**, Theorem 4], the ideal (P, f) is maximal if $m(f) = 1$ and $A(f)$ intersects every element of $A(P)$. Let f_1, f_2 be any two such functions, so that $M_1 = (P, f_1)$ and $M_2 = (P, f_2)$ are maximal ideals containing P. If

$$A(d) = A(f_1) \cap A(f_2),$$

then $M = (P, d)$ is a maximal ideal containing P, and $M_1 \subset M$, $M_2 \subset M$, so that

$$M_1 = M_2 = M.$$

More concrete constructions of nonmaximal prime ideals are given below in terms of maximal free ideals.

THEOREM 3. *If M is a maximal free ideal of R, then*

$$P^* = \bigcap_{k=1}^{\infty} M^k$$

is a prime ideal, and is the largest nonmaximal prime ideal contained in M.

Proof. Since every finitely generated ideal of R is principal, P^* is easily seen to be the set of all $f \in R$ expressible in the form $h_k d_k^k$, with $d_k \in M$, $k = 1, 2, \cdots$. Thus, if $f \in M$, $f \in P^*$ if and only if $m(f/e) = \infty$ whenever e divides f and $e \in R - M$, (whence $f/e \in M$). Suppose f_1, f_2 are not in P^*. Clearly, $f_1 f_2$ is not in P^* except possibly when both f_1 and f_2 are in M. In this case, there exist e_i dividing f_i, with $e_i \in R - M$ such that $m(f_i/e_i) < \infty$, $(i = 1, 2)$. Since M is prime, $e_1 e_2 \in R - M$ and $m(f_1 f_2/e_1 e_2) \leq m(f_1/e_1) + m(f_2/e_2) < \infty$. So $f_1 f_2$ is not in P^*, whence P^* is a prime ideal.

The second part of the Theorem is a direct consequence of Theorem 1 (b).

We proceed now to identify the remainder of the class P_M of prime ideals contained in M. This is done by considering the rates of growth of the functions $0_n(f)$ on the filter $A(M)$. Results of Bourbaki [1] are used without further acknowledgement.

DEFINITION 4. If $f, g \in M$, and there is an $e \in M$ such that

$$A^*(e) \subset A^*(f) \cap A^*(g)$$

with

$$0_n(f{:}A^*(e)) \geq 0_n(g{:}A^*(e)),$$

then $f \geq g$ $(g \leq f)$.

It is easily seen that the relation "$\geq$" is reflexive and transitive. Moreover:

LEMMA 1. *If $f, g \in M$, either $f \geq g$ or $g \geq f$.*

Proof. Let

$$A(d) = A(f) \cap A(g),$$

and let

$$A_1 = [z \in A^*(d) \mid 0_n(f:\{z\}) \geq 0_n(g:\{z\})],$$
$$A_2 = [z \in A^*(d) \mid 0_n(f:\{z\}) < 0_n(g:\{z\})].$$

Since $A_1 \cap A_2$ is empty, $A_1 \cup A_2 = A^*(d)$; and since M is prime, one and only one of $A_1, A_2 \in M$. Hence $f \geq g$ or $g \geq f$.

DEFINITION 5. Suppose $f, g \in M$.

(a) If there exist positive integers N_1, N_2 such that $f^{N_1} \geq g$ and $g^{N_2} \geq f$, then $f \sim g$.

(b) If $f \geq g^N$ for all positive integers N or if $f = 0$, then $f \gg g$ ($g \ll f$).

LEMMA 2. (a) *The relation '$\sim$' is an equivalence relation.*

(b) *The relation '$\gg$' is transitive.*

(c) *If $f, g \in M$, one and only one of $f \sim g$, $f \gg g$, $f \ll g$ holds.*

Proof. The relations (a) and (b) follow easily from the observations that

$$0_n(f^N) = N \cdot 0_n(f), \text{ and if } f \geq g \text{ then } f^N \geq g^N.$$

It is clear that at most one of the relations (c) can hold. By Lemma 1, $f \geq g$ or $g \geq f$. Suppose $f \geq g$ and not $f \sim g$; then $f \geq g^N$ for all N, whence $f \gg g$. Similarly, if $g \geq f$.

LEMMA 3. *Let f be an element of a prime ideal P of P_M. If $g \geq f$, or $g \sim f$, then $g \in P$.*

Proof. Suppose first that $g \geq f$. Then, as is evident from the construction in Lemma 1, we can write

$$f = f_1 d_1, \; g = g_1 d_2,$$

where

$$A^*(d_1) = A^*(d_2), \quad 0_n(d_2) \geq 0_n(d_1),$$

and f_1, g_1 are not in M. Hence $d_1 \in P$; and, since d_2 is a multiple of d_1, d_2 and $g \in P$. If $g \sim f$, then $g^N \geq f$, for some N. By the above, $g^N \in P$. But P is a prime ideal, so $g \in P$.

THEOREM 4. (a) *Let Ω be any subset of M, and let*

$$P_\Omega = [f \in M \mid f \gg g, \; \textit{for all } g \in \Omega].$$

Then P_Ω is a prime ideal.

(b) *If P is a prime ideal, then $P = P_\Omega$*, where $\Omega = M - P$.

Proof. (a) Note first that if $g_1, g_2 \in M$ and $g_1 g_2 \neq 0$

$$A = A^*(g_1) \cap A^*(g_2),$$

then

$$0_n(g_1 - g_2 : A) = \min \{0_n(g_1 : A),\ 0_n(g_2 : A)\}.$$

If $g_1 \in M$, $g_2 \in R$, $g_1 g_2 \neq 0$, then

$$0_n(g_1 g_2 : A^*(g_1)) = 0_n(g_1 : A^*(g_1)) + 0_n(g_2 : A^*(g_1)).$$

It now follows from the lemmas above that P is an ideal. The primality of P follows from the observation that

$$P_g = [f \in M \mid f >> g]$$

is a prime ideal, and that P_Ω is an intersection of a descending chain (under set inclusion) of ideals of this form.

(b) If P is a prime ideal, the relations $f \in P$, $g \in M - P$, imply that $f >> g$, by Lemma 3.

COROLLARY. *The ideals of P_M are linearly ordered under set inclusion.*

By the Theorem above, every element of P_M is the upper class of a Dedekind cut (under $<<$). If P contains a least element f, then

$$P = P_f^+ = [g \in M \mid g >> f \text{ or } g \sim f].$$

If $M - P$ has a greatest element g, then $P = P_g$ as defined in the proof of the theorem. It is clear that P_M contains the greatest lower bound and least upper bound of any set of elements.

Note, moreover that $P_{f_1} = P_{f_2}$ ($P_{f_1}^+ = P_{f_2}^+$) if and only if $f_1 \sim f_2$.

LEMMA 4. *The set $P^* - \{0\}$ has no countable cofinal or coinitial subset. Moreover, if $\{f_{1,n}\}$, $\{f_{2,n}\}$ are two sequences of nonzero elements of P^*, such that*

$$f_{1,n+1} >> f_{1,n} >> f_{2,m} >> f_{2,m+1}, \qquad \textit{for all } n, m,$$

then there is an $f \in P^*$ such that

$$f_{1,n} \gg f \gg f_{2,m}, \qquad \textit{for all } n, m.$$

Proof. See [1, p. 123, exercise 8].

The author is indebted to Dr. P. Erdös and Dr. L. Gillman for the following Theorem.

THEOREM 5. *The set P_M has power at least $2^{\aleph_1}$.*

Proof. It is implicit in arguments of Hausdorff and Sierpinski [10, p. 62] that every set satisfying Lemma 4 contains a subset similar to the lexicographically ordered set S of ω_1-sequences of 0's and 1's, each having at most countably many 1's By [11], S is dense in the set of all dyadic ω_1-sequences, which has power $2^{\aleph_1}$. Since the set P_M is complete, card $(P_M) \geq 2^{\aleph_1}$.

Since card $(P_M) \leq 2^c$, where c is the cardinal number of the continuum, we have:

COROLLARY. *If $2^{\aleph_1} = 2^c$, in particular if $\aleph_1 = c$, then* card $(P_M) = 2^c$.

4. Residue class rings of prime ideals. We adopt the following definition of Krull [7, p. 110]:

DEFINITION 6. An integral domain D such that if $f, g \in D$, then f divides g or g divides f, is called a *valuation ring.*

It is easily seen that a valuation ring possesses a unique maximal ideal, consisting of all its nonunits.

THEOREM 6. *The residue class ring R/P of a prime ideal P of R is a valuaring whose unique maximal ideal is principal.*

First, we prove a lemma.

LEMMA 5. *If $P \in P_M$, then f is singular modulo P if and only if $f \in M$.*

Proof. Consider the equation

$$fX \equiv 1 \pmod{P}.$$

If $f \in M$, the equation clearly has no solution since $A(f) \cap A(p)$ is nonempty for all $p \in P$ (see [6, Theorem 4]).

On the other hand, if f is not in M, there is a $p \in P$ such that $A(f) \cap A(p)$ is empty. Let $A^*(p) = \{a_n\}$, with $0_n(p) = l_n$, in which case $f(a_n) \neq 0$. The equation in question has a solution if and only if there exists a $g \in R$ such that

(i) $g(a_n) = \{f(a_n)\}^{-1}$,

and

(ii) $(fg)^{(k)}(a_n) = 0, \; k = 1, \cdots, l_n$.

Since

$$(fg)^{(k)} = fg^{(k)} + \sum_{i=1}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}, \qquad \text{where } \binom{k}{i} = \frac{k!}{i!(k-i)!},$$

(ii) is satisfied if

(iii) $g^{(k)}(a_n) = -\{f(a_n)\}^{-1} \sum_{i=1}^{k} \binom{k}{i} f^{(i)}(a_n)\, g^{(k-i)}(a_n)$.

Such a g can be constructed by (2.2), whence

$$fg \equiv 1 \pmod{P}.$$

Proof of Theorem 6. By Lemma 5, every element of $R - M$ is a unit, so we may assume that $f, g \in M$. Let

$$A(d) = A(f) \cap A(g),$$

so that $A(f/d) \cap A(g/d)$ is empty. Clearly, at least one of $f/d, g/d \in R - M$, and hence is a unit modulo P. So R/P is a valuation ring.

If, in particular, f is chosen to be in $M - M^2$, f/d cannot be in M, so g is a multiple (modulo P) of f. Therefore the unique maximal ideal M/P of R/P is generated by f, and hence is principal.

If $P \neq P^*$, R/P possesses the nonmaximal prime ideals P_1/P, where P_1 is a nonmaximal prime ideal of R properly containing P. Moreover:

THEOREM 7. *The residue class ring R/P of a nonmaximal prime ideal P is Noetherian if and only if $P = P^*$.*

Proof. Every nonzero element of $M - P^*$ is in $M^k - M^{k-1}$, for some unique positive integer k. Hence every nonzero ideal of R/P^* is of the form (f^k), where $f \in M - M^2$.

If $f \in P - P^*$, construct f_k such that

$$A^*(f_k) = A^*(f)$$

and

$$0_n(f_k) = \max\{0_n(f) - k,\ 1\}.$$

Then f_{k+1} is a proper divisor (modulo P) of f_k. Hence the ideal generated by all the f_k does not have finite basis.

The residue class ring R/P^* is concretely identified below by the use of the structure theory of complete local rings [2] of Cohen. First we make a definition.

DEFINITION 7. (a) If the nonunits of a Noetherian ring D with unit form a maximal ideal M such that

$$\bigcap_{k=1}^{\infty} M^k = (0),$$

D is called a *local ring.*

(b) If $f_1, \cdots, f_n$ is a minimal basis for M such that $f_1, \cdots, f_i$ generate a prime ideal $(i = 1, \cdots, n)$, S is called a *regular* local ring.

(c) Using the powers of M as a system of neighborhoods of 0, (thereby topologizing D), we call D *complete* if every Cauchy sequence in D has a (unique) limit.

THEOREM 8. *The residue class ring R/P^* is isomorphic with the ring $K\{z\}$ of all formal power series over K.*

Proof. By Theorems 3, 4, 6, R/P^* is a local ring and is trivially regular since M/P^* is principal. Cohen [2, Theorem 15] has shown that every regular, complete, local ring, whose unique maximal ideal is principal, and such that D/M is isomorphic to K, is isomorphic to $K\{z\}$. By [6, Theorem 6],

$$(R/P^*)/(M/P^*) \cong R/M \cong K.$$

The proof is completed by the following Lemma.

LEMMA 6. *The residue class ring R/P^* is complete.*

Proof. Let $\{f_k\}$ be any Cauchy sequence in R/P^*. We may assume without loss of generality that $f_{k+1} - f_k \in M^k$, since a Cauchy sequence has at most one limit. Let

$$A_k = \{a_k, a_{k+1}, \cdots\} \in A(M),$$

with all a_k distinct. Let

$$B_k = A_k \cap A(f_{k+1} - f_k).$$

Clearly, $B_k \in A(M)$, and $\bigcap_{k=1}^{\infty} B_k$ is empty. Hence, we may construct by (2.2) an $f \in R$ such that

$$f(z) = f_1(z) \qquad \text{for } z \in B_1,$$

and

$$f^{(k)}(z) = f_k^{(k)}(z) \qquad \text{for } z \in B_{k+1}.$$

Then

$$f_k \equiv f \pmod{M^k},$$

whence

$$L_{k\to\infty} f_k = f.$$

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PURDUE UNIVERSITY

COMPLETELY CONTINUOUS NORMAL OPERATORS WITH PROPERTY L

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1. Introduction. Two matrices A and B are said to have property L if it is possible to arrange their characteristic roots

$$A: \lambda_1, \lambda_2, \cdots, \lambda_n$$

$$B: \mu_1, \mu_2, \cdots, \mu_n$$

in such a way that for every α, the characteristic roots of $\alpha A + B$ are given by $\alpha \lambda_i + \mu_i$. In [1] this property is investigated, and among other things a conjecture of Kac is confirmed by showing that if A and B are hermitian, then they commute. In [2] this is generalized by replacing "hermitian" by "normal".

In this note we launch the project of generalizing such results to (complex) Hilbert space. However, since it is not clear how to formulate the problem for general operators (especially in the presence of a continuous spectrum), we shall content ourselves with the completely continuous case. For self-adjoint operators we obtain a fully satisfactory generalization (Theorem 1). For the more general case of normal operators we find ourselves obliged to make an extra assumption roughly to the effect that nonzero characteristic roots are paired only to nonzero roots. In the finite-dimensional case such an assumption would be harmless; indeed, by adding suitable constants to A and B, we could even arrange to have all the characteristic roots of A and B nonzero. It would nevertheless be of interest to determine whether this blemish can be removed from Theorem 2.

2. Remarks. Before we state the results, some remarks are in order. The number λ is a characteristic root of A if $A - \lambda I$ has a nonzero null space. If A is a completely continuous normal operator, its characteristic roots are either finite in number or form a sequence approaching zero. We have an orthogonal decomposition of the Hilbert space:

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$$H = H(0) \oplus H(\lambda_1) \oplus H(\lambda_2) \oplus \cdots,$$

where A acts on $H(\lambda_i)$ as a multiplication by λ_i. The dimension of $H(\lambda_i)$ is called the multiplicity of the characteristic root λ_i; it is finite except possibly for the characteristic root 0.

Now even though A and B are both to be normal, $\alpha A + B = C$ need not (a priori) be normal. We must accordingly give further attention to the meaning of the multiplicity of a characteristic root ν of C. For our purposes virtually any reasonable definition would do; we select the following one. We note that the null spaces of the operators $C - \nu I$, $(C - \nu I)^2$, $\cdots$ form an ascending chain, and we form their union; the dimension of this union is the multiplicity of ν. Note that this agrees with customary usage in the finite-dimensional case.

We shall need the (easily proved) additivity of the multiplicity. In detail: suppose H is an orthogonal direct sum of two closed subspaces both invariant under C; then the multiplicity of ν in the whole space is the sum of its multiplicities in the two subspaces.

3. Results. We are now ready to define property L. We do this in a way that is adequate for the proof, although it does not treat A and B symmetrically.

Let A and B be completely continuous normal operators. Let there be given two sequences λ_i, μ_i of complex numbers. We say that A and B have *property L* (relative to the two sequences) provided:

(1) The λ's constitute precisely the nonzero characteristic roots of A, each counted as often as its multiplicity.

(2) If, for a certain α and ν, there are k values of i such that $\nu = \alpha\lambda_i + \mu_i$, then $\alpha A + B$ has ν as a characteristic root at least of multiplicity k.

THEOREM 1. *Let A and B be completely continuous self-adjoint operators with property L. Then A and B commute.*

THEOREM 2. *Let A and B be completely continuous normal operators with property L, relative to the sequences λ_i and μ_i. Suppose further that the μ's are all nonzero. Then A and B commute.*

4. Proof. The two theorems can conveniently be proved simultaneously. We can suppose that λ_1 is a characteristic root of maximum absolute value, that is, $|\lambda_1| = ||A||$. For brevity write $\lambda = \lambda_1$, $\mu = \mu_1$. By an application of the definition of property L, with $\alpha = 0$, we see that μ is a characteristic root of B. We are going to prove that there exists a nonzero vector x with

$$Ax = \lambda x, \ Bx = \mu x.$$

If $\mu \neq 0$, we are ready to proceed. If $\mu = 0$, then by hypothesis both A and B are self-adjoint. We replace B by $A + B$ which is again self-adjoint; this replaces μ by

$$\lambda + \mu = \lambda \neq 0.$$

So in any event we are entitled to assume that μ is nonzero.

Let $H(\mu)$ be the (finite-dimensional) characteristic subspace of B for the characteristic root μ, and K the orthogonal complement; let E and F be the projections on $H(\mu)$ and K. We note that $B - \mu I$ is nonsingular on K; let S be defined as its inverse on K and as 0 on $H(\mu)$. Thus we have

(1) $$SF(B - \mu I) = F.$$

Next we consider $E(A - \lambda I)E$ as an operator on $H(\mu)$, and we are going to prove that it is singular. Suppose the contrary and define R to be its inverse on $H(\mu)$, 0 on K. Then R will satisfy

(2) $$R(B - \mu I) = 0, \ R(A - \lambda I)E = E, \ RF = 0.$$

Choose $\alpha \neq 0$ so that

(3) $$\|\alpha SF(A - \lambda I)(F - RAF)\| < 1.$$

By hypothesis, the operator $\alpha A + B$ has $\alpha\lambda + \mu$ as a characteristic root, say with characteristic vector $y \neq 0$. We have

(4) $$\alpha(A - \lambda I)y + (B - \mu I)y = 0.$$

Write $y = Ey + Fy$ in (4), apply R, and then use (2); we find that $Ey = -RAFy$, and so

(5) $$y = Ey + Fy = (F - RAF)y.$$

Next apply SF to (4), and use (1) and (5):

(6) $$Fy = -\alpha SF(A - \lambda I)(F - RAF)y.$$

On contemplating (6) in conjunction with (3) we see that Fy must be 0. But then $y = 0$ by (5). This contradiction shows that we were in error in supposing $E(A - \lambda I)E$ to be nonsingular on $H(\mu)$. Consequently we can find in $H(\mu)$

a nonzero vector x annihilated by $E(A-\lambda I)E$. Then since $Ex = x$, we have $EAx = \lambda x$. Form the orthogonal decomposition

$$(7) \qquad Ax = EAx + FAx = \lambda x + FAx.$$

But

$$||Ax|| \leq |\lambda| \, ||x||,$$

since the norm of A is $|\lambda|$. Hence in (7) we must actually have $Ax = \lambda x$. Also $Bx = \mu x$ since x is in $H(\mu)$, and we have fulfilled our initial objective.

Let M be the orthogonal complement of x. It follows from the additivity of multiplicity (see above) that when the operator $\alpha A + B$ is confined to M, the multiplicity of its characteristic root $\alpha\lambda + \mu$ is diminished by precisely 1, while all other characteristic roots have unchanged multiplicity. Thus A and B, confined to M, satisfy property L relative to the sequences λ_i and μ_i for $i \geq 2$. The procedure may now be repeated to get within M another joint characteristic vector for A and B. In this way we proceed down the nonzero characteristic roots of A. Finally we are left with the null space of A, which of course commutes with whatever is left of B. Hence A and B commute.

5. Remark. As soon as we know that A and B commute (and hence can be simultaneously put in diagonal form), we can assert that they satisfy property L symmetrically, and indeed various stronger statements are obvious consequences of simultaneous diagonal form.

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SOME RANDOM WALKS ARISING IN LEARNING MODELS I

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Introduction The present paper presents an analysis of certain transition operators arising in some learning models introduced by Bush and Mosteller [2]. They suppose that the organism makes a sequence of responses among a fixed finite set of alternatives and there is a probability p_s^n at moment n that response s will occur. They suppose further that the probabilities $p_s^{(n+1)}$ are determined by the p_s^n, the response s_n made after moment n, and the outcome or event r_n that follows response s_n. We shall examine in detail the one-dimensional models which occur in their theory. These models can be described in simplest form as follows: There exist two alternatives A_1 and A_2, and two possible outcomes r_1 and r_2, for each experiment. There exists a set of Markoff matrices F_{ij} which will apply where choice i was made and outcome r_j occurs. Let p represent the initial probability of choosing alternative A_2, and $1 - p$ the probability of choosing A_1. Depending on the choice and outcome, the vector $(p, 1 - p)$ is transformed by the appropriate F_{ij} into a new probability vector which represents the new probabilities of preference of A_2 and A_1, respectively, by the organism. The psychologist is interested in knowing the limiting form of the probability choice vector $(p, 1 - p)$.

The mathematical description of the simplest process of this type can be formulated as follows: A particle on the unit interval executes a random walk subject to two impulses. If it is located at the point x, then $x \longrightarrow F_1 x = \sigma x$ with probability $1 - \phi(x)$, and $x \longrightarrow F_2 x = 1 - \alpha + \alpha x$ with probability $\phi(x)$. The actual limiting behavior of x depends on the nature of $\phi(x)$. The transition operator representing the change of the distribution describing the position of the particle is given by

$$(TF)(x) = \int_0^{x/\sigma} [1 - \Phi(t)]\, dF + \int_0^{(x-1+\alpha)/\alpha} \Phi(t)\, dF.$$

We introduce an additional operator, acting on continuous functions, and given by

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$$U\pi(t) = (1 - \phi(t))\,\pi(\sigma t) + \phi(t)\,\pi(1 - \alpha + \alpha t).$$

It turns out that T is conjugate to U; hence knowing the behavior of U one obtains much information about T. This interplay shall be exploited considerably. The operator T is not weakly completely continuous, nor does it possess any kind of compactness property; thus none of the classical ergodic theorems apply to this type [3]. The limiting behavior of T^nF depends very sensitively on the assumptions made about the operators F_i and the probabilities $\phi(x)$.

Section 1 treats the case where $\phi(x) = x$. This causes the boundaries 0 and 1 to be absorbing states, and thus the limiting distribution concentrates only at these points. However, the concentration depends on the initial distribution. By examining the corresponding U in detail, we have been able to obtain much additional knowledge. For example, we have shown that if π is m times continuously differentiable then $(U^n\pi)^{(r)}$ converges uniformly for each $0 \leq r \leq m - 1$. It is worth emphasizing that the knowledge of the convergence of the distributions does not imply the uniform convergence of $U^n\pi$ for any continuous function π. Additional arguments are needed for this conclusion. In this connection, we finally remark that R. Bellman, T. Harris, and H. N. Shapiro [1] have analyzed only this case independently. They did not point out the connection between the operators T and U. The methods they used to establish the convergence of T^nF are probabilistic. Our paper in § 1 overlaps with theirs in some of the theorems, notably 6, 8, 9, 12, and 15; our results subsume theirs, and their proofs are entirely different from ours. Section 2 considers the case where $\phi(x)$ is monotone increasing and

$$|\phi(x) - \phi(y)| \leq u < 1.$$

This leads to the ergodic phenomonon, or steady-state situation, where the limiting distributions are independent of the starting distributions.

In § 3, we examine the situation $\phi(x) = 1 - x$. This corresponds to completely reflecting boundaries, and of course the ergodic phenomenon holds. Other interesting properties of the operators are also developed. We consider in § 4 the case where $\phi(x)$ is linear and monotonic decreasing. Section 5 introduces a further possibility where we allow the particle to stand still with certain probability. This type has been statistically examined by M. M. Flood [5]. In § 6 we investigate the general ergodic type where $\phi(x)$ is not necessarily linear. The arguments here combine both abstract analysis and probabilistic reasoning involving recurrent event theory. Furthermore, it is worth emphasizing, the proofs given in § 6 apply without any modifications to the case where we allow any finite number of impulses acting on the particle. In a future paper we shall present the extension of this model to the circumstance where changes in time occur continuously and the possible motion of the particle has a continuous or infinite discrete range of values.

The last section studies some of the properties of the limiting distribution in the ergodic types. It is shown in all circumstances that the limiting distribution is either singular or absolutely continuous, and the actual form depends on the value of $\alpha + \sigma$.

Most of the analysis carries over to higher dimensional models where more alternatives are allowed. In a subsequent paper we shall present this theory with other generalizations. We finally note that this paper represents a combination of abstract analysis and probability; it is hoped that the methods used will be useful for future investigations of this type.

It has been brought to my attention by the referee that the material of [**6**], [**7**], [**8**], and [**9**] relate closely to the content of this paper. This techniques seem to be different.

1. A particle undergoes a random walk on the unit interval subject to the following law: If the particle is at x, then after unit time $x \longrightarrow \alpha + (1-\alpha)x$ with probability x, and $x \longrightarrow \sigma x$ with probability $1-x$, where $0 < \alpha, \sigma < 1$. If $F(x)$ represents the cumulative distribution describing the location of x at the beginning of the time interval, with the understanding that $F(x) \equiv 1$ for $x \geq 1$ and $F(x) = 0$ for $x \leq 0$, then the new distribution locating the position of the particle at the end of the time interval is given by

$$G(x) = TF = \int_0^{x/\sigma} (1-t)\,dF(t) + \int_0^{(x-\alpha)/(1-\alpha)} t\,dF(t). \tag{1}$$

Indeed, the probability $dG(x)$ that after unit time the particle is located at x can materialize in two ways; namely, the particle was at x/σ and moved with probability $1 - x/\sigma$ to x, or it jumped with probability $(x-\alpha)/(1-\alpha)$ from $(x-\alpha)/(1-\alpha)$ to x during the unit time interval. This yields

$$dG(x) = \left(1 - \frac{x}{\sigma}\right) dF\left(\frac{x}{\sigma}\right) + \frac{x-\alpha}{1-\alpha}\; dF\left(\frac{x-\alpha}{1-\alpha}\right),$$

which easily implies the conclusion of equation (1).

Equation (1) represents the transition law for the particular Markoff process on hand.

The transformation T is easily seen to furnish a linear bounded mapping of the space of functions of bounded variation (V) on the unit interval into itself. Furthermore, T takes distributions into distributions and is of norm 1. This section investigates the behavior of T^n for large n with the aim of determining limiting properties of T^n.

We consider the following additional mapping U applied to the space of of continuous functions defined on the unit interval $C[0, 1]$:

$$(2) \qquad (U\pi)(t) = (1-t)\,\pi(\sigma t) + t\pi(\alpha + (1-\alpha)t).$$

The operator U has a probabilistic interpretation which we shall speak about later; but its direct relevance to T is given in Theorem 1. The inner-product notation

$$(\pi, F) = \int_0^1 \pi(t)\, dF(t)$$

will be extensively used.

THEOREM 1. *The conjugate map U^* to U is T.*

Proof. It is necessary to verify that $(U\pi, F) = (\pi, TF)$ for any continuous function $\pi(t)$ and any distribution $F(t)$ with $F(t) \equiv 1$ for $t \geq 1$ and $F(t) = 0$ for $t \leq 0-$. Indeed,

$$(U\pi, F) = \int (1-t)\pi(\sigma t)\, dF(t) + \int t\pi(\alpha + (1-\alpha)t)\, dF(t).$$

By a change of variable, we get

$$(U\pi, F) = \int \left(1 - \frac{t}{\sigma}\right) \pi(t)\, dF\left(\frac{t}{\sigma}\right) + \int \pi(t)\, \frac{t-\alpha}{1-\alpha}\, dF\left(\frac{t-\alpha}{1-\alpha}\right)$$

$$= \int \pi(t)\, dG(t) \text{ where } G(t) = TF.$$

The value of Theorem 1 is that, by studying the iterates of U^n, we deduce corresponding results about the conjugate operators T^n. We proceed now to study this operator U. To be complete, we should denote the operator by $U_{\sigma,\alpha}$, but where no ambiguity arises we shall drop the subscripts. Let W denote the isometry

$$W\pi(t) = \pi(1-t).$$

Clearly $W^{-1} = W$. We now observe the identity

(3) $$U_{1-\alpha,\, 1-\sigma} = W U_{\sigma,\,\alpha} W.$$

The mapping $(\sigma, \alpha) \longrightarrow (1-\alpha,\, 1-\sigma)$ of the parameter space into itself has the effect of mapping the triangle of the unit square bounded above by $1-\alpha-\sigma=0$ into the other triangle located in the unit square. This isomorphism property (3) enables us to restrict our attention to the case where $1-\alpha-\sigma \geq 0$. Corresponding theorems valid for the other circumstances, where $1-\alpha-\sigma < 0$, are deduced easily by virtue of (3) and will be summarized at the end of this section. From now on in § 2, unless explicitly stated otherwise, we shall assume that $1-\alpha-\sigma \geq 0$.

The next two theorems, which we state for completeness, are immediate from (2).

THEOREM 2. *The operator U preserves the values at* 0 *and* 1.

THEOREM 3. *The operator U is positive; that is, it transforms positive continuous functions into positive continuous functions.*

In particular, if $\pi_1(t) \geq \pi_2(t)$, for all t, then $U\pi_1 \geq U\pi_2$.

THEOREM 4. *If $\pi, \pi', \cdots, \pi^{(n)} \geq 0$, then $U\pi, (U\pi)', \cdots, (U\pi)^{(n)} \geq 0$.*

Proof. A simple calculation yields

(4) $$(U\pi)^{(n)} = (1-t)\sigma^n \pi^{(n)}(\sigma t) + t(1-\alpha)^n \pi^{(n)}(\alpha + (1-\alpha)t)$$
$$+ n(1-\alpha)^{n-1} \pi^{(n-1)}(\alpha + (1-\alpha)t) - n\sigma^{n-1} \pi^{(n-1)}(\sigma t).$$

Since

$$\sigma t < t < \alpha + (1-\alpha)t,$$

we conclude since $\pi^{(n-1)}(t)$ is monotonic increasing that

$$\pi^{(n-1)}(\alpha + (1-\alpha)t) \geq \pi^{(n-1)}(\sigma t) \geq 0.$$

The assumption that $1-\alpha \geq \sigma$ implies that $(1-\alpha)^{n-1} \geq \sigma^{n-1}$. As $\pi^{(n)}(t) \geq 0$, it follows that $(U\pi)^{(n)} \geq 0$. The same conclusion and argument apply to $(U\pi)^{(i)}$ for $0 \leq i \leq n-1$.

In particular, U transforms positive monotonic convex functions into functions of the same kind. Although in the proof of Theorem 4 we assumed the existence of derivatives, the argument can be carried through routinely at the expense of elegance, by use of the general definitions of convexity and monotonicity.

THEOREM 5. *If* $c \geq \pi^{(i)}(t) \geq 0$ *for* $0 \leq i \leq n$, *then* $(U^r\pi)^{(i)}(1) \leq K_i$ *for* $0 \leq i \leq n$ *and hence* $(U^r\pi)^{(i)}(t) \leq K_i$.

Proof. The proof is by induction. By Theorem 2, the theorem is trivially true for $i = 0$. Suppose we have established the result for the ith derivative with $0 \leq i \leq n - 1$. Equation (4) yields

$$(5) \qquad (U\pi)^{(n)}(1) - \pi^{(n)}(1) = c_1(\alpha)\pi^{(n-1)}(1) - c_2(\sigma)\pi^{(n-1)}(\sigma) + [(1-\alpha)^n - 1]\,\pi^{(n)}(1),$$

where $c_1(\alpha)$ and $c_2(\sigma)$ are constants depending only on α and σ respectively, and on n. If

$$\pi^{(n)}(1) > M(\alpha, \sigma, c),$$

where M is a constant sufficiently large, then (5) yields

$$(U\pi)^{(n)}(1) < \pi^{(n)}(1).$$

Since $c_1(\alpha)$ and $c_2(\sigma)$ do not depend on k, and by the induction hypotheses

$$|(U^k\pi)^{n-1}(x)| \leq M$$

uniformly in k and x, we find in general that when $(U^k\pi)^{(n)}(1)$ becomes larger than $M(\alpha, \sigma, c)$, then

$$(U^{k+1}\pi)^{(n)}(1) < (U^k\pi)^{(n)}(1).$$

Consequently, the iterates $(U^k\pi)^{(n)}(1)$ for $k \geq k_0$ are bounded by

$$M(\alpha, \sigma, c) + c_1(\alpha)M + c_2(\sigma)M.$$

This trivially implies the conclusion of Theorem 5.

The proof of the next theorem is due originally to R. Bellman. We present it for completeness.

THEOREM 6. *There exists at most one continuous solution of* $U\pi = \pi$ *for which* $\pi(0) = 0$ *and* $\pi(1) = 1$.

Proof. (By contradiction.) Let π_1 and π_2 denote two solutions with the prescribed boundary conditions. Put $\pi_0 = \pi_1 - \pi_2$; then $\pi_0(0) = \pi_0(1) = 0$. Let t_0 be a point where π_0 achieves its maximum. Since

$$\pi(t_0) = (1 - t_0)\pi(\sigma t_0) + t_0\pi(\alpha + (1 - \alpha)t_0),$$

we deduce that σt_0 is also a maximum point. Iterating, we find by continuity that $\pi(0) \equiv 0$ is the maximum value of $\pi(t)$. A similar argument shows that $0 = \min \pi(t)$, which implies that $\pi_1 = \pi_2$.

THEOREM 7. *For any function* $\pi(t) = t^r$ *with* $\infty > r \geq 1$, $U^n(t^r)$ *converges uniformly as* $n \longrightarrow \infty$.

Proof. Clearly $t \geq t^r > p(t)$, where

$$p(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq t_0; \\ \dfrac{t - t_0}{1 - t_0} & \text{for } t_0 \leq t \leq 1; \end{cases}$$

and t_0 is close to 1 with r fixed. Since Ut is convex by Theorem 4, and the values at 0 and 1 are fixed, we find that $t \geq Ut$. Hence

$$U^n t \geq U^{n+1} t \geq 0,$$

and $\lim U^n t = \theta(t)$ for every t. Since $\theta(t)$ is convex, and by Theorem 5 the derivatives of $U^n t$ at 1 are uniformly bounded, we conclude that $\theta(t)$ is continuous. By Dini's theorem the convergence of $U^n t$ to $\theta(t)$ is uniform. Obviously, $U\theta = \theta$. On the other hand, if t_0 is close to 1 then $(Up)'(1) < p'(1)$ (see the proof of Theorem 5). Since Theorem 4 guarantees the convexity of Up, and the slope at 0 is 0, it follows that $Up \leq p$, and hence $U^n p \leq U^{n+1} p$; therefore $\lim U^n p = \phi(t)$. Again, $\phi(t)$ is a continuous fixed point, and therefore by Theorem 6 we infer that $\phi(t) = \theta(t)$. On account of $U^n t \geq U^n t^r \geq U^n p$, we deduce that $\lim U^n t^r = \phi(t)$ with the convergence being uniform.

We denote this unique fixed point of U by $\phi_{\sigma,\alpha}(t)$, or by $\phi(t)$ whenever no ambiguity arises.

THEOREM 8. *The iterates U^n converge strongly (that is, $U^n\pi$ converges uniformly for any continuous function π).*

Proof. The constant functions are fixed points of U^n. Consequently by Theorem 7, $U^n q$ converges uniformly for any function $q(t)$ in the linear space L spanned by the functions $(1, t^r)$. The set L is dense in the space of continuous functions. Moreover, as $\|U^n\| = 1$, by a well-known theorem of Banach, $U^n q$ converges strongly when applied to any continuous function $q(t)$.

The actual limit is easily seen to be given by

$$\lim_{n\to\infty} U^n q(t) = q(1)\,\phi_{\sigma,\alpha}(t) + q(0)[1 - \phi_{\sigma,\alpha}(t)]. \tag{6}$$

This is an immediate consequence of the fact that the fixed points of U consist of the two-dimensional space spanned by the function 1 and $\phi_{\sigma,\alpha}$. Equation (6) shows that two functions q_1 and q_2 which agree at 0 and 1 have the same limit. This enables us to show:

THEOREM 9. *If $q(t)$ is any bounded function continuous at 0 and 1, then $U^n q$ converges strongly.*

Proof. Let $q(t)$, in addition to being continuous at 0 and 1, possess finite derivatives at 0 and 1. Then clearly there exist two continuous functions $h_1(t)$ and $h_2(t)$ with

$$h_1(t) \geq q(t) \geq h_2(t),$$

where $h_1(0) = h_2(0)$ and $h_1(1) = h_2(1)$. We conclude the result from this using the argument of Theorem 7 and equation (6). If now $q(t)$ is only continuous at 0 and 1, then we can find for any ϵ a $q_\epsilon(t)$ satisfying the properties assumed about $q(t)$ in the first part of the proof with $|q(t) - q_\epsilon(t)| \leq \epsilon$. As $\|U^n\| = 1$, the conclusion of the theorem now follows by a standard argument.

THEOREM 10. *If $|\pi^{(i)}(t)| \leq c_i$ for $0 \leq i \leq m$, then $|U^n \pi^{(i)}(t)| \leq c_i$ for $0 \leq i \leq m$.*

Proof. The proof is by induction. For $r = 0$, the result is trivial since U preserves positivity, and the constant functions are fixed points of U. Suppose we have established the result for $r = m - 1$. We note that

$$U\pi^{(m)} = (1-t)\sigma^m \pi^{(m)}(\sigma t) + t(1-\alpha)^{(m)}\pi^{(m)}(\alpha + (1-\alpha)t)$$

$$+ m(1-\alpha)^{m-1}\pi^{(m-1)}(\alpha + (1-\alpha)t) - m\sigma^{m-1}\pi^{(m-1)}(\sigma t).$$

This easily yields that

$$\max_t |U\pi^{(m)}(t)| \le \lambda \max |\pi^{(m)}(t)| + C \max |\pi^{(m-1)}(t)|,$$

where

$$\lambda = \max_t [(1-t)\sigma^m + t(1-\alpha)^m] < 1.$$

Therefore,

$$\max_t |(U^k\pi)^{(m)}(t)| \le \lambda \max |(U^{k-1}\pi)^{(m)}(t)| + C \max_t |(U^{k-1}\pi)^{m-1}(t)|$$

$$\le \lambda \max |(U^{(k-1)}\pi)^{(m)}(t)| + K$$

by our induction hypothesis. Iterating this last inequality gives that

$$\max_t |(U^k\pi)^{(m)}(t)| \le \sum_{i=0}^{k-1} \lambda^i K + \lambda^k \max_t |\pi^{(m)}(t)| \le M.$$

This establishes the theorem.

THEOREM 11. *If* $q(t)$ *belongs to* C^n *(n continuous derivatives), then*

$$\lim_{m\to\infty} [U^m q(t)]^{(r)}$$

converges uniformly for $0 \le r \le n-1$.

Proof. We prove the theorem only for $r=1$, for the other cases are similar. On account of Theorem 10, the uniform boundedness of $(U^m q)^{(2)}$ implies the equi-continuity of $U^m q^{(1)}$. Thus we can select a subsequence converging uniformly since $U^m q^{(1)}$ are also uniformly bounded. Let

$$\Psi(t) = \lim_{i\to\infty} U^{m_i} q^{(1)}.$$

Since $\lim U^{m_i} q$ converges uniformly to a unique limit $\theta(t)$, we obtain that $\theta'(t) = \Psi(t)$. As $\theta'(t)$ is independent of the subsequence chosen, the conclusion of the theorem easily follows.

THEOREM 12. *The fixed point* $\phi_{\sigma,\alpha}$ *is analytic for* $0 \le t \le 1$ *with* $\phi_{\sigma,\alpha}^{(r)} \ge 0$.

Proof. Let $p(t)$ denote a function infinitely differentiable with $p^{(r)}(t) \geq 0$ and $p(0) = 0$, $p(1) = 1$. By virtue of Theorem 11 and Theorem 4 we deduce that

$$\lim_{n \to \infty} (U^n p)^{(r)} = \phi_{\sigma,\alpha}^{(r)} \geq 0.$$

Therefore $\phi_{\sigma,\alpha}$ is absolutely monotonic and hence, by a well-known theorem, is analytic.

At this point it seems desirable to summarize the analogous results of Theorems 2 through Theorem 12 for the case where $\alpha + \sigma \leq 1$. We enumerate the corresponding theorems.

THEOREM 4′. *If* $(-1)^{i-1} \pi^{(i)}(t) \geq 0$ *for* $i = 0, 1, 2, \cdots, n$, *and* $\pi(t) \geq 0$, *then* $(-1)^{i-1}(U\pi)^{(i)}(t) \geq 0$.

In particular, positive increasing concave functions are transformed into functions of the same kind.

THEOREM 5′. *If* $C \geq \pi(t) \geq 0$ *and* $C \geq (-1)^{i-1} \pi^{(i)}(t) \geq 0$ *for* $1 \leq i \leq n$, *then* $0 \leq (-1)^{i-1}(U^r\pi)^{(i)}(0) \leq K_i$, *and hence* $|U^r \pi^{(i)}(t)| \leq K_i$ *for* $1 \leq i \leq n$.

Theorem 6 remains unchanged and is valid independent of the conditions on α and σ, provided only they lie in the open unit interval.

Theorem 7 holds with a modification of the proof where $p(t)$ is replaced by the concave function

$$p(t) = \left\{ \begin{array}{ll} 1 & \text{for } 1 \geq t \geq t_0 \\ \\ \dfrac{1}{t_0}\, t & \text{for } 0 \leq t \leq t_0 \end{array} \right\},$$

and the functions t^r are replaced by $1 - (1-t)^r$. These also constitute, with the constant function, a family of functions whose linear span is dense in $C[0, 1]$. This enables us to infer the validity of Theorem 8. Theorems 9, 10, and 11, with suitable changes in their statements which we leave for the reader, are established by simple appropriate modifications similar to that indicated above for Theorem 7. The unique solution $\phi_{\sigma,\alpha}$ for this situation, where $\alpha + \sigma \leq 1$, is completely monotonic and hence analytic. In the remainder of this section the theorems are established without any specification as to the value of $\alpha + \sigma$.

THEOREM 13. *The functions*

$$\phi_m(t) = \sum_{n=m}^{\infty} U^n(t(1-t))$$

converge geometrically to 0.

Proof. It is immediate from (6) that

$$U^n(t(1-t)) = \Psi_n(t)$$

tends uniformly to zero. Since the derivative at 0 and 1 of $t(1-t)$ is 1 and -1, we conclude by Theorem 11 that for n sufficiently large there exists an $n_0(\lambda)$ such that

$$U^{n_0}(t(1-t)) \leq \lambda t(1-t)$$

with $\lambda < 1$. Let kn_0 denote the last integer k for which $kn_0 \leq m$. We obtain

$$0 \leq \phi_m(t) \leq \phi_{kn_0}(t) \leq \frac{\lambda^k}{1-\lambda} \sum_{i=0}^{n_0-1} U^i(t(1-t)) \leq C\lambda^k \leq C\rho^{(n_0+1)k} < C\rho^m,$$

where

$$\rho = \lambda^{1/(n_0+1)} < 1.$$

THEOREM 14. *If* $q(t)$ *is continuous,* $|q'(1)| < \infty$ *and* $|q'(0)| < \infty$, *then* $\lim U^n[q(t)]$ *converges geometrically.*

Proof. We first establish the result for special functions t^r with $1 \leq r \leq \infty$. A simple calculation shows that

$$-Ct(1-t) \leq U(t^r) - t^r \leq Ct(1-t).$$

For $n < m$, we obtain upon continued application of U and summation that

$$-C \sum_{i=m}^{n} U^i(t(1-t)) \leq U^n(t^r) - U^m(t^r) \leq C \sum_{i=m}^{n} U^i(t(1-t)).$$

The conclusion now follows from Theorem 13. The general function $q(t)$, satisfying the hypothesis of Theorem 14, can be bounded from above and below by two polynomials $P_1(t)$ and $P_2(t)$ which agree at 0 and 1. The result now follows directly from this fact and the first part of this proof.

We observe easily the identity

$$Ut - t = (\alpha + \sigma - 1)\, t(1-t).$$

Applying successively U and adding, we obtain

$$(7) \qquad \phi_{\sigma,\alpha} = \lim_{n\to\infty} U^m t = t + (\alpha+\sigma-1) \sum_{n=1}^{\infty} U^n_{\sigma,\alpha} t(1-t).$$

This is useful for purposes of calculation.

Some remarks describing the dependence of $\phi_{\sigma,\alpha}$ on σ and α are in order. We consider the following identity:

$$(8) \qquad U^n_{\sigma,\alpha} - U^n_{\sigma',\alpha'} = \sum_{i=0}^{n-1} U^i_{\sigma,\alpha}(U_{\sigma,\alpha} - U_{\sigma',\alpha'})\, U^{n-i-1}_{\sigma',\alpha'}.$$

If $f(t)$ is any function with bounded derivatives, then we obtain by the mean-value theorem that

$$|(U_{\sigma,\alpha} - U_{\sigma',\alpha'})f| \leq |(1-t)[f(\sigma t) - f(\sigma' t)] + t[f(\alpha + (1-\alpha)t) - f(\alpha' + (1-\alpha')t)]|$$

$$\leq C(|\sigma - \sigma'| + |\alpha - \alpha'|)\, t(1-t).$$

Applying equation 8 to $f(t) = \phi_{\sigma',\alpha'}$, and remembering that inequalities are preserved by Theorem 2, we obtain

$$|U^n_{\sigma,\alpha}\phi_{\sigma',\alpha'} - \phi_{\sigma',\alpha'}| \leq C(|\sigma-\sigma'| + |\alpha - \alpha'|) \sum_{i=0}^{n-1} U^i(t(1-t)).$$

Allowing n to go to ∞, we have easily that

$$|\phi_{\sigma,\alpha} - \phi_{\sigma',\alpha'}| \leq K(|\sigma - \sigma'| + |\alpha - \alpha'|),$$

where $K(\eta)$ is finite, provided that $0 < \eta < \alpha, \alpha' \sigma, \sigma' < 1 - \eta < 1$.

It is worthwhile to discuss the nature of $\phi_{\sigma,\alpha}$ for (σ, α) lying on the boundary of the unit square. First, we observe by direct verification that when $\alpha + \sigma = 1$, then $\phi_{\sigma,\alpha}(x) = x$. Next let $\alpha = 0$ and $\sigma < 1$; then

$$U\phi = (1-x)\phi(\sigma x) + x\phi(x).$$

Therefore, if ϕ is a fixed point with $\phi(0)=0$ and $\phi(1)=1$, then for $x \neq 1$ we have that $\phi(x)=\phi(\sigma x)$, and hence $\phi(x)\equiv\phi(0)=0$ $(0 \leq x < 1)$ provided that ϕ is continuous at 0. Similarly, when $\sigma = 1$ and $\alpha < 1$ then the only fixed point ϕ continuous at 1 and satisfying $\phi(0)=0$, $\phi(1)=1$, is $\phi(x)\equiv 1$ for $0 < x \leq 1$. On the other two boundaries of the unit square the solutions are easily calculated and turn out as follows: If $0 < \sigma < 1$ is arbitrary and $\alpha \leq 1$, then

$$\phi_{\sigma,1} = 1 - \prod_{r=0}^{\infty}(1-\sigma^r x),$$

while when $\sigma = 0$, $0 < \alpha < 1$, then

$$\phi_{\sigma,\alpha} = \prod_{r=0}^{\infty} L^r x,$$

where $L^0 = I$ and the operation L applied to x gives $\alpha + (1-\alpha)x$. Finally for $\alpha = 0$, $\sigma \neq 1$ the operator U reduces to the identity mapping. We now investigate the dependence of $\phi_{\sigma,\alpha}$ on σ and α as we allow σ and α to tend to the boundary. We limit our attention for definiteness to studying the case where $(\sigma,\alpha)\to(\sigma_0,0)$ with $\sigma_0 < 1$, and we show that $\phi_{\sigma,\alpha}$ converges pointwise to 0 for $0 \leq x < 1$, and $\phi_{\sigma,\alpha}(1)\equiv 1$ otherwise. Moreover, the convergence is uniform in any interval $0 \leq x \leq 1-\delta < 1$. Let $(\sigma_n, \alpha_n) \longrightarrow (\sigma_0, 0)$; then without loss of generality we may assume that $1-\sigma_n-\alpha_n > 0$. Therefore the ϕ_{σ_n,α_n} are convex, monotonic increasing and positive, with $\phi_{\sigma_n,\alpha_n}(0)=0$. Also, for any interior interval $0 \leq x \leq 1-\sigma < 1$, the first derivatives $\phi'_{\sigma_n,\alpha_n}$ are uniformly bounded. Since this implies the ϕ_{σ_n,α_n} are equi-continuous over the subinterval, and as $0 \leq \phi_{\sigma_n,\alpha_n} \leq 1$, we can select a subsequence which may be denoted as ϕ_{σ_r,α_r} converging to $\Psi(t)$ uniformly, for any interval of the form $0 \leq x \leq 1-\delta < 1$. As

$$\phi_{\sigma_r,\alpha_r}(1) = 1,$$

we get $\Psi(1)=1$ and similarly $\Psi(0)=0$. The uniform convergence of ϕ_{σ_r,α_r} guarantees the continuity of Ψ at zero.

Put

$$U_r = U_{\sigma_r,\alpha_r},\ U_0 = U_{\sigma_0,0} \text{ and } \phi_r = \phi_{\sigma_r,\alpha_r}.$$

We consider the following identity:

$$\Psi - U_0\Psi = (\Psi - \phi_r) + (\phi_r - U_r\Psi) + (U_r\Psi - U_0\Psi) = I_1 + I_2 + I_3 .$$

We take a fixed $x < 1$; then trivially $|I_1| = |\Psi - \phi_r| \leq \epsilon$ when r is sufficiently large. Also

$$|I_2| = |\phi_r - U_r\Psi| = |U_r\phi_r - U_r\Psi| = |(1-x)[\phi_r(\sigma_r x) - \Psi(\sigma_r x)]$$

$$+ x[\phi_r(\alpha_r + (1-\alpha_r)x) - \Psi(\alpha_r + (1-\alpha_r)x)]| .$$

But for $x = x_0 < 1$ fixed, we observe that $\alpha_r + (1-\alpha_r)x_0$ varies in an interval $\leq 1-\delta$ as $\alpha_r \longrightarrow 0$, and the same applies to $\sigma_r x$. The uniform convergence of $\phi_r \longrightarrow \Psi$ inside $0 \leq x \leq 1-\delta$ yields $|I_2| \leq \epsilon$. By construction, $|I_3| \leq \epsilon$ for r large. Thus we infer the equality $\Psi = U_0\Psi$ for $0 \leq x < 1$, and by direct verification for $x = 1$. However, the fixed point to the equation $U_0\Psi = \Psi$ with $\Psi(0) = 0$, $\Psi(1) = 1$ and Ψ continuous at 0 is $\Psi(x) = 1$ for $0 \leq x < 1$ and $\Psi(1) = 1$. Thus the limit function Ψ is the same for every subsequence of ϕ_{σ_n, a_n}, and hence we deduce that ϕ_{σ_n, a_n} converges pointwise. We furthermore note that Ψ is independent of $\sigma_0 < 1$. A similar analysis applies to the case where $(\sigma, \alpha) \longrightarrow (1, \alpha)$ $(\alpha > 0)$. The continuity properties of the solution for the other two boundaries yield to simpler analysis. Summarizing, we have established the following theorem:

THEOREM 15. *The fixed points $\phi_{\sigma,a}$ satisfy the following continuity properties: If $0 < \eta < \alpha$, $\alpha' \leq 1$ and $0 \leq \sigma, \sigma' \leq 1-\eta$, then*

$$|\phi_{\sigma,a} - \phi_{\sigma',a'}| \leq K(\eta)[|\sigma - \sigma'| + |\alpha - \alpha'|].$$

If $(\sigma, \alpha) \longrightarrow (\sigma_0, 0)$ with $\sigma_0 < 1$, then $\phi_{\sigma,a}(x) \longrightarrow 0$ pointwise for $0 \leq x < 1$ and $\phi_{\sigma,a}(1) \equiv 1$. If $(\sigma, \alpha) \longrightarrow (1, \alpha_0)$ with $\alpha_0 > 0$, then $\phi_{\sigma,a}(x) \longrightarrow 1$ pointwise for $0 < x \leq 1$.

Finally, a word concerning convergence of $U^n\pi$ for π continuous when the parameter values lie on the boundary. When $\alpha = 0$, $\sigma < 1$, then $U^n\pi$ converges pointwise. The same conclusion holds when $\alpha > 0$ and $\sigma = 1$. On the other two boundaries the convergence is uniform for $U^n\pi$. We omit the proofs.

We now return to the study of the operator T.

THEOREM 16. *For any distribution the iterates T^nF converge in the sense* *of distributions to the distribution*

$$G(x) = I_1(x)\int \phi_{\sigma,\alpha}\, dF + I_0(x)\int (1-\phi_{\sigma,\alpha})\, dF,$$

where $I_0(x)$ and $I_1(x)$ are the distributions concentrating fully at 0 *and* 1 *respectively.*

Proof. From the convergence of $U^n\pi$ for any continuous function π and Theorem 1 follows the weak*convergence of T^nF. This is equivalent to the convergence of T^nF in the sense of distributions. The actual form of

$$\lim_{n\to\infty} T^nF = G$$

as given in the theorem follows directly from (6).

By choosing the distribution $F = I_{x_0}$, we obtain from Theorem 6 that $\phi_{\sigma,\alpha}(x_0)$ represents the probability with which the limiting distribution concentrates at 1, or in other words - as can be easily shown - the probability with which the particle beginning at x_0 will converge to 1. This furnishes a probability interpretation to the fixed point of the operator U which is different from a constant.

In connection with Theorem 8, we remark that $U^n\pi$ cannot converge for an arbitrary Lebesgue measureable bounded function. In fact, if we assume that $U^n\pi$ converges for every bounded measureable function $\pi(t)$, then T^nF would converge weakly if F were absolutely continuous. Since the space of all integrable functions $L[0, 1]$ is weakly complete, and T maps distributions into distribution, we could find a fixed point $TF = F$ with F absolutely continuous and total variation 1. However, in view of (16) the only fixed distributions which exist concentrate only at 0 and 1, and hence cannot be absolutely continuous.

Finally, we present a slight application of Theorem 14. We show that the expected position of the particle converges geometrically for any starting distribution, although the iterated distributions converge slowly to the limiting distribution. The expected position of the particle is given by

$$\int_0^1 x\, dF(x) = (x, F),$$

where F is the cumulative distribution describing the position. The expected position at the nth step is given by

$$(x, T^nF) = (U^nx, F).$$

On account of Theorem 14, U^nx converges geometrically, which establishes the assertion. The same conclusion applies to all the moments. This observation is very useful for computational and estimation purposes.

Finally, we note that the spectrum of the operator T cannot consist of the isolated point 1. Otherwise, by standard techniques one can show that $U^n\pi$ converges for any measurable bounded function π.

2. **In this second model the random walk** is described as follows: If the particle is at x, then $x \longrightarrow \alpha + (1-\alpha)x$ with probability $\phi(x)$ and $x \longrightarrow \sigma x$ with probability $1-\phi(x)$, where

$$|\phi(x) - \phi(y)| \leq \mu < 1.$$

The analogous transition operator to (1) becomes

$$(9) \qquad G(x) = TF = \int_0^{x/\sigma} (1-\phi(t))(dF(t)) + \int_0^{(x-\alpha)/(1-\alpha)} \phi(t)\, dF(t),$$

with the same understanding concerning F applying as before. Let

$$(10) \qquad U\pi = [1-\phi(t)]\,\pi(\sigma t) + \phi(t)\,\pi(\alpha + (1-\alpha)t).$$

In this section, we take $0 < \alpha, \sigma < 1$; the case where boundary values for α and σ are considered is easy to handle but not of great interest. The spaces on which they operate are the same as in §1. Again, in a similar manner to Theorem 1, we obtain:

THEOREM 17. *The operator T is conjugate to the operator U.*

We now further assume that $\phi(t)$ is monotonic increasing. This model includes the important case where $\phi(t) = \lambda + \mu t$, where $\lambda + \mu \leq 1$; and whenever $\lambda + \mu = 1$ then $\lambda > 0$.

THEOREM 18. *The operator U preserves positivity and positive monotonic increasing functions.*

Proof. Direct verification.

Since the hypothesis on $\phi(t)$ implies either $\phi(1) < 1$ or $\phi(0) > 0$, we analyze the case where $\phi(1) < 1$. The other circumstance can be treated in an analogous manner. Furthermore, we now assume that if $\phi(0) = 0$, then $\phi'(0)$ exists and is finite.

THEOREM 19. *If $\pi(t)$ is monotonic increasing bounded and positive, then $U^n\pi$ converges uniformly to a constant.*

The proof can be carried out easily using the techniques employed above.

The hypothesis on $\phi(t)$ easily yields the fact that the only continuous fixed points of $U\pi = \pi$ are constant functions. The proof is similar to the proof used in Theorem 6. This fact directly connects with the result of Theorem 21 below. First, we complete the proof of convergence of $U^n\pi$ for any continuous function $\pi(t)$.

THEOREM 20. *The operators $U^n\pi$ converge uniformly for any continuous function.*

Proof. Since $\|U^n\| = 1$, and the space of all monotonic positive continuous functions spans a dense subset of the set of all continuous functions, the theorem follows by a well-known theorem of Banach.

THEOREM 21. *For any distribution F, the distribution T^nF converge as distributions to a unique distribution G for which $TG = G$ which is independent of F.*

Proof. The weak*convergence of T^nF follows directly from Theorem 20 and Theorem 16. To complete the proof we must establish that if $\lim T^nF = G$ and $\lim T^nH = K$, then $G = K$. Indeed, let Ψ denote any continuous function. We have that

$$(11)\quad (\Psi, G-K) = \lim_{n\to\infty} (\Psi, T^n(F-H)) = \lim_{n\to\infty} (U^n\Psi, F-H) = a\left(\int dF - \int dH\right) \equiv 0,$$

as F and H are distributions. Hence

$$\int \Psi(t)\, dF(t) = \int \Psi(t)\, dK(t)$$

for any continuous function Ψ, and therefore $G = K$.

It seems extremely difficult to determine the complete nature of this unique fixed distribution. We shall say more about it in a future section. We denote it by $F_{\sigma,\alpha}$.

THEOREM 22. *The distributions $F_{\sigma,\alpha}$ is a continuous function of σ, α; that is, if $(\sigma_n, \alpha_n) \longrightarrow (\sigma, \alpha)$ with $0 < \sigma$, $\alpha < 1$, then $F_{\sigma_n,\alpha_n} \longrightarrow F_{\sigma,\alpha}$ at every point of continuity of $F_{\sigma,\alpha}$.*

Proof. Let $(\sigma_n, \alpha_n) \longrightarrow (\sigma, \alpha)$; by Helly's theorem we can choose a subsequence $F_r = F_{\sigma_{n_r} \alpha_{n_r}}$ converging to the distribution F at every continuity point. Write T_r for $T_{\sigma_{n_r} \alpha_{n_r}}$ and T for $T_{\sigma,\alpha}$. Let $\pi(t)$ denote any fixed continuous function. We consider the quantity

$$(\pi, F - TF) = (\pi, F - F_r) + (\pi, F_r) - (\pi, TF_r) + (\pi, TF_r - TF).$$

Since $F_r \longrightarrow F$ as distributions, we find for r sufficiently large that $|(\pi, F - F_r)| < \epsilon$. Now we note that

$$|(\pi, F_r) - (\pi, TF_r)| = |(\pi, T_r F_r) - (\pi, TF_r)| = |(U_r \pi - U\pi, F_r)|.$$

Since $U = U_{\sigma_{n_r} \alpha_{n_r}}$ converges strongly to $U = U_{\sigma,\alpha}$, as is trivial to verify, it follows that U_r converges uniformly to $U\pi$. Whence, as F_r are distributions, we infer that

$$|(U_r \pi - U\pi, F_r)| \le \max_t |U_r \pi - U\pi| < \epsilon$$

when r is chosen large enough. Evidently, with r large we get as before that

$$|(\pi, T(F_r - F))| = |(U\pi, F_r - F)| \le \epsilon .$$

Therefore we obtain for r large that $|(\pi, F - TF)| \le 3\epsilon$, and hence $(\pi, F) = (\pi, TF)$. Since π is any continuous function, we infer $F = TF$ and therefore $F = F_{\sigma,\alpha}$ by Theorem 21. Consequently, as any limit distribution of F_{σ_n,α_n} must be $F_{\sigma,\alpha}$ the conclusion of Theorem 22 is now immediate.

3. The model considered in this section is with $\phi(x) = 1 - x$. In this case ϕ is monotonic decreasing. The operator U becomes

$$(12) \qquad U\pi(t) = t\pi(\sigma t) + (1-t)\pi(1-\alpha+\alpha t).$$

Note that we have replaced α by $1-\alpha$. This is only for convenience in Theorem 28, and does not restrict any generality. In this model the closer the particle moves to the ends 0 and 1 the greater probability there is of moving back into the interior. The situation described here is of completely reflecting boundaries. Again it is easy to show that the only continuous fixed points $U\pi = \pi$ are the constant function. Therefore, we shall find as in § 2 that the distributions describing the position of the particle converge to a limit distribution independent of the initial distribution. We first proceed to analyze convergence properties of $U^n\pi$. In this case it is no longer true that U preserves the class of positive monotonic functions. Only positivity is conserved by the mapping U. However, a new quality as described in Theorem 23 serves here well.

Throughout this section in order to avoid trivial changes of proof and different results at times, we suppose that $0 < \alpha, \sigma < 1$.

THEOREM 23. *If $\pi(t)$ has a continuous derivative, then*

$$\max_t |(U\pi)'(t)| \leq \max_t |\pi'(t)|,$$

with equality holding if and only if $\pi(t)$ is linear.

Proof. By direct computation, we obtain

$$U\pi'(t) = t\sigma\pi'(\sigma t) + (1-t)\alpha\pi'(1-\alpha+\alpha t) + \pi(\sigma t) - \pi(1-\alpha+\alpha t).$$

Hence, with the aid of the mean-value theorem we get

$$(13) \quad \max_t |U\pi'(t)| \leq \max_t |t\sigma\pi'(\sigma t) + (1-t)\alpha\pi'(1-\alpha+\alpha t)|$$

$$+ (\sigma t - (1-\alpha) - \alpha t) \left| \frac{\pi(\sigma t) - \pi(1-\alpha+\alpha t)}{\sigma t - (1-\alpha) - \alpha t} \right|$$

$$\leq \max_t [t\sigma + (1-t)\alpha + 1 - \alpha - (\sigma - \alpha)t] \max_t |\pi'(t)| = \max_t |\pi'(t)|.$$

If equality holds, then let t_0 denote a point where

$$\max_t |\pi'(t)| = |\pi'(t_0)|.$$

It follows easily from (13) that

$$(14) \quad \max |\pi'(t)| = |\pi'(\sigma t_0)| = |\pi'(1-\alpha+\alpha t_0)| = \left| \frac{\pi(\sigma t_0) - \pi(1-\alpha+\alpha t_0)}{\sigma t_0 - (1-\alpha) - \alpha t_0} \right|.$$

This yields that $\pi(t)$ is linear for $\sigma t_0 \leq t \leq 1 - \alpha + \alpha t_0$, or otherwise somewhere between σt_0 and $1 - \alpha + \alpha t_0$ the slope has greater magnitude than the slope of the chord subtended by $\pi(t)$ at these points. Equation (14) yields also that σt_0 and $(1 - \alpha + \alpha t_0)$ are maximum points of $\pi'(t)$. Repeating this argument successively then shows that equality in (13) requires $\pi(t)$ to be linear.

THEOREM 24. *If $\pi(t)$ belongs to C^m ($\pi(t)$ possesses m continuous derivatives), then* $\max_t |(U^n\pi)^{(r)}(t)|$ *is uniformly bounded in n for each r* $(0 \leq r \leq m)$.

Proof. The proof is similar to that of Theorem 10.

THEOREM 25. *If $\pi(t)$ possesses two continuous derivatives, and $\sigma \neq \alpha$, then $U^n\pi$ converges uniformly to a constant.*

Remark. The reason why the two cases $\sigma = \alpha$ and $\sigma \neq \alpha$ are distinguished, and necessarily so, will be explained later.

Proof. In view of Theorem 23 and Theorem 24, the first and second derivatives of $U^n\pi$ are uniformly bounded. Thus $U^n\pi$ and $(U^n\pi)'$ constitute equicontinuous families of functions. We can thus select a subsequence n_i such that $U^{n_i}\pi$ converges uniformly to $\phi(t)$, and $(U^{n_i}\pi)'$ converges uniformly to $\phi'(t)$. It follows trivially that $U^{n_i+1}\pi$ tends uniformly to $U\phi$ and

$$U^{n_i+2}\pi \longrightarrow U^2\phi.$$

Moreover, by virtue of Theorem 23,

$$(15) \qquad \max_t |(U^{n_i}\pi)'| \geq \max_t |(U^{n_i+1}\pi)'| \geq \max_t |(U^{n_{i+1}}\pi)'|.$$

Hence

$$\lim_{i\to\infty} \max_t |(U^{n_i}\pi)'| = \lim_{i\to\infty} \max_t |(U^{n_i+1}\pi)'| = \lim_{i\to\infty} \max_t |(U^{n_i+2}\pi)'|.$$

Therefore, by the uniform convergence of the derivatives, we secure

$$\max_t |\phi'(t)| = \max_t |(U\phi)'(t)| = \max_t |(U^2\phi)'(t)|.$$

Invoking Theorem 23 yields that $\phi(t)$ and $U\phi(t)$ are linear. However, if $\alpha \neq \sigma$ and $\phi(t)$ contains a term with t, then $U\phi$ is quadratic. This impossibility forces $\phi(t)$ to be identically a constant. Let i be chosen sufficiently large so that

$$|U^{n_i}\pi - c| \leq \epsilon.$$

Then

$$|U^{n_i+1}\pi - c| \leq t|U^{n_i}\pi(\sigma t) - c| + (1-t)|U^{n_i}\pi(1-\alpha+\alpha t) - c| < \epsilon.$$

Repeating this argument shows that

$$|U^{n_i+p}\pi - c| \leq \epsilon$$

for any p. This establishes that $U^n\pi$ converges uniformly to c.

THEOREM 26. *If $\pi(t)$ is continuous and $\sigma \neq \alpha$, then $U^n\pi$ converges uniformly.*

Proof. The space of all functions with two continuous derivatives spans linearly a dense subset of the space of all continuous functions. Since $||U^n|| = 1$, we obtain the result using Theorem 25 and a well-known theorem of Banach.

In the next two theorems we establish the uniform convergence of $U^n\pi$ for the case where $1 > \sigma = \alpha > 0$. We note in this case the interesting fact that U applied to a polynomial does not increase its degree. Particularly,

$$Ux^n = [\alpha^n - n\alpha^{n-1}(1-\alpha)]x^n + P_{n-1}(x),$$

where $P_{n-1}(x)$ denotes a polynomial of degree $n-1$.

THEOREM 27. *If $P(t)$ is any polynomial, then U^kP converges uniformly to a constant and the convergence is geometric.*

Proof. The proof is by induction on the degree of the polynomial. Clearly if P is a constant $= c$ then $U^kP \equiv c$. Suppose we have shown for any polynomial P_{n-1} of degree $\leq n-1$ that the iterates U^kP_{n-1} converge uniformly. To complete the proof, it is enough to verify that U^kx^n converges uniformly. Let

$$\lambda = \alpha^n - n\alpha^{n-1}(1-\alpha);$$

then $|\lambda| < 1$ since $1 > \alpha > 0$. We obtain

$$Ux^n = \lambda x^n + P_{n-1}(x).$$

Repeating, we get, for $k \geq 1$,

$$U^k x^n = \lambda^k x^n + \sum_{r=0}^{k-1} \lambda^r U^{k-r-1} P_{n-1}.$$

This last sum is of the form

$$c_k = \sum_{r=0}^{k} a_r b_{k-r},$$

with $\sum |a_r| < \infty$, and $\lim_{k=\infty} b_k(x)$ exists. It is a well-known theorem that $\lim c_k(x)$ exists uniformly whenever

$$b_k(x) = U^{k-1} P_{n-1}$$

converges uniformly. Thus, $U^k x^n$ converges uniformly to a fixed point which must be a constant function. Finally we note that in this case where $\sigma = \alpha$ (the rate of learning, so to speak, is the same regardless of the outcome of the experiment), then $U^n P$ for any polynomial converges geometrically. The proof can be carried through easily by induction.

This yields the fact that the expected position converges geometrically to a limiting expected position with similar statements applying to higher moments.

THEOREM 28. *If $\pi(t)$ is continuous and $\sigma = \alpha > 0$, then $U^n \pi$ converges uniformly.*

Proof. Similar to Theorem 26, since the set of all polynomials is dense.

We now note the important example that when $\alpha = \sigma = 0$ it is no longer true that $U^n \pi$ converges. It is easily verified that in this case $U^{2n}\pi$ and $U^{2n+1}\pi$ converge separately but that a periodic phenomenon occurs otherwise. The argument of Theorem 27 breaks down in this case as the quantity λ is -1. We only mention that other difficult convergence behavior occurs when α, σ traverse the boundary of the unit square for this model. In particular, when $\alpha = 1$ and $\sigma < 1$ it is not hard to show that $U^n_{\sigma,\alpha}\pi$ does not necessarily converge for every continuous function π, and even for the circumstance where π is a polynomial. The case where $\sigma = \alpha = 1$ produces for U the identity operator for which the convergence of U^n is trivial. For $\alpha < 1$ and $\sigma = 1$ we can conclude again a lack of convergence. However, when $\alpha = 0$ and $1 > \sigma > 0$, or $\sigma = 0$ and $1 > \alpha > 0$, then $U^n_{\sigma,\alpha}\pi$ converges for every continuous function π.

We return now to the hypothesis $0 < \alpha, \sigma < 1$.

THEOREM 29. *If $\pi(t)$ belongs to C^m, then $(U^k\pi)^{(r)}(t)$ converges uniformly for $0 \le r \le m$.*

Proof. This follows easily from Theorems 24, 26, and 28. Let

Let

$$TF = \int_0^{x/\sigma} t\, dF(t) + \int_0^{(x+\alpha-1)/\alpha} (1-t)\, dF(t).$$

This represents the transition law for the distribution describing the position of of the particle for this model. By arguments analogous to those employed in the preceding sections, we can establish the following theorems, using the conjugate relationship between T and U.

THEOREM 30. *For any distribution F the distributions T^nF converge as distributions to a unique distribution $F_{\sigma,\alpha}$ for which $TF_{\sigma,\alpha} = F_{\sigma,\alpha}$, which is independent of F.*

THEOREM 31. *The distributions $F_{\sigma,\alpha}$ constitute a continuous family of distributions in the sense of Theorem 22.*

Again it seems very difficult to determine any more explicit information about $F_{\sigma,\alpha}$.

4. The model examined here is such that $1 - \phi(x) = \lambda x + \mu$, with $\lambda + \mu \le 1$ and at least $1 > \lambda$ or $0 < \mu$. The operator U has the form

$$U\pi = (\lambda x + \mu)\,\pi(\sigma x) + (1 - \lambda x - \mu)\,\pi(1 - \alpha + \alpha x). \tag{16}$$

Of course, as before, $0 < \alpha, \sigma < 1$. Convergence questions for $U^n\pi$ turn out to be very elementary in this case in view of the following theorem which is easily proven.

THEOREM 32. *If $\pi(x)$ has a bounded derivative, then*

$$\max_x |(U\pi)'(x)| \le a \max_x |\pi'(x)|$$

with $a < 1$.

An immediate consequence of Theorem 32 is that $(U^k\pi)'$ converges geometrically to 0. Let T denote the transition operator of distributions for this model. In the standard way, we obtain:

THEOREM 33. *For any distribution F the distributions T^nF converge to the distribution $F_{\sigma,\alpha}$ which is a continuous function of (σ, α),* and $TF_{\sigma,\alpha} = F_{\sigma,\alpha}$. *Moreover, $F_{\sigma,\alpha}$ is independent of F.*

5. **This section is devoted to some variations** of the preceding models. A new feature added first is that we allow in addition to the two impulses of motions towards the two fixed points 0 and 1 by the transformations

$$F_1x = \sigma x \text{ and } F_2x = 1 - \alpha + \alpha x$$

the possibility of a third motion where the particle stands still with certain probability. These models are particularly important in learning problems, and much statistical investigation on this type has been done by M. M. Flood [5]. They are referred to as the pure models. The mathematical description of the first model of this type is as follows: A particle x on the unit interval is subject to three random impulses: (1) $x \longrightarrow \sigma x$ with probability $\pi_1(1-x)$; (2) $x \longrightarrow 1-\alpha+\alpha x$ with probability $\pi_2 x$; and (3) $x \longrightarrow x$ with probability $(1-\pi_1)(1-x)+(1-\pi_2)x$, where $0 \leq \pi_1, \pi_2 \leq 1$. This is similar to model I where absorption takes place at the boundaries 0 and 1. The operator analogous to (2) becomes

$$U\pi = \pi_1(1-x)\pi(\sigma x) + [(1-\pi_1)(1-x) + (1-\pi_2)x]\,\pi(x) + \pi_2 x\,\pi(1-\alpha+\alpha x). \tag{17}$$

Again, let T denote the transition operator which maps the distribution locating the particle into the corresponding distribution at the end of the experiment. Theorem 1 is valid for this setup, and T is consequently conjugate to U. It is easy to verify that U fulfills the conditions of Theorems 2 and 3 and also preserves the property of monotone increasing functions. Furthermore, we obtain:

THEOREM 34. *If π, π' and $\pi'' \geq 0$, then $(U\pi)'' \geq 0$ if and only if*

$$(1-\sigma)\pi_1 + \pi_2(\alpha - 1) \geq 0,$$

and otherwise $U\pi$ preserves with π and $\pi' \geq 0$ the property of concavity.

Proof. The proof can be carried through by direct computation.

We remark that the remainder of the analogue to Theorem 4 does not carry over under the condition stated in Theorem 34. Moreover, noting that we have here changed α into $1-\alpha$ as compared to § 2, we obtain for $\pi_1 = \pi_2 = 1$ the condition of § 1 for preservation of convexity, and so on.

The analogues of Theorems 5, 6, 7, and 8 easily extend to this model by the same methods, and we obtain that $U^n\pi$ converges uniformly to a limit given by

$$(18) \qquad [1-\phi_{\sigma,\alpha,\pi_1,\pi_2}(x)]\,\pi(0) + \phi_{\sigma,\alpha,\pi_1,\pi_2}(x)\,\pi(1),$$

where $\phi_{\sigma,\alpha,\pi_1,\pi_2}$ is the unique continuous fixed point of $U\phi = \phi$ with $\phi(0) = 0$ and $\phi(1) = 1$. The entire theory of geometric convergence, continuity of ϕ as a function of σ, α, π_1, and π_2, and the form of the limiting distribution of the particle established for the model of § 1 remains valid with slight changes in the proofs. The general conclusion is that introducing a probability of standing still has no effect on the convergence of the distributions or its limiting form provided only the essential feature of absorbing boundaries still persists. Finally, in this connection we remark that for special boundary values of the parameters π_1 and π_2 the motion may become a drift to one or other of the end points; for example, $\pi_1 = 0$, $\pi_2 > 0$.

6. We treat in this section, the following general nonlinear one-dimensional learning model. The particle moves with probability $\phi(x)$ from x to $1-\alpha+\alpha x$ and with probability $1-\phi(x)$ from x to σx. The function is only continuous with the additional important requirement for this case that $\phi(x) \geq \delta > 0$ and $1-\phi(x) > \delta > 0$ for all x in the unit interval. This excludes the types of models discussed in §§ 1 and 3, but includes some subcases of the examples investigated in §§ 2 and 4. However, in those cases we obtained much stronger results about the rate of convergence of derivatives, and so on. The transition operators become

$$(20) \qquad TF = \int_0^{x/\sigma} [1-\phi(t)]\,dF(t) + \int_0^{(x-1+\alpha)/\alpha} \phi(t)\,dF(t),$$

and T is adjoint to

$$(21) \qquad (U\pi)(t) = (1-\phi(t))\,\pi(\sigma t) + \phi(t)\,\pi(1-\alpha+\alpha t).$$

We shall show that $U^n\pi$ converges uniformly for any continuous function $\pi(t)$. The proof of this fact shall be based on the following highly intuitive proposition. Let an experiment be repeated with only two possible outcomes, success or failure at each trial. Suppose further that the probability of success p_n at the nth trial depends on the outcome of the previous trial, but that these conditional probabilities satisfy $p_n \geq \eta > 0$; that is, regardless of the previous number of failures the conditional probability of success is always at least $\eta > 0$. Then the recurrent event of a success run of length r with r fixed is a certain event; that is, with probability 1 it will occur in finite time. This result can be deduced in a standard way using the theory of recurrent events [**4**].

We turn back now to the examination of $U^n\pi$. Let

$$F_1 x = \sigma x \quad \text{and} \quad F_2 x = 1 - \alpha + \alpha x$$

and by Fx denote the operation that either F_1 or F_2 is applied. We note the important obvious fact that

$$(22) \qquad |F^r x - F^r y| \leq \lambda^r \, |x - y|,$$

with $0 < \lambda < 1$, where F^r denotes r applications of F_1 and F_2 in some order acting on x and y in the same way.

Next, we need the important lemma:

LEMMA. *If* $|\phi^{(m)}(t)| \leq K$ *for* $m = 0, 1, \cdots,$ *and* $|\pi^{(m)}(t)| \leq K_1$, *then* $|U^n \pi^{(m)}(t)| \leq K_2$ *uniformly in n and t.*

Proof. The proof is similar to that of Theorem 24.

Now let $\pi(t)$ denote a continuously differentiable function. Consider the following identity:

$$(23) \quad U^n\pi(x) - U^n\pi(y) = (1-\phi(x))(1-\phi(y))[U^{n-1}\pi(F_1x) - U^{n-1}\pi(F_1y)]$$
$$+ \phi(x)\,\phi(y)[U^{n-1}\pi(F_2x) - U^{n-2}\pi(F_2y)]$$
$$+ (1-\phi(y))\,\phi(x)[U^{n-1}\pi(F_2x) - U^{n-1}\pi(F_1y)]$$
$$+ \phi(y)\,(1-\phi(x))\,[U^{n-1}\pi(F_1x) - U^{n-1}\pi(F_2y)].$$

We continue to apply this identity to the factors $U^{n-1}\pi(\cdot) - U^{n-1}\pi(\cdot)$; and when any term of the form $U^m\pi(F^r w) - U^n\pi(F^r z)$ is achieved, then that factor is allowed to stand without any further reduction. All other terms are reduced to expressions involving as factors $\pi(\cdot) - \pi(\cdot)$. Thus we obtain

$$U^n\pi(x) - U^n\pi(y) = I_1^n + I_2^n,$$

when I_1 consists of terms of the form

$$\sum p_k [U^{m_k}\pi(F^r w_k) - U^{m_k}\pi(F^r z_k)],$$

and $\sum p_k \leq 1$ while I_2 consists of the remaining terms. We now conceive of the following probability model. Let two particles undergo the random walk described by this model starting from x and y, respectively. We say a success occurs if the same impulse activates both particles, and otherwise failure occurs. The probability of success is given initially by

$$\phi(x)\ \phi(y) + [1-\phi(x)][1-\phi(y)] \geq 2\delta^2 > 0,$$

and it is easily seen that each p_k, where p_k is the conditional probability of success occurring on the kth trial, satisfies

$$p_k \geq 2\delta^2 > 0.$$

Consequently, a success run of length r is certain to happen in finite time. In particular as $n \longrightarrow \infty$, $I_2^n \longrightarrow 0$, since I_2^n is bounded by twice the probability of no success run in n trials times K. On the other hand, in view of the lemma and equation (22) we secure that $I_1^n \leq C\lambda^r$. Therefore,

$$\overline{\lim_{n\to\infty}}\ |U^n\pi(x) - U^n\pi(y)| \leq C\lambda^r,$$

which can be made arbitrarily small as $r \longrightarrow \infty$. Hence, if

$$\lim U^n\pi(y) = a$$

exists for a single y, then

$$\lim_{n\to\infty} U^n\pi(x) = a$$

for every x. Since a subsequence can be found so that

$$\lim_{i\to\infty} U^{n_i}\pi(x) = a$$

for one x and hence for all x, an argument used in the close of the proof of

Theorem 25 shows that

$$\lim_{n \to \infty} U^n \pi(x) = a.$$

The lemma easily implies that the convergence is uniform. Using the fact that $\| U^n \| = 1$, we can sum up the conclusions for this nonlinear model as follows:

THEOREM 35. *If $\pi(t)$ is continuous, then $\lim_{n \to \infty} U^n \pi$ exists uniformly converging to a constant limit.*

THEOREM 36. *If $\phi(t)$ belongs to C^m, and $\pi(t)$ is in C^m, then*

$$\lim_{n \to \infty} (U^n \pi)^{(m)}(t) = 0$$

with convergence uniform in t.

THEOREM 37. *For any distributions F, $T^n F$ converges to a distribution $F_{\sigma,\alpha}$ independent of F with $TF_{\sigma,\alpha} = F_{\sigma,\alpha}$ and $F_{\sigma,\alpha}$ continuous with respect to σ, α.*

This last theorem follows on account of the conjugate relationship of T and U.

Finally, we note that the method used in this section can be employed to analyze the random walks with any number of impulses

$$F_i x = (1 - \alpha_i) m_i + \alpha_i x.$$

7. In the present section we investigate the nature of the limiting distribution obtained in the various models. In the case where the boundaries were absorbing states as in §§1 and 5, we find that the limiting distribution is discrete and concentrates at the two ends 0 and 1. The weight at 1 depends on the starting distribution F and is given by

$$\int_0^1 \phi_{\sigma,\alpha}(x)\, dF(x),$$

where $\phi_{\sigma,\alpha}$ is the unique continuous fixed point of $U\phi = \phi$ with $\phi(0) = 0$ and $\phi(1) = 1$. Many properties of $\phi_{\sigma,\alpha}$ are developed in those sections. In all the other types the ergodic property was seen to hold and the limiting distribution was independent of the initial distribution. Let us deal with the following general type. The random walk is given by $x \longrightarrow F_1 x = \sigma x$ with probability $1 - \phi(x)$, and $x \longrightarrow F_2 x = 1 - \alpha + \alpha x$ with probability $\phi(x)$, where $1 - \delta \geq \phi(x) \geq \delta > 0$. The relevant operators are given by equations (20) and (21). Let the limiting distribution be denoted by $F_{\sigma,\alpha}$.

We now distinguish two cases: (a) $\sigma \geq 1 - \alpha$ and (b) $\sigma < 1 - \alpha$. Let us examine case (b) first. We note that the union of the image sets $F_1[0, 1] + F_2[0, 1]$ of F_1 and F_2 applied to the unit interval does not overlap with the open subinterval $(\sigma, 1 - \alpha)$. Any two applications of F_1 and F_2 leaves empty the two additional open intervals $(\sigma^2, (1 - \alpha)\sigma)$ and $(\sigma(1 - \alpha), (1 - \alpha)^2)$. Proceeding in this way, we find that the limit of the total set covered by n applications of $F_i (i = 1, 2)$ in any arrangement is a Cantor set C. It is easily seen that $F_{\sigma,\alpha}$ must concentrate its full probability on this set C.

Now let

$$\pi_{t_0}(x) = \begin{Bmatrix} 1 & \text{if } x = t_0 \\ 0 & \text{if } x \neq t_0 \end{Bmatrix}.$$

We show that $U^n \pi_{t_0}(x)$ converges uniformly to zero. Note that $U\pi_{t_0}(t)$ is zero for every t except at most one value of t; namely, $F_1^{-1} t_0$ or $F_2^{-1} t_0$. Of course, if $\sigma < t_0 < 1 - \alpha$, then neither inverse exists for that t_0; and otherwise only one exists and

$$| U\pi_{t_0} | \leq \max_x [\phi(x), 1 - \phi(x)] \leq 1 - \delta.$$

Similarly, $U^n \pi_{t_0} \leq (1 - \delta)^n$, from which the assertion follows. We now observe that

$$(\pi_{t_0}, F_{\sigma,\alpha}) = (\pi_{t_0}, T^n F_{\sigma,\alpha}) = (U^n \pi_{t_0}, F_{\sigma,\alpha}) \longrightarrow 0.$$

Consequently, the probability of $F_{\sigma,\alpha}$ at t_0 is zero for any t_0 with $0 \leq t_0 \leq 1$. Summing up, we have established:

THEOREM 38. *If $\sigma < 1 - \alpha$, then the limiting distribution $F_{\sigma,\alpha}$ is a singular distribution (probability zero at every point) spread on a Cantor-like set.*

We turn now to examine case (a) where $\sigma \geq 1 - \alpha$. We note first that at least one of the two mappings F_1^{-1} or F_2^{-1} is defined for every x in the unit interval. Let $\pi(t)$ denote any continuous positive function defined on the unit interval so that $\pi(t) \geq \eta > 0$ for some subinterval $t_0 - h \leq t \leq t_0 + h$ $(h > 0)$.

Since at least F_1^{-1} or F_2^{-1} exists at t_0 (say F_1^{-1}), we obtain $F_1^{-1} t_0 = t_1$. We construct t_2 from t_1 in the same way and continue this for n steps, obtaining $t_n = F^{-n} t_0$, where F^{-n} denotes a specific order of application of F_1^{-1} or F_2^{-1} a total of n times. Let F^n denote the reverse order of the operators obtained by passing from t_0 to t_n. We note that

$$| F^n x - F^n y | \leq \lambda^n | x - y | \leq \lambda^n ,$$

where $\lambda < 1$. Choose n so large that $\lambda^n < h$; then for every x we get that

$$| F^n x - F^n t_n | = | F^n x - t_0 | < h .$$

Consequently, as

$$1 > 1 - \delta \geq \phi(x) \geq \delta > 0,$$

$U^n \pi$ is positive for all x since $F^{-n}[t_0 - h, \ t_0 + h]$ covers the entire unit interval and $\pi(t) \geq \eta > 0$ on this initial interval which is spread out by the term in U^n involving F^n. We have thus shown:

THEOREM 39. *If $\sigma > 1 - \alpha$, the operator U is strictly positive; that is, for each positive continuous function $\pi(t)$ there exists an n depending upon π so that $U^n \pi$ is strictly positive.*

Now let $\pi_{t_0}(t)$ be defined as before. Again we establish that $U^n \pi_{t_0}$ converges uniformly to zero. To this end we observe that $U \pi_{t_0}$ has at most two possible values at $F_1^{-1} t_0$ and $F_2^{-1} t_0$ given by $1 - \phi(F_1^{-1} t_0)$ and $\phi(F_2^{-1} t_0)$, respectively, while $U \pi_{t_0} \equiv 0$ elsewhere. Also, $U^2 \pi_{t_0}$ has at most four possible values and the maximum value that could be achieved for $U^2 \pi_{t_0}$ is

$$\max \{ [1 - \phi(F_1^{-1} t_0)][1 - \phi(F_1^{-2} t_0)], \ \phi(F_2^{-1} t_0) \phi(F_2^{-2} t_0),$$

$$[1 - \phi(F_1^{-1} t_0)] \phi(F_2^{-1} F_1^{-1} t_0) + \phi(F_2^{-1} t_0) [1 - \phi(F_1^{-1} F_2^{-1} t_0)] \}.$$

To secure a bound for the maximum of $U^n \pi_{t_0}$, let us consider the same repeated-experiment model set up in the previous section. The conditional probabilities of success p_n at the nth trial satisfy the uniform inequalities $1 > 1 - \eta \geq p_n \geq \eta > 0$, where success in this case is taken to be an application of the impulse F_1 to the particle. It is readily seen by standard inequalities that the probability of securing k $(k \leq n)$ successes converges uniformly to zero as $n \longrightarrow \infty$. Moreover, it follows directly that $\max_k$ (probability of k successes) is a bound for

$U^n\pi_{t_0}$, and hence $U^n\pi_{t_0} \longrightarrow 0$. We deduce as before that $F_{\sigma,\alpha}$ has probability zero for every t. Thus the cumulative distribution of F is continuous. Let $F = F_1 + F_2$, where F_1 is absolutely continuous and F_2 is singular. Observing that the transition operator transforms absolutely continuous measures into absolutely continuous measures and singular measures into singular measures, we find that $TF_1 = F_1$ and $TF_2 = F_2$. However, as the fixed distribution is unique, we deduce that either F_1 or F_2 vanishes.

THEOREM 40. *If $\sigma \geq 1 - \alpha$, then the unique distribution $F_{\sigma,\alpha}$ is either absolutely continuous or singular. Furthermore, $F_{\sigma,\alpha}$ has positive measure in every open interval.*

Proof. We have demonstrated all the conclusions of the theorem but the last. Let $\pi(t)$ denote a continuous positive function bounded by 1, and zero outside an open interval I, and 1 on a closed subinterval I' of I. By virtue of Theorem 39 there exists an n such that $U^n\pi \geq \delta > 0$ for all t. We note that

$$(\pi, F_{\sigma,\alpha}) = (\pi, T^nF_{\sigma,\alpha}) = (U^n\pi, F_{\sigma,\alpha}) \geq \delta > 0.$$

But

$$\int_I dF_{\sigma,\alpha} \geq (\pi, F_{\sigma,\alpha}) \geq \delta > 0,$$

and the proof of the theorem is complete.

We close with the conjecture that when $\sigma \geq 1 - \alpha$, then $F_{\sigma,\alpha}$ is always absolutely continuous. An example where this is the case is furnished by $\phi(x) \equiv 1/2$, $\sigma = 1/2 = 1 - \alpha$, where $F_{\sigma,\alpha}(x) = x$.

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ON UNIFORM DISTRIBUTION MODULO A SUBDIVISION

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1. Let Δ be a subdivision of the interval $(0, \infty)$: $\Delta = (z_0, z_1, \cdots)$, where

$$0 = z_0 < z_1 < \cdots \quad \text{and} \quad \lim_{n \to \infty} z_n = \infty .$$

For $z_{n-1} \le x < z_n$, put

$$[x]_\Delta = z_{n-1}, \quad \delta(x) = z_n - z_{n-1}, \quad \langle x \rangle_\Delta = \frac{x - [x]_\Delta}{\delta(x)}, \quad \phi(x) = n + \langle x \rangle_\Delta ,$$

so that $0 \le \langle x \rangle_\Delta < 1$. Let $\{x_k\}$ be an increasing sequence of positive numbers. If the sequence $\{\langle x_k \rangle_\Delta\}$ is uniformly distributed over $[0, 1]$, in the sense that the proportion of the numbers $\langle x_1 \rangle_\Delta, \cdots, \langle x_k \rangle_\Delta$ which lie in $[0, \alpha)$ approaches α as $k \longrightarrow \infty$, for each $\alpha \in [0, 1)$, then we shall say that the sequence $\{x_k\}$ is *uniformly distributed modulo* Δ. If Δ is the subdivision Δ_0 for which $z_n = n$, this reduces to the ordinary concept of uniform distribution (mod 1), since then $[x]_\Delta = [x]$, $\delta(x) = 1$ for all x, and $\langle x \rangle_\Delta = x - [x]$ is the fractional part of x. Even in other cases, the generalization is more apparent than real, since the uniform distribution of one sequence (mod Δ) is equivalent to the uniform distribution of another sequence (mod 1). But most of the known theorems concerning uniform distribution (mod 1) are not applicable to the sequences $\{\langle x_k \rangle_\Delta\}$, if Δ is not Δ_0, for in such theorems x_k is ordinarily taken to be the value $f(k)$ of a function whose derivative exists and is monotonic for positive x. Here, on the other hand, $\langle x_k \rangle_\Delta \equiv \phi(x_k)$ (mod 1), and ϕ, although a continuous polygonal function, is not necessarily everywhere differentiable; and unless $\delta(x)$ is assumed monotonic, ϕ' is not monotonic even over the set on which it exists. This lack of monotonicity introduces serious difficulties; it is the object of the present work to show how they can be dealt with in certain cases.

For brevity, "uniformly distributed" will be abbreviated to "u.d.". The symbols "$\uparrow$", "$\nearrow$", "$\downarrow$" and "$\searrow$" indicate monotonic approach: increasing, non-decreasing, decreasing, and non-increasing, respectively.

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2. Put

$$N(\alpha, x) = \sum_{\substack{x_k \leq x \\ \langle x_k \rangle_\Delta < \alpha}} 1, \quad N(x) = N(1, x);$$

then $\{x_k\}$ is u.d. (mod Δ) if and only if, for each $\alpha \in [0, 1)$,

$$\lim_{x \to \infty} \frac{N(\alpha, x)}{N(x)} = \alpha.$$

THEOREM 1. *A necessary condition that* $\{x_k\}$ *be u.d.* (mod Δ) *is that*

$$N(z_{n+1}) \sim N(z_n)$$

as $n \longrightarrow \infty$.

For suppose that $\{x_k\}$ is u.d. (mod Δ). Then since

$$N\left(\frac{1}{2}, \frac{z_n + z_{n+1}}{2}\right) - N(1/2, z_n) = N\left(\frac{z_n + z_{n+1}}{2}\right) - N(z_n),$$

we have

$$\frac{1}{2} \sim \frac{N(1/2, (z_n + z_{n+1})/2)}{N((z_n + z_{n+1})/2)} = \frac{N(1/2, z_n)}{N((z_n + z_{n+1})/2)} + \frac{N((z_n + z_{n+1})/2) - N(z_n)}{N((z_n + z_{n+1})/2)}$$

$$= \frac{N(1/2, z_n)}{N(z_n)} \cdot \frac{N(z_n)}{N((z_n + z_{n+1})/2)} + 1 - \frac{N(z_n)}{N((z_n + z_{n+1})/2)}$$

$$= 1 + \frac{N(z_n)}{N((z_n + z_{n+1})/2)} \left(\frac{N(1/2, z_n)}{N(z_n)} - 1 \right) \sim 1 - \frac{1}{2} \frac{N(z_n)}{N((z_n + z_{n+1})/2)}$$

as $n \longrightarrow \infty$, and so

$$N(z_n) \sim N\left(\frac{z_n + z_{n+1}}{2}\right).$$

In the same way it can be shown that

$$N\left(\frac{z_n + z_{n+1}}{2}\right) \sim N(z_{n+1}),$$

and consequently $N(z_n) \sim N(z_{n+1})$.

3. The following theorem, due in a slightly different form to Fejér (see [1, p.88-89]), expresses the fact that if f is sufficiently smooth and $[f(x)]$ is constant over increasingly long intervals as x increases, such that the length of the n-th interval is of smaller order of magnitude than the total length of all preceding intervals, then $f(k)$ is u.d. (mod 1):

Suppose that $f(x)$ has the following properties:

(i) *f is continuously differentiable for $x > x_0$,*

(ii) *$f(x) \uparrow \infty$ as $x \uparrow \infty$,*

(iii) *$f'(x) \searrow 0$ as $x \uparrow \infty$,*

(iv) *$xf'(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.*

Then $f(k)$ is u.d. (mod 1).

The following theorem uses the same general idea:

THEOREM 2. *Suppose that, for a given subdivision Δ and a sequence $\{x_k\}$, $N(z_n) - N(z_{n-1}) \longrightarrow \infty$ as $n \longrightarrow \infty$. Then $\{x_k\}$ is u.d.* (mod Δ) *if the following conditions are satisfied:*

(i) *$N(z_{n-1}) \sim N(z_n)$ as $n \longrightarrow \infty$,*

(ii) *except possibly on a sequence of intervals $[z_{n_t-1}, z_{n_t})$ such that*

$$(1) \qquad \sum_{t=1}^{m} (N(z_{n_t}) - N(z_{n_t-1})) = o(N(z_{n_m})),$$

the relation

$$\max (x_k - x_{k-1}) \sim \min (x_k - x_{k-1})$$

holds as $n \longrightarrow \infty$, the maximum and minimum being taken independently, for given $n \neq n_1, n_2, \cdots$, over all k for which at least one of x_{k-1} and x_k is in $[z_{n-1}, z_n]$.

Give the name δ_n to the interval $[z_{n-1}, z_n]$, and put

$$N(\alpha, \delta_n) = N(z_{n-1} + \alpha(z_n - z_{n-1})) - N(z_{n-1}),$$

$$N(\delta_n) = N(1, \delta_n) = N(z_n) - N(z_{n-1}).$$

It will be shown that

$$\lim_{\substack{n \to \infty \\ n \neq n_1, n_2, \cdots}} \frac{N(\alpha, \delta_n)}{N(\delta_n)} = \alpha;$$

in other words, that in the limit the x_k's which lie in $\delta_n \neq \delta_{n_t}$ are u.d. there. This implies the theorem, for using it, (1), and (i) we have, for $x \in \delta_n$,

$$\frac{N(\alpha, x)}{N(x)} = \frac{1}{N(x)}\left\{\sum_{\nu=1}^{n-1} N(\alpha, \delta_n) + N(\min(x, z_{n-1} + \alpha(z_n - z_{n-1}))) - N(z_{n-1})\right\}$$

$$= \frac{1}{N(x)} \sum{}^{\circ} N(\alpha, \delta_\nu) + o(1)$$

$$= \frac{\sum^{\circ} (\alpha + o(1))\, N(\delta_\nu)}{\sum^{\circ} N(\delta_\nu) + o\left(\sum^{\circ} N(\delta_\nu)\right) + N(x) - N(z_{n-1})} + o(1)$$

$$= \frac{\alpha}{1 + o(1)} + o(1) = \alpha + o(1),$$

where $\sum^{\circ}$ denotes summation from $\nu = 1$ to $\nu = n - 1$, $\nu \neq n_1, n_2, \cdots$.

To prove (2), suppose that $n \neq n_1, n_2, \cdots$, that $z_{n-1} \in (x_{k_n}, x_{k_n+1}]$, and that

$$\min_{k_n \le k \le k_{n+1}} (x_k - x_{k-1}) = X_n.$$

Then for $k_n \le k \le k_{n+1}$, we have $x_k - x_{k-1} = (1 + \epsilon_{kn}) X_n$, where ϵ_{kn} is a positive quantity tending to zero as $n \longrightarrow \infty$. Put

$$\epsilon_n = \max_{k_n \le k \le k_{n+1}} \epsilon_{kn},$$

and put $\Delta x_k = x_k - x_{k-1}$. Now if

$$x_{k_n+t} \le z_{n-1} + \alpha(z_n - z_{n-1}) < x_{k_n+t+1},$$

then

$$\alpha(z_n - z_{n-1}) = (x_{k_n+1} - z_{n-1}) + \sum_{k=k_n+2}^{k_n+t} \Delta x_k + (z_{n-1} + \alpha(z_n - z_{n-1}) - x_{k_n+t})$$

$$= \sum_{s=1}^{t} \Delta x_{k_n+s} + \epsilon_n' X_n,$$

where $\epsilon_n' = O(1)$ as $n \longrightarrow \infty$. But

$$tX_n \leq \sum_{s=1}^{t} \Delta x_{k_n+s} \leq tX_n + t\epsilon_n X_n \leq tX_n + u\epsilon_n X_n,$$

where $u = N(z_n) - N(z_{n-1})$. Hence

$$\alpha \frac{z_n - z_{n-1}}{X_n} - \epsilon_n' - u\epsilon_n \leq t \leq \alpha \frac{z_n - z_{n-1}}{X_n} - \epsilon_n'.$$

Similarly,

$$\frac{z_n - z_{n-1}}{X_n} - \epsilon_n' - u\epsilon_n \leq u \leq \frac{z_n - z_{n-1}}{X_n} - \epsilon_n',$$

so that

$$\frac{\alpha(z_n - z_{n-1})/X_n - \epsilon_n' - u\epsilon_n}{(z_n - z_{n-1})/X_n - \epsilon_n'} \leq \frac{t}{u} \leq \frac{(z_n - z_{n-1})/X_n - \epsilon_n'}{(z_n - z_{n-1})/X_n - \epsilon_n' - u\epsilon_n}.$$

Since $N(z_n) - N(z_{n-1}) \longrightarrow \infty$ as $n \longrightarrow \infty$, also $(z_n - z_{n-1})/X_n \longrightarrow \infty$, and so

$$\frac{\alpha + o(1) - u\epsilon_n X_n/(z_n - z_{n-1})}{1 + o(1)} \leq \frac{t}{u} \leq \frac{\alpha + o(1)}{1 + o(1) - u\epsilon_n X_n/(z_n - z_{n-1})}.$$

But since

$$uX_n \leq \sum_{k=k_n+1}^{k_{n+1}} \Delta x_k \leq z_n - z_{n-1},$$

$uX_n = O(z_n - z_{n-1})$; thus

$$\frac{\alpha + o(1)}{1 + o(1)} \leq \frac{t}{u} \leq \frac{\alpha + o(1)}{1 + o(1)},$$

and therefore

$$\frac{N(\alpha, \delta_n)}{N(\delta_n)} = \frac{t}{u} \sim \alpha.$$

This completes the proof.

In case $\Delta = \Delta_0$ and $x_k = f(k)$, it is easily seen that the hypotheses of Fejér's theorem imply two of the hypotheses of Theorem 2, namely that $N(z_n) -$

$N(z_{n-1}) \uparrow \infty$ and $N(z_{n-1}) \sim N(z_n)$ as $n \longrightarrow \infty$. But I do not know whether Theorem 2 includes Fejér's theorem; the most that I can show is that the exceptional sequence $\{z_{n_t}\} = \{n_t\}$ mentioned in (ii) of Theorem 2 is in this case of density zero, which does not imply (1) for all functions f satisfying the hypotheses of Fejér's theorem. Certainly, however, Theorem 2 deals with cases not covered by the following direct extension of Fejér's theorem, since it does not require the monotonicity of either $z_n - z_{n-1}$ or Δx_k.

THEOREM 3. *The sequence* $\{x_k\}$ *is u.d.* (mod Δ) *if the following conditions are satisfied*:

(i) $z_n - z_{n-1} \geq z_{n-1} - z_{n-2}$ *for* $n = 2, 3, \cdots,$

(ii) $\Delta x_k \downarrow 0$ *as* $k \uparrow \infty$,

(iii) $N(z_{n-1}) \sim N(z_n)$ *as* $n \longrightarrow \infty$.

We sketch the proof. Let ψ be the continuous polygonal function such that $\psi(x_k) = k$; then $0 \leq \psi(x) - N(x) < 1$. Let $\{\epsilon_k\}$ be such that $\epsilon_k = o(\Delta x_k)$ and $0 < \epsilon_k < \Delta x_k/2$ for $k = 1, 2, \cdots$. Define ψ_1 as follows:

$$\psi_1(x) = \frac{1}{2\epsilon_k} \int_{x-\epsilon_k}^{x+\epsilon_k} \psi(t)\, dt \quad \text{for} \quad x \in \left[x - \frac{1}{2}\Delta x_k, x_k + \frac{1}{2}\Delta x_{k+1}\right]$$

$$(k = 2, 3, \cdots).$$

Then ψ_1 is continuously differentiable, and is identical with ψ except at the corners of ψ, where it is smooth. For $0 \leq \alpha \leq 1$, $n = 1, 2, 3, \cdots$, put

$$\rho(n + \alpha) = \psi_1(z_{n-1} + \alpha(z_n - z_{n-1}));$$

ρ is continuously differentiable except at $x = 1, 2, \cdots$. A function ρ_1 can now be defined in terms of ρ, just as ψ_1 was determined from ψ, so that ρ_1 is everywhere continuously differentiable, and ρ_1 differs from ρ only on an interval about $x = n (n = 1, 2, \cdots)$ whose length ϵ_n' is of lower order of magnitude than Δx_{k_n} if $z_n \in [x_{k_n - 1}, x_{k_n})$. If $x = n + \alpha$ is such that

$$\rho_1(x) = \rho(x), \quad \psi_1(z_{n-1} + \alpha(z_n - z_{n-1})) = \psi(z_{n-1} + \alpha(z_n - z_{n-1})),$$

and

$$z_{n-1} + \alpha(z_n - z_{n-1}) \in (x_{k-1}, x_k),$$

then

$$\rho_1'(x) = \frac{z_n - z_{n-1}}{\Delta x_k};$$

it follows that $\rho_1'(x) \nearrow \infty$. Moreover, since

$$\frac{\rho_1(n+1)}{\rho_1(n)} \sim \frac{\psi(z_n)}{\psi(z_{n-1})} \sim \frac{N(z_n)}{N(z_{n-1})} \longrightarrow 1,$$

it follows that $\rho_1'(x)/\rho_1(x) \longrightarrow 0$ as $x \longrightarrow \infty$. But if f is the function inverse to ρ_1, these facts imply that $f(x) \uparrow \infty$, $f'(x) \searrow 0$, and $xf'(x) \longrightarrow \infty$ as $x \uparrow \infty$. Since $f(k) \longrightarrow x_k$ as the arbitrary numbers ϵ_k and ϵ_n' approach zero, the conclusion follows from Fejér's theorem.

A trivial variation of Theorem 3 has, instead of (i) and (ii), the hypotheses

(i′) $z_n - z_{n-1} \uparrow \infty$,

(ii′) $\Delta x_{k-1} \geq \Delta x_k$ for $k = 2, 3, \cdots$.

For then it will still be true that $\rho_1'(x) \nearrow \infty$ as $x \uparrow \infty$.

4. It follows from Theorem 2 (and also from the variation of Theorem 3 just mentioned) that *if* $z_n - z_{n-1} \nearrow \infty$ *in such a way that* $z_{n-1} \sim z_n$, *the sequence* $\{k\theta\}$ *is u.d.* (mod Δ) *for each* $\theta > 0$. In this section we examine the distribution of $\{k\theta\}$ (mod Δ) when $\delta(x) \searrow 0$. This is a problem of a very different kind from the earlier one; the result is expressed in the following metric theorem:

THEOREM 4. *If* $\delta(x) \searrow 0$ *and* $\delta(x) = O(x^{-1})$ *then* $\{k\theta\}$ *is u.d.* (mod Δ) *for almost all* $\theta > 0$.

The proof depends on a principle used in an earlier paper [2]:

If C *and* ϵ *are positive constants and* $\{f_k\}$ *is a sequence of real-valued functions such that*

$$(3) \qquad \left| \int_a^b e^{i(f_j(x) - f_k(x))}\, dx \right| \leq \frac{C}{\max(1, |j-k|^{\epsilon})}, \qquad (j, k = 1, 2, \cdots),$$

then $\{f_k(x)\}$ *is u.d.* (mod 1) *for almost all* $x \in (a, b)$.

This will be applied with $f_k(x) = \phi(kx)$, where ϕ is the function defined in § 1; it was noted there that the u.d. (mod Δ) of $\{x_k\}$ is equivalent to the u.d. (mod 1) of $\{\phi(x_k)\}$. Let a and b be arbitrary positive numbers with $a < b$, and put

$$J_{jk} = \int_a^b e^{i(f_j(x) - f_k(x))}\, dx ;$$

since J_{kj} and J_{jk} are complex conjugates, it suffices to consider the case $j > k$. For fixed j and k, denote by $\xi_0, \cdots \xi_r$ all the numbers of the form z_m/j or z_m/k in the interval (a, b), so named that $\xi_0 < \cdots < \xi_r$. Then the function

$$f_j(x) - f_k(x) = \left(\frac{j}{\delta(jx)} - \frac{k}{\delta(kx)}\right) x - \left(\frac{[jx]_\Delta}{\delta(jx)} - \frac{[kx]_\Delta}{\delta(kx)}\right) = xA(x) + B(x)$$

is linear in each interval $[\xi_{l-1}, \xi_l)$, $A(x)$ and $B(x)$ being certain constants A_l and B_l there. Hence

$$J_{jk} = \sum_{l=1}^{r} \int_{\xi_{l-1}}^{\xi_l} e^{i(A_l x + B_l)}\, dx = \sum_{l=1}^{r} \frac{e^{i(A_l \xi_l + B_l)} - e^{i(A_l \xi_{l-1} + B_l)}}{iA_l}.$$

Since f is continuous,

$$A_l\, \xi_l + B_l = A_{l+1}\, \xi_l + B_{l+1},$$

and so for $1 \le t \le r$,

$$\sum_{l=1}^{t} [e^{i(A_l \xi_l + B_l)} - e^{i(A_l \xi_{l-1} + B_l)}] = e^{i(A_t\, \xi_t + B_t)} - e^{i(A_1\, \xi_0 + B_1)}$$

Thus, using the relation

$$\sum_{m=1}^{n} a_m b_m = \sum_{m=1}^{n-1} \left(\sum_{\mu=1}^{m} a_\mu\right) (b_m - b_{m+1}) + b_n \sum_{\mu=1}^{n} a_\mu ,$$

we have

$$J_{jk} = \frac{1}{i} \sum_{t=1}^{r-1} (e^{i(A_t \xi_t + B_t)} - e^{i(A_1\, \xi_0 + B_1)}) \left(\frac{1}{A_t} - \frac{1}{A_{t+1}}\right)$$

$$+ (e^{i(A_r \xi_r + B_r)} - e^{i(A_1\, \xi_0 + B_1)}) \frac{1}{iA_r} ,$$

and so

$$|J_{jk}| \leq 2 \sum_{t=1}^{r-1} \left| \frac{1}{A_t} - \frac{1}{A_{t+1}} \right| + \frac{2}{|A_r|}. \tag{4}$$

By the facts that $\xi_t \geq a > 0$, $\delta(x) \searrow 0$ as $x \longrightarrow \infty$, and

$$A_t = \frac{j}{\delta(j\xi_{t-1})} - \frac{k}{\delta(k\xi_{t-1})},$$

it is clear that

$$A_t > C(j-k) > 0$$

for $t = 1, 2, \cdots, r$, so that (3) will follow from (4) if it can be shown that for some c, $\epsilon > 0$, the inequality

$$\sum_{t=1}^{r-1} \left| \frac{1}{A_t} - \frac{1}{A_{t+1}} \right| < \frac{c}{(j-k)^\epsilon}$$

holds. Moreover, writing

$$C_t = \frac{1}{A_t} - \frac{1}{A_{t+1}}$$

and

$$\sum_{t=1}^{r-1} |C_t| = \sum_{t=1}^{r} C_t - 2\sum{}' C_t = \frac{1}{A_1} - \frac{1}{A_r} - 2 \sum{}' C_t,$$

where $\sum'$ is the sum over those t for which $C_t < 0$, we see that it suffices to show that

$$\sum{}' |C_t| < \frac{c}{(j-k)^\epsilon}.$$

We consider three cases. Suppose first that t is such that $\xi_{t+1} = z_m/j$ for some m, but that for no l is $\xi_{t+1} = z_l/k$. Then

$$A_t = \frac{j}{\delta(z_{m-1})} - \frac{k}{\delta(k\xi_t)}, \quad A_{t+1} = \frac{j}{\delta(z_m)} - \frac{k}{\delta(k\xi_t)},$$

so that $A_{t+1} \geq A_t$, and the term C_t does not occur in $\sum'$. If

$$\xi_{t+1} = z_m/j = z_l/k,$$

then $z_m > z_l$ and

$$C_t = \frac{1}{j/\delta(z_{m-1}) - k/\delta(z_{l-1})} - \frac{1}{j/\delta(z_m) - k/\delta(z_l)}$$
$$\geq \frac{-k(1/\delta(z_l) - 1/\delta(z_{l-1}))}{(j/\delta(z_{m-1}) - k/\delta(z_{l-1}))(j/\delta(z_m) - k/\delta(z_l))}.$$

Finally, if $\xi_{t+1} = z_l/k$ for some l, but $\xi_{t+1} \neq z_m/j$ for every m, then

$$C_t = \frac{-k(1/\delta(z_l) - 1/\delta(z_{l-1}))}{(j/\delta(j\xi_{t+1}) - k/\delta(z_{l-1}))(j/\delta(j\xi_{t+1}) - k/\delta(z_l))}.$$

Thus, writing $\delta(x^+)$ and $\delta(x^-)$ for $\lim_{\xi\to x^+}\delta(\xi)$ and $\lim_{\xi\to x^-}\delta(\xi)$, we have

$$\sum{}' |C_t| \leq k \sum{}'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j/\delta(j\xi_{t+1}^-) - k/\delta(z_{l-1}))(j/\delta(j\xi_{t+1}^+) - k/\delta(z_l))}$$
$$= \sum{}'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j/\delta(jz_l^-/k) - k/\delta(z_{l-1}))(j/\delta(jz_l^+/k) - k/\delta(z_l))},$$

where $\sum''$ denotes summation with respect to l with $z_l/k \in (a, b)$. But

$$\delta(jz_l^-/k) \leq \delta(z_{l-1})$$

and

$$\delta(jz_l^+/k) \leq \delta(z_l),$$

and so

$$\sum{}' |C_t| \leq k \sum{}'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j-k)^2/\delta(z_{l-1})\delta(z_l)}$$
$$= \frac{k}{(j-k)^2} \sum{}'' \{\delta(z_{l-1}) - \delta(z_l)\} \leq \frac{2k\,\delta(ka)}{(j-k)^2}.$$

If now $\delta(x) = O(1/x)$, then

$$\sum{}' |C_t| = O\left(\frac{1}{(j-k)^2}\right),$$

and the proof is complete.

5. **The** preceding result can be generalized considerably by using the following transfer theorem:

THEOREM 5. *Suppose that* $\{x_k\}$ *is u.d.* (mod Δ), *where* $\Delta = \{z_n\}$, *and that* f *is a function which is differentiable except possibly at the points* $z_1, z_2, \cdots$, *such that* $f(x) \uparrow \infty$ *as* $x \uparrow \infty$ *and*

$$\inf_{x \in (z_{n-1}, z_n)} f'(x) \sim \sup_{x \in (z_{n-1}, z_n)} f'(x). \tag{5}$$

Then the sequence $\{x_k^*\} = \{f(x_k)\}$ *is u.d.* (mod Δ^*), *where* $\Delta^* = \{f(z_n)\}$.

Put

$$N(\alpha, x) = \sum 1, \; N(1, x) = N(x), \; N^*(\alpha, x) = \sum{}^* 1, \; N^*(1, x) = N^*(x),$$

where $\sum$ denotes summation with $x_k \le x$ and $\langle x_k \rangle_\Delta < \alpha$ and $\sum^*$ denotes summation with $x_k^* \le x$, $\langle x_k^* \rangle_{\Delta^*} < \alpha$. Since f is an increasing function,

$$N^*(f(x)) = \sum_{f(x_k) \le f(x)} 1 = \sum_{x_k \le x} 1 = N(x).$$

By assumption, the relation

$$\lim_{x \to \infty} \frac{N(\alpha, x)}{N(x)} = \alpha$$

holds for $\alpha \in [0, 1]$. So we need only show that $N^*(\alpha, f(x)) \sim N(\alpha, x)$ as $x \longrightarrow \infty$, and by Theorem 1 it suffices to prove this as x runs through the sequence $\{z_n\}$. But

$$N(\alpha, z_n) = \sum_{m=1}^{n} \{N(z_{m-1} + \alpha(z_m - z_{m-1})) - N(z_{m-1})\},$$

and so

$$N^*(\alpha, f(z_n)) = \sum_{m=1}^{n} \{N^*(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) - N^*(z_{m-1}^*)\}$$

$$= N(\alpha, z_n) + \sum_{m=1}^{n} \{N^*(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\}.$$

Thus the problem reduces to showing that

$$\sum_{m=1}^{n} \{N^*(z^*_{m-1} + \alpha(z^*_m - z^*_{m-1})) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\} = o(N(\alpha, z_n)),$$

or what is the same thing, that

$$(6) \sum_{m=1}^{n} \{N(f^{-1}(z^*_{m-1} + \alpha(z^*_m - z^*_{m-1}))) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\} = o(N(z_n)).$$

Put

$$f^{-1}(z^*_{m-1} + \alpha(z^*_m - z^*_{m-1})) = u_m(\alpha),$$
$$z_{m-1} + \alpha(z_m - z_{m-1}) = v_m(\alpha).$$

If it can be shown that

$$(7) \qquad |u_m(\alpha) - v_m(\alpha)| < \epsilon_m(z_m - z_{m-1}),$$

where $\epsilon_m \longrightarrow 0$ as $m \longrightarrow \infty$, then for every $\epsilon > 0$,

$$\sum_{m=1}^{n} \{N(u_m(\alpha)) - N(v_m(\alpha))\}$$
$$= O\left(\sum_{m=1}^{n} \{N(v_m(\alpha) + \epsilon(z_m - z_{m-1})) - N(v_m(\alpha))\}\right)$$
$$= O(N(\epsilon, z_n)) = O(\epsilon N(z_n)),$$

which implies (6).

Now

$$u_m(0) = v_m(0), u_m(1) = v_m(1),$$

and

$$u_m(\alpha) - v_m(\alpha) = f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1})))$$
$$- (z_{m-1} + \alpha(z_m - z_{m-1}));$$

hence

$$u_m'(\alpha) - v_m'(\alpha) = \frac{f(z_m) - f(z_{m-1})}{f'\{f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1})))\}} - (z_m - z_{m-1}).$$

To maximize $u_m(\alpha) - v_m(\alpha)$, we must have

$$f(z_m) - f(z_{m-1}) - (z_m - z_{m-1}) f'\{f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1})))\} = 0.$$

There is a $Z_0 \in (z_{m-1}, z_m)$ such that

$$\frac{f(z_m) - f(z_{m-1})}{z_m - z_{m-1}} = f'(Z_0),$$

and a corresponding $\alpha_0 \in (0, 1)$ such that

$$f(z_{m-1}) + \alpha_0(f(z_m) - f(z_{m-1})) = f(Z_0),$$

(so that $u_m'(\alpha_0) - v_m'(\alpha_0) = 0$) for which

$$|u_m(\alpha) - v_m(\alpha)| \le |u_m(\alpha_0) - v_m(\alpha_0)| = |Z_0 - v_m(\alpha_0)|$$

for all $\alpha \in (0, 1)$. But

$$v_m(\alpha_0) = z_{m-1} + \frac{f(Z_0) - f(z_{m-1})}{f(z_m) - f(z_{m-1})}(z_m - z_{m-1})$$

$$= z_{m-1} + \frac{f(Z_0) - f(z_{m-1})}{f'(Z_0)},$$

so that

$$Z_0 - v_m(\alpha_0) = Z_0 - z_{m-1} - \frac{f(Z_0) - f(z_{m-1})}{f'(Z_0)}$$

and

$$|u_m(\alpha) - v_m(\alpha)| \le \sup_{Z \in \delta_m}\left(|Z - z_{m-1}| \left|1 - \frac{f(Z) - f(z_{m-1})}{(Z - z_{m-1}) f'(Z)}\right|\right),$$

whence

$$\left|\frac{u_m(\alpha) - v_m(\alpha)}{z_m - z_{m-1}}\right| \le \sup_{\substack{Z \in \delta_m \\ W \in \delta_m}} \left|1 - \frac{f'(W)}{f'(Z)}\right|,$$

and this last upper bound is $o(1)$ as $m \longrightarrow \infty$. Thus (7) holds, and the proof is complete.

If the f of Theorem 5 is taken to be an arbitrary increasing polygonal function, with vertices on the abscissas $x = z_1, z_2, \cdots$, then the condition (5) on the derivative is trivially satisfied. Such a transformation merely represents a change of scale inside each interval δ_n, and the distribution modulo Δ of any sequence $\{x_k\}$ is identical with the distribution of $\{f(x_k)\}$ modulo Δ^*.

In case f' is monotone, (5) can be replaced by the simpler condition

$$(5') \qquad f'(z_{n-1}) \sim f'(z_n) \qquad \text{as } n \longrightarrow \infty.$$

Combining this version of Theorem 5 with Theorem 4, we have:

THEOREM 6. *The sequence $\{f(k\theta)\}$ is u.d.* (mod Δ) *for almost all $\theta > 0$ if $f(x) \uparrow \infty$, f' is monotonic, and*

$$f^{-1}(z_n) - f^{-1}(z_{n-1}) \searrow 0,$$

$$f^{-1}(z_n) - f^{-1}(z_{n-1}) = O\left(\frac{1}{f^{-1}(z_n)}\right),$$

$$f'(f^{-1}(z_n)) \sim f'(f^{-1}(z_{n-1})),$$

where f^{-1} is the function inverse to f.

COROLLARY. *The sequence $\{\alpha^k\}$ is u.d.* (mod Δ) *for almost all $\alpha > 1$ if $z_n = g(n)$, where g is an increasing function with monotonic logarithmic derivative such that*

$$(8) \qquad \frac{g'(x)}{g(x)} = O(x^{-1/2}).$$

For writing α^k as $e^{k \log \alpha}$, we see that we can take the f of Theorem 6 to be the exponential function, and the conditions displayed there become

$$\log z_n - \log z_{n-1} \searrow 0,$$

$$\log z_n - \log z_{n-1} = O\left(\frac{1}{\log z_n}\right),$$

$$z_n \sim z_{n-1}.$$

Of these, the third is implied by the first. Since

$$\frac{d}{dx} \log g(x) \searrow 0,$$

it is clear that $\log g(n) - \log g(n-1) \searrow 0$. From the extended law of the mean,

$$\frac{G(x) - G(x-1)}{H(x) - H(x-1)} = \frac{G'(X)}{H'(X)}, \quad X \in (x-1, x),$$

it follows that if $G'(x) = O(H'(x))$, then

$$G(x) - G(x-1) = O(H(x) - H(x-1)).$$

Taking

$$G(x) = \log g(x), \quad H(x) = \log e^{\sqrt{x}} = \sqrt{x},$$

we have by (8) that

$$\log g(n) - \log g(n-1) = O(n^{-1/2}).$$

But it also follows from the relation $G'(x) = O(H'(x))$ that $G(x) = O(H(x))$; hence

$$\log g(x) = O(x^{1/2}), \quad n^{-1/2} = O((\log g(n)^{-1}),$$

and the proof is complete.

For sufficiently smooth g, (8) can be replaced by the condition $g(x) = O(\exp \sqrt{x})$.

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DERIVATIVES OF INFINITE ORDER

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1. Introduction. The major purpose here is to reexamine, chiefly from the standpoint of summation by Borel's exponential means, a number of problems concerning the existence and form of

$$\lim_{n \to \infty} f^{(n)}(x),$$

for x a real variable in an interval. Several articles have been contributed on this topic [**5, 6, 11, 16**], all of which take the limit process involved to be ordinary convergence. In one [5], however, Boas and Chandrasekharan point to the desirability of interpreting the limit process in a more general sense and state without proof that one of their results (the case $\alpha = 1$, $\lambda_n = 1$ for all n, of Theorem 4 below) can be established by their method for any (presumably linear) summation method T having the property that, as $n \longrightarrow \infty$,

(1) T-lim s_n exists and equals s implies T-lim s_{n-1} exists and equals s.

Borel's method of exponential means, like his integral method, possesses property (1) although, curiously, not its converse, as Hardy [cf. 9, pp. 183, 196] pointed out. Methods satisfying both (1) and its converse include ordinary convergence and the summation methods of Abel, Cesàro, Euler, Hölder, and, when regular (see below), Voronoi-Nörlund.

It is not clear from [5] just how their proof of the cited result (that $f^{(n)}(x) \longrightarrow g(x)$ dominatedly in (a, b) implies $g(x) = ke^x$) can really be carried over to *all* linear summation methods of type (1). Since the transform $\{F_m(x)\}$, m discrete or continuous, of the sequence $\{f^{(n)}(x)\}$ converges dominatedly, it follows that

$$\lim_{m \to \infty} \int_c^x F_m(t)\,dt = \int_c^x g(t)\,dt, \text{ uniformly for } c,\ x \text{ in } (a, b).$$

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But further argumentation is needed to justify interchanging (in the left member) the integral and whatever limit process may be involved in defining $F_m(x)$ in terms of $\{f^{(n)}(x)\}$, which would seem to be the next step in the proof. Where $F_m(x)$ is a *finite* linear combination of $f(x), \cdots, f^{(m)}(x)$, as in the Cesàro, Euler, Hölder, and Voronoi-Nörlund methods, this is trivial. In the Abel and Borel methods, for example, however, the transforms involve infinite series. The usual difficulties incident to an interchange of limits therefore intrude themselves at this point of the argument. Perhaps this difficulty can be overcome; but [5] does not suggest how.

In the case of Borel's exponential means these difficulties can be avoided and more complete results obtained otherwise by rather simple arguments which get to the heart of the problem more directly. Borel's exponential means provide a natural tool for working with the problems at hand; for, when applied to the sequence $\{f^{(n)}(x)\}$, they give rise to the Taylor expansion of $f(x)$. Repeated use can then be made of the property that the value to which the Taylor series of an analytic function converges is independent of the point around which the expansion is taken, since the hypotheses of most of the theorems below either assume or imply that $f(x)$ is analytic.

A sequence $\{s_n\}$, $n = 0, 1, 2, \cdots$, is said to be B_α-*summable* to the value s if

$$(2) \qquad \alpha \lim_{r\to\infty} e^{-r} \sum_{n=0}^{\infty} \frac{s_n}{(\alpha n)!} r^{\alpha n} = s.$$

When (2) is satisfied, it is also written as

$$(3) \qquad B_\alpha\text{-}\lim_{n\to\infty} s_n = s.$$

This method is *regular* (sometimes called *permanent*) in the sense that any sequence $\{s_n\}$ converging in the ordinary sense to a value s is also B_α-summable and to the same value s.

If $\alpha = 1$, the definition (2) describes summation by Borel's exponential means. B_1-summation is denoted simply as B-summation, and, when $\alpha = 1$, (3) is written B-lim $s_n = s$.

B_α-summation possesses property (1) when α is a positive integer, since B-summation does: Let B_α-lim $s_n = s$ and define t_k to be αs_n when $k = \alpha n$ and to be 0 otherwise. Then B-lim $t_k = s$ and, upon α applications of (1), B-lim $t_{k-\alpha} = s$. But this last is the same as asserting B_α-lim $s_{n-1} = s$, completing the proof.

2. Borel limits of the sequence of derivatives. We shall establish the following result.

THEOREM 1. *If $f(x)$ is analytic in the real interval (a, b), and if*

$$B\text{-}\lim_{n \to \infty} f^{(n)}(x_0) = ke^{x_0}$$

for a single x_0 in (a, b), then

$$B\text{-}\lim_{n \to \infty} f^{(n)}(x) = ke^{x}$$

for each x in (a, b). The convergence is uniform if the interval (a, b) is finite.

Proof. The function $f(x)$ can be represented by its Taylor series in (a, b), being analytic in that interval. Thus

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(t - x_0)^n \qquad \text{for } t, x_0 \text{ in } (a, b). \tag{4}$$

The power series has an infinite radius of convergence in t for x_0 in (a, b), since the existence of the Borel limit of $f^{(n)}(x_0)$ may be written (with $r = t - x_0$)

$$\lim_{t \to \infty} e^{-(t-x_0)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(t - x_0)^n = ke^{x_0}. \tag{5}$$

Thus $f(t)$, t in (a, b), possesses a unique analytic extension $\phi(t)$, and this function is an entire function. Thus (5) can be written as

$$\lim_{t \to \infty} e^{-t}\phi(t) = k. \tag{6}$$

Expanding $\phi(t)$ about an arbitrary point x in (a, b), multiplying both sides of (6) by e^x, and placing $r = t - x$ completes the proof of the theorem, except for the part dealing with uniform convergence.

To prove that the convergence is uniform when (a, b) is finite, let $\epsilon > 0$ be given and find t_0 (whose existence is assured by (6)) such that

$$|e^{-t}\phi(t) - k| < \epsilon \qquad \text{for } t > t_0.$$

Then

$$| e^{-(t-x)} \phi(t) - ke^{x} | < \epsilon e^{x} < \epsilon e^{b}$$

for $t > t_0$ and all x in (a, b), and

$$\left| e^{-(t-x)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (t-x)^n - ke^{x} \right| < \epsilon e^{b}$$

for $t > t_0$ and all x in (a, b).

Hence, putting $r = t - x$, we get

$$\left| e^{-r} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} r^n - ke^{x} \right| < \epsilon e^{b}$$

for $r > t_0 - a$ and all x in (a, b). This completes the proof.

An examination of this proof makes it clear that the point x_0 and the interval (a, b) do not have to be required to be real. What is essential is to have the quantity $t - x_0$ become positively infinite through real values, to conform to the definition of Borel summation. With this in mind, we can rephrase Theorem 1 in the following somewhat more general form:

THEOREM 1.′ *If $f(x + iy_0)$, regarded as a function of the real variable x, is analytic for $a < x < b$, y_0 fixed, and if*

$$B\text{-}\lim_{n \to \infty} f^{(n)}(x_0 + iy_0) \text{ exists and equals } ke^{x_0 + iy_0}$$

for a single x_0 in (a, b), then

$$B\text{-}\lim_{n \to \infty} f^{(n)}(x + iy_0) \text{ exists and equals } ke^{x + iy_0}$$

for each x in (a, b). The convergence is uniform if the interval (a, b) is finite.

This theorem enables one to pass from a fixed point $z_0 = x_0 + iy_0$ in the complex plane to any other point in a certain interval on the horizontal line passing through z_0. But what about points z not on this line? The proof of Theorem 1 is not adequate to cover this situation, since it must be shown that the limit in (6) exists and has the value k as $r = t - z$ becomes positively infinite through real values. (Here the complex value z replaces the real number x.) This is required by the very definition of Borel summation. In turn, moreover, this necessitates establishing that the limit (6) exists and equals k as t becomes infinite to the right, not only on the given horizontal line $y = y_0$, but also on other horizontal lines. This can be done in certain circumstances.

THEOREM 1′. *Let $f(z)$ be analytic in S, a horizontal half-strip, quadrant, or half-plane, opening to the right:*

$$z = x + iy, \ x = a, \ c < y < d.$$

Let $f(z) = O(e^z)$ as z becomes infinite in S. Suppose that

$$B\text{-}\lim_{n \to \infty} f^{(n)}(z_0) = ke^{z_0}$$

for a single z_0 in S. Then

$$B\text{-}\lim_{n \to \infty} f^{(n)}(z) \text{ exists and equals } ke^z$$

for all z in S. If c and d are finite, then the convergence is uniform in $c + \delta \leq y \leq d - \delta$ for any positive δ. If S is a quadrant or half-plane, then the convergence is uniform in any half-strip in its interior.

Proof. In the preliminary discussion, it has been noted that only one issue needs be settled in order to extend the proof of Theorem 1 to this theorem as well: That is the existence and value of the limit in (6) as $t - z$, z an arbitrary point in S, becomes positively infinite through real values, where the imaginary parts of z and z_0 may be unequal. This limit, for z arbitrary in S, does exist and have the value k under the assumption made here that $f(z) = O(e^z)$ as $z \longrightarrow \infty$ in S. This follows from Montel's theorem [**15**, p. 170], after that theorem has been expressed in terms of the horizontal strips involved here, rather than the vertical strips used in [**15**]. The conclusion concerning uniformity is also a consequence of this formulation of Montel's theorem.

THEOREM 2. *If $f(x)$ belongs to a Denjoy-Carleman quasi-analytic class in the (open) interval (a, b) and if*

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x_0) = ke^{x_0}$$

for a single x_0 in the open interval (a, b), then $f(x)$ is analytic in (a, b) (and

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x) = ke^{x}$$

for all x, $a < x < b$).

Proof. It is sufficient to prove the first half of the conclusion, the analyticity of $f(x)$; the other half is then a consequence of Theorem 1.

As in the previous proof, the Borel summability of the sequence $\{f^{(n)}(x_0)\}$ implies that the right hand member of (4) has an infinite radius of convergence, and so defines an entire function $\phi(t)$. Expanding $\phi(t)$ in a Taylor series about the point x_0 in (a, b) shows that

$$\phi^{(n)}(x_0) = f^{(n)}(x_0) \qquad (n = 0, 1, 2, \cdots) .$$

The analyticity of $f(x)$ in (a, b) is a consequence of the following result of Bang [1, p. 84], as quoted in [6]: "... If $f(x)$ belongs to a quasianalytic class on $a < x < b$ and $g(x)$ is analytic, then $f^{(n)}(x_0) = g^{(n)}(x_0)$ for all n and $a < x_0 < b$ implies $f(x) \equiv g(x) \cdots$." This completes the proof.

The next theorem provides a simple set of necessary and sufficient conditions on the structure of $f(x)$ as well as on that of $g(x)$. That these conditions are not sufficient if convergence is used instead of Borel summation is shown by the example

$$f(x) = ke^{x} + \sin x .$$

The Borel limit of the sequence of derivatives exists and equals ke^{x} for all x, whereas the (convergence) limit of this sequence does not even exist. Analyticity is not assumed in the necessity part of the theorem, but is inferred as in Theorem 1 of [5].

THEOREM 3. *A set of necessary and sufficient conditions that*

$$B\text{-}\lim_{n\to\infty} f^{(n)}(x) = g(x)$$

for each x in (a, b), where $g(x)$ is finite, is (i) *that $f(x)$ coincide in (a, b)*

with an entire function $\phi(x)$, *having the property that*

$$\phi(x) = ke^x + o(e^x),$$

as x *becomes infinite, and* (ii) *that*

$$g(x) = ke^x, \ x \ in \ (a, b).$$

Proof of sufficiency. Here

$$\phi(t) = ke^t + o(e^t); \quad \phi(t) = f(t), \text{ for } t \text{ in } (a, b),$$

and $\phi(t)$ is an entire function. Then

$$e^{-(t-x)}\phi(t) = e^{-(t-x)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (t-x)^n, \ x \text{ in } (a, b).$$

By hypothesis,

$$\lim_{t \to \infty} e^{-(t-x)} \phi(t) = ke^x,$$

whence, with $r = t - x$,

$$\lim_{r \to \infty} e^{-r} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} r^n = ke^x,$$

completing the proof of sufficiency.

Necessity. Putting $r = t - x$, we can write the assumption of Borel summability as follows:

$$\lim_{t \to \infty} e^{-(t-x)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (t-x)^n = g(x) \qquad \text{for each } x \text{ in } (a, b).$$

This implies that the radius of convergence of the power series above is infinite for each x in (a, b). Hence $f(t)$ is analytic in (a, b), as a consequence of a theorem of Pringsheim [13] for which a complete proof was supplied first by Boas [4] and again later by Zahorski [17]. In fact, $f(t)$ has as analytic continuation an entire function, $\phi(t)$. Then

$$\lim_{t\to\infty} e^{-(t-x)}\,\phi(t) = g(x) \qquad \text{for each } x \text{ in } (a, b),$$

whence

$$\lim_{t\to\infty} e^{-t}\,\phi(t) = e^{-x}\,g(x) \qquad \text{for each } x \text{ in } (a, b).$$

The left side is independent of x since $\phi(t)$ is, and this is the case because the values of an analytic function do not depend on the point in the region of analyticity around which the function is expanded. Hence the right side must be a constant k. This completes the proof.

3. Subsequences of $\{f^{(n)}(x)\}$. For the proof of the theorem below, the following lemma is needed. The proof given first is due to Julian H. Blau.

LEMMA 1. *If a sequence of polynomials, $\{P_n(x)\}$, defined in the closed interval $[c, d]$, each of which is of degree at most β, has a limit $h(x)$ in $[c, d]$, then this limit is likewise a polynomial of degree at most β.*

Proof of lemma (by induction). Let each $P_n(x)$ be written as a polynomial in $x - c$.

(i) The lemma is obvious for $\beta = 0$.

(ii) Assume that the result is valid for all integers γ, $0 \le \gamma < \beta$. Let $\{P_n(x)\}$ be a convergent sequence of polynomials of degree at most $\gamma + 1$. Then

$$P_n(x) - P_n(c) \longrightarrow h(x) - h(c).$$

The left side is divisible by $x - c$, giving a sequence $\{Q_n(x)\}$ of polynomials of degree at most γ, and

$$Q_n(x) = \frac{P_n(x) - P_n(c)}{x - c} \longrightarrow \frac{h(x) - h(c)}{x - c} \qquad (x \neq c).$$

From the induction hypothesis, the right member is a polynomial of degree at most γ. Hence $h(x)$ is a polynomial of degree at most $\gamma + 1$. This completes the induction.

The referee suggests the following alternative proof of the lemma: If $P_n(x)$ converges pointwise, so does $\Delta^{\beta+1} P_n(x)$; but these differences are all zero, and so $\Delta^{\beta+1} h(x) = 0$ (for all spans). It is well known that the polynomials of degree $\leq \beta$ are characterized among measurable functions by the property of having vanishing $(\beta + 1)$th differences; and $h(x)$ is even of the first Baire class.

He also comments that the lemma is well known, but that, like the author, he can think of no specific reference.

The case $\alpha = 1$, $\lambda_n = 1$ (all n) of Theorem 4 below is proved in the opening remarks of [5]. Theorem 3 of [5] is also included in Theorem 4 below, which gives somewhat more precise information than is formulated in the statement of Theorem 3 of [5], even for the case $\alpha = 1$, which is the case analyzed in Theorem 3 of [5]. The proof below is fashioned after that of the latter theorem.

THEOREM 4. *Let $\{\lambda_n\}$ be a given sequence of constants; let α be a fixed positive integer; and let*

$$\lim_{n\to\infty} \frac{f^{(\alpha n)}(x)}{\lambda_n} = g(x) \text{ dominatedly in } a \leq x \leq b. \tag{7}$$

Then the following statements are true for $a \leq x \leq b$.

(i) *If*

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = 0,$$

then $g(x) = 0$ almost everywhere. If (7) holds uniformly, then $g(x) \equiv 0$.

(ii) *If*

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = L \neq 0,$$

L finite, then $Lg^{(\alpha)}(x) = g(x)$.

(iii) *If the sequence $\{\lambda_{n-1}/\lambda_n\}$ has an infinite limit-point, then $g(x) = P_{\alpha-1}(x)$, where $P_{\alpha-1}(x)$ is a polynomial whose degree does not exceed $\alpha - 1$.*

(iv) *If the sequence $\{\lambda_{n-1}/\lambda_n\}$ has at least two limit-points, of which at least one is finite, then $g(x) \equiv 0$.*

Proof. The common hypothesis gives the following extension of (3) of

[5] in all four cases, since the sequence obtained by integrating a dominatedly convergent sequence converges uniformly [**10**, p. 290, p. 304], whence successive termwise integrations are valid for x, c in $[a, b]$:

$$\lim_{n \to \infty} \left[\frac{\lambda_{n-1}}{\lambda_n} \left\{ \frac{f^{(an-a)}(x)}{\lambda_{n-1}} - \frac{f^{(an-a)}(c)}{\lambda_{n-1}} \right\} \right.$$

$$\left. - \frac{1}{\lambda_n} \frac{f^{(an-1)}(c)}{(\alpha - 1)!} (x-c)^{\alpha-1} - \cdots - \frac{f^{(an-a+1)}(c)}{\lambda_n} (x-c) \right] \tag{8}$$

$$= \int_c^x \int_c^{x_\alpha} \cdots \int_c^{x_2} g(x_1) dx_1 \cdots dx_\alpha .$$

Moreover,

$$\frac{f^{(an-a)}(x)}{\lambda_{n-1}} \longrightarrow g(x), \quad \frac{f^{(an-a)}(c)}{\lambda_{n-1}} \longrightarrow g(c),$$

since $s_n \longrightarrow s$ implies $s_{n-1} \longrightarrow s$.

To prove (i), note that the first term of the left member of (8) approaches zero. Then, from Lemma 1, the combined remaining terms have as their collective limit a polynomial $P_{\alpha-1}(x)$ whose degree does not exceed $\alpha - 1$. Differentiating both sides of (8) $\alpha - 1$ times, under these circumstances, shows that $\int_c^x g(t)dt$ is constant for all x in $[a, b]$, whence $g(x) = 0$ almost everywhere, as asserted in the first part of (i). If (7) holds uniformly, then $g(x)$ is continuous and hence identically zero.

To prove (ii), note that (8) becomes, as above,

$$L\{g(x) - g(c)\} - P_{\alpha-1}(x) = \int_c^x \int_c^{x_\alpha} \cdots \int_c^{x_2} g(x_1) dx_1 \cdots dx_\alpha .$$

Differentiating both sides α times with respect to x completes the proof of (ii).

To prove (iii), rewrite (8) by using λ_{n-1}/λ_n as a factor of all the terms within the brackets and not just of the terms in the braces. Then the (new) expression inside the brackets must approach zero (since the right member of (8) is finite) as n becomes infinite through a subsequence for which the corresponding λ_{n-1}/λ_n becomes infinite. Using Lemma 1 again shows that

$$g(x) - g(c) - P_{\alpha-1}(x) = 0;$$

and, of course, $g(c)$ can be absorbed in $P_{\alpha-1}(x)$, completing the proof of (iii).

To prove (iv), consider first the case in which there are exactly two limit-points, one of which is zero. The presence of the zero limit-point implies (by use of an appropriate subsequence of $\{\lambda_{n-1}/\lambda_n\}$ in the proof of (i)) that $g(x) = 0$ almost everywhere. The other limit-point may be finite or infinite. If finite, the same modification is introduced into the proof of (ii), showing $g(x)$ to be continuous. If infinite, (iii) applies directly, again showing $g(x)$ to be continuous. Hence, in this case, $g(x) \equiv 0$.

In the remaining ("general") case of (iv), there is a finite nonzero limit-point L, whence, modifying (ii) as above, we obtain

$$Lg^{(\alpha)}(x) = g(x) \tag{9}$$

and *either* another finite nonzero limit-point M, implying

$$Mg^{(\alpha)}(x) = g(x)$$

with $L \neq M$, *or* an infinite limit-point, in which eventuality $g(x)$ is a polynomial whose degree does not exceed $\alpha - 1$, from (iii). Comparing either of these alternatives for $g(x)$ with (9) shows that $g(x) = 0$.

This completes the proof of (iv) and of the theorem.

Theorem 4 (iv) does not exclude the possibility that

$$\liminf \left| \frac{\lambda_{n-1}}{\lambda_n} \right|$$

may be zero. For the case $\alpha = 1$, therefore, it overlaps—and partially generalizes—Theorem 3(i) of [5] in which it is assumed, instead of (7), that

$$\frac{f^{(n)}(x)}{\lambda_n} \longrightarrow g(x)$$

uniformly in $[a, b]$, as in Theorem 4(i) here, in order to infer that $g(x) \equiv 0$.

This casts further light on the significance of counter-examples connected with Theorem 3(i) of [5] (which is the case $\alpha = 1$ of Theorem 4(i) above).

One is due to Boas and Chandrasekharan [5], another to Bang [1], described also in the final paragraph of [6]. Each exhibits a sequence $\{f^{(n)}(x)/\lambda_n\}$ converging dominatedly to $g(x)$ in $[a, b]$ with $\lim(\lambda_{n-1}/\lambda_n) = 0$ and $g(x)$ *not* identically zero there, although, of course, it is zero almost everywhere. In their examples, in fact, $g(x)$ is zero except for a single point.

In addition to the examples due to these authors, Philip Davis has called attention to earlier constructions [2a; 3; **7**, pp. 38-42; **8**; **12**, p. 244; **14**] of functions differentiable infinitely often on an interval and analytic on that interval except for one or more interior points at which the successive derivatives increase arbitrarily rapidly. Taking λ_n to be the nth derivative at a singular point converts these constructions into examples of the phenomenon described above.

R. P. Boas, who transmitted Davis's information to the author, added a reference to another exposition [2b] of S. Bernstein's examples.

Theorem 4(iv) shows, *i. a.*, that it is impossible to construct similar counter-examples in which the condition on the λ_n's is weakened to

$$\liminf \left| \frac{\lambda_{n-1}}{\lambda_n} \right| = 0$$

with $\lim(\lambda_{n-1}/\lambda_n)$ nonexistent.

This last remark can be inferred also from a consideration of formula (3) of [5], which is valid for dominatedly convergent sequences and which reads as follows:

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} \left\{ \frac{f^{(n-1)}(x)}{\lambda_{n-1}} - \frac{f^{(n-1)}(c)}{\lambda_{n-1}} \right\} = \int_c^x g(t)\,dt,\ a < c < b.$$

Choose c to be a point such that $g(c) \neq 0$, x a point at which $g(x) = 0$. The right member is zero, since $g(x) = 0$ almost everywhere. Thus

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n}\ g(c) = 0,\ g(c) \neq 0,$$

whence

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = 0.$$

When $\lambda_n = 1$ for all n, Theorem 4 (of which only part (ii) is now relevant) can be extended readily to certain summation methods. Consider the transformation

$$(10) \qquad T:\ t_r(x) = \sum_{n=0}^{\infty} c_n(r) s_n(x),$$

where r is continuous or discrete.

DEFINITION. The transformation T of (10) will be said to be of *dominated type* in the interval (a, b) with respect to a sequence of Lebesgue integrable functions $\{s_n(x)\}$, defined in (a, b), if the infinite series (10) taking the sequence $\{s_n(x)\}$ into $t_r(x)$ converges dominatedly (in the sense that all its partial sums are uniformly less, in absolute value, than a fixed Lebesgue integrable function) in (a, b) for each sufficiently large r.

Any row-finite or row-bounded matrix transformation is of dominated type with respect to all sequences of Lebesgue integrable functions. This includes all Hausdorff and Voronoi-Nörlund methods, in particular Cesàro's and Euler's. All regular (or even merely convergence-preserving) transformations given by (10) are of dominated type with respect to any sequence of Lebesgue integrable functions dominated as a whole by a single Lebesgue integrable function.

LEMMA 2. *Let T be a summation method of dominated type with respect to the sequence of Lebesgue integrable functions $\{s_n(x)\}$ in (a, b). Suppose that $\{s_n(x)\}$ is dominatedly T-summable in (a, b) to $s(x)$. Then*

$$(11) \qquad T\text{-}\lim \int_c^x s_n(t)\,dt = \int_c^x s(t)\,dt,$$

uniformly for c, x in (a, b).

Proof. The transformation T being of dominated type, it follows [**10**, pp. 290, 304] as in the justification of (8), that

$$\sum_{n=0}^{\infty} c_n(r) \int_c^x s_n(t)\,dt = \int_c^x \sum_{n=0}^{\infty} c_n(r) s_n(t)\,dt,$$

uniformly for c, x in (a, b), for each sufficiently large r. In turn, the right member approaches the right member of (11) uniformly for c, x in (a, b) as

$r \longrightarrow \infty$, since the integrand approaches $s(t)$ dominatedly. The left member is the T-transform of the integral of $s_n(t)$. Hence the lemma is established.

THEOREM 5. *Let T be a summation method satisfying* (1) *and of dominated type with respect to the sequence $\{f^{(\alpha n)}(x)\}$, x in (a, b), where α is a fixed positive integer. If*

$$T\text{-}\lim f^{(\alpha n)}(x) = g(x),$$

dominatedly in (a, b), as $n \longrightarrow \infty$, then $g(x)$ satisfies the differential equation $g^{(\alpha)}(x) = g(x)$ in (a, b).

Proof. By α applications of Lemma 2 we obtain

$$\lim_{r\to\infty} \sum_{n=0}^{\infty} c_n(r)\,[f^{(\alpha n-\alpha)}(x) - f^{(\alpha n-\alpha)}(c)]$$

$$-\lim_{r\to\infty} \sum_{n=0}^{\infty} c_n(r)\left[f^{(\alpha n-\alpha+1)}(c)\,\frac{x-c}{1!} + \cdots + f^{(\alpha n-1)}(c)\,\frac{(x-c)^{\alpha-1}}{(\alpha-1)!}\right]$$

$$= \int_c^x \int_c^{x_\alpha} \cdots \int_c^{x_2} g(x_1)\,dx_1 \cdots dx_\alpha$$

uniformly for c, x in (a, b). Lemma 2 actually gives the existence and value of the limit of the difference of the two sums, rather than the difference of the limits of the individual sums, as written above. However, once the existence of the first limit above is established, that of the second is immediate.

Writing $\alpha n - \alpha$ as $\alpha(n-1)$, we see from (1) that the first limit exists and is $g(x) - g(c)$. Lemma 1, with $\beta = \alpha - 1$, shows that the second limit, whose existence is now assured, is a polynomial in $x - c$ of degree at most $\alpha - 1$, say $P_{\alpha-1}(x-c)$, vanishing for $x = c$. Then

$$g(x) - g(c) - P_{\alpha-1}(x-c) = \int_c^x \int_c^{x_\alpha} \cdots \int_c^{x_2} g(x_1)\,dx_1 \cdots dx_\alpha .$$

Continuity and then α-fold differentiability follow from this equation. Differentiating α times completes the proof.

Some open questions. If $\lim f^{(\alpha n)}(x)$, $n \longrightarrow \infty$, α a fixed positive integer, exists, and is finite for each x in (a, b), then must the convergence necessarily be dominated or perhaps even bounded or uniform? If this is not the case for general indefinitely differentiable functions, would it be true for $f(x)$ in a quasi-analytic class? If not then, what if $f(x)$ is analytic? If $\alpha = 1$, then the answer to the first (and hence to all) of these questions is affirmative. If the answer to any of these questions is affirmative for other α, it would then follow, from Theorem 4(ii), that the limit, $g(x)$, satisfies the differential equation $g^{(\alpha)}(x) = g(x)$. Similar questions can be framed for more general sequences of λ_n's.

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SOME EXTENSION THEOREMS FOR CONTINUOUS FUNCTIONS

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1. Introduction. In a recent paper, J. Dugundji proved [**11**, Th. 4.1] that every convex subset Y of a locally convex topological linear space has the following property:

(1) If X is a metric space, A a closed subset of X, and f a continuous function from A into Y, then f can be extended to a continuous function from X into Y.

Let us call a topological space Y which has property (1) an *absolute extensor for metric spaces*, and let *absolute extensor for normal* (*or paracompact, etc.*) *spaces* be defined analogously. According to Dugundji's theorem above, the supply of spaces which are absolute extensors for metric spaces is quite substantial, and it becomes reasonable to ask the following question:

(2) Suppose that Y is an absolute extensor for metric spaces. Under what conditions is it also an absolute extensor for normal (or paracompact, etc.) spaces?

Most of this paper (§§2-6) will be devoted to answering this question and related questions. The related questions arise in connection with the concepts of absolute retract, absolute neighborhood retract, and absolute neighborhood extensor (in §2 these are all defined and their interrelations and significance explained), and it is both convenient and natural to answer all the questions simultaneously. Assuming that the space Y of (2) is metrizable, we are able to answer these questions completely (thereby solving some heretofore unsolved problems of Arens [**2**, p. 19] and Hu [**18**]) in Theorems 3.1 and 3.2 of §3; §§4 and 5 are devoted to proving these theorems. In §6 we show by an example that things can go completely awry if Y is not assumed to be metrizable.

Our final section (§7), which is also based on Dugundji's [**11**, Th. 4.1], deals with *simultaneous* extensions of continuous functions. It is entirely independent of §§2-6, and is the only part of this paper which might interest those readers who are interested only in metric spaces.

We conclude this introduction with a summary of some of the less familiar

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or possibly ambiguous terms used in this paper. All our *normal* spaces are assumed to be Hausdorff. A *perfectly normal* space is a normal space in which every closed subset is a G_δ (i.e., the intersection of countably many open sets). A covering $\mathfrak{V}$ of *a* topological space X is called *locally finite* [**10**, p. 66] if every x in X has a neighborhood which intersects only finitely many $V \in \mathfrak{V}$. A topological space X is *paracompact* [**10**, p. 66] if it is Hausdorff, and if to every open covering $\mathfrak{U}$ of X there corresponds a locally finite open covering $\mathfrak{V}$ of X such that every $V \in \mathfrak{V}$ is a subset of some $U \in \mathfrak{U}$. (Every paracompact space is normal [**10**, Th. 1], every metric space is paracompact [**22**, Cor. 1], and a Hausdorff space is paracompact if and only if it is fully normal [**22**, Th. 1 and Th. 2].) A metrizable space is *topologically complete* if it can be given a complete metric which agrees with the topology. A topological space is *σ-compact* if it is the union of countably many compact subsets.

2. Definitions and interrelations. Let us begin this section by formally defining the concepts which were mentioned in the introduction, and which will be the objects of investigation of most of this paper. For convenience, we will use the following abbreviations:

AE = absolute extensor

ANE = absolute neighborhood extensor

AR = absolute retract

ANR = absolute neighborhood retract

DEFINITION 2.1. A topological space Y is called an AE (resp. ANE) *for metric spaces* if, whenever X is a metric space and A is a closed subset of X, then any continuous function from A into Y can be extended to a continuous function from X (resp. some neighborhood of A in X) into Y. Similarly if "metric" is replaced by the name of some other kind of space in the above.

DEFINITION 2.2. A topological space Y is called an AR (resp. ANR) *for metric spaces* if, whenever Y is a closed subset of a metric space X, there exists a continuous function from X (resp. some neighborhood of Y in X) onto Y which keeps Y pointwise fixed. Similarly if "metric" is replaced by the name of some other kind of space in the above.

REMARK. Observe that if Y is an AE (resp. ANE) for a certain class of spaces, then Y is *a fortiori* an AR (resp. ANR) for this class of spaces.

The concepts defined in Definition 2.2 are essentially due to Borsuk [**4** and **5**], who proved [**5**, p. 227] that every finite simplicial complex is an *ANR* for compact metric spaces; this was, in fact, Borsuk's motive for introducing *ANR*'s. More recently, Hanner [**17**] generalized that result by showing that every locally finite simplicial complex is an *ANR* for separable metric spaces. Finally this result was generalized still further by Dugundji [**12**, Th. 5.2], who proved that every simplicial complex with J.H.C. Whitehead's *CW* topology is an *ANE* for metric spaces.

The following propositions summarize the known relations between the various concepts defined above. Propositions 1 and 3 are due to Hu [**18**], and parts of Proposition 2 are essentially due to Dugundji [**11**] and Hanner [**16**].

PROPOSITION 2.3 (Hu). *Let Y be a separable metric space. Then Y is an AR* (resp. *ANR*) *for metric spaces if and only if Y is an AR* (resp. *ANR*) *for separable metric spaces.*

Proof. This follows at once from [18, Th. 3.1].

PROPOSITION 2.4. *Let Y be a metric space. Then Y is an AR* (resp. *ARN*) *for metric spaces if and only if Y is an AE* (resp. *ANE*) *for metric spaces. This assertion remains true if "metric" is everywhere replaced by "paracompact", or "normal", or "perfectly normal".*

Proof. The "if" assertions are clear (see the Remark after Definition 2.2), so let us turn to the "only if" assertions. Here the metric case was proved by Dugundji [**11**, Th. 7.1]; to prove the results in the other cases, we shall use the method employed by Hanner in his proof of the normal case [**16**, Th. 3.1 and Th. 3.2].

Let X and Y be topological spaces, A a closed subset of X, and $f: A \longrightarrow Y$ a continuous function. Let $X \cup Y$ denote the disjoint union of X and Y, and let Z be the identification space which we get from $X \cup Y$ by identifying $x \in A$ with $f(x) \in Y$. To prove our results, it is sufficient, as in Hanner's proof of the normal case, to show that if X and Y are both paracompact (resp. normal, perfectly normal), then so is Z. For normal spaces this was proved by Hanner [**16**, Lem. 3.3], and for perfectly normal spaces the proof is almost the same as that for normal spaces; this leaves paracompact spaces, where our proof depends on the following two facts. The first of these is a characterization of paracompact spaces which the author will prove in another paper, and the second is an immediate consequence of the first.

(1) *If Z is a T_1-space, then Z is paracompact if and only if it has the following property: If E is a Banach space, and if $\widetilde{u}$ is a l.s.c.*[1] *function from Z to the space $\mathcal{C}(E)$ of nonempty, closed, convex subsets of E, then there exists a continuous $u: Z \longrightarrow E$ such that $u(z) \in \widetilde{u}(z)$ for every z in Z.*

(2) *Let X be a paracompact space, E a Banach space, $\widetilde{w}: X \longrightarrow C(E)$ a l.s.c. function, and A a closed subset of X. Then any continuous $v: A \longrightarrow E$ such that $v(x) \in \widetilde{w}(x)$ for every x in A can be extended to a continuous $w: X \longrightarrow E$ such that $w(x) \in \widetilde{w}(x)$ for every x in X.*

We shall also need the following elementary facts about Z. Let g be the natural mapping from $X \cup Y$ onto Z, and denote $g|X$ by h and $g|Y$ by k; also denote $k(Y)$ by Y'. As observed by Hanner, k is a homeomorphism onto Y', and $h|X - A$ is a homeomorphism onto $Z - Y'$. It follows that a function u with domain Z is continuous if and only if $u|Y'$ and uh are both continuous.

Suppose now that X and Y are paracompact, and let us prove that Z is also paracompact. Since Z is certainly T_1, we need only show that Z has the property in (1). Suppose, therefore, that E is a Banach space, and $\widetilde{u}: Z \longrightarrow \mathcal{C}(E)$ a l.s.c. function; we must find a continuous $u: Z \longrightarrow E$ such that $u(z) \in \widetilde{u}(z)$ for every z in Z. Now Y' is paracompact, and $\widetilde{u}|Y'$ is l.s.c.; hence, by (1), there exists a continuous $r: Y' \longrightarrow E$ such that $r(z) \in \widetilde{u}(z)$ for every z in Y'. Let $\widetilde{w} = \widetilde{u}h$, let $h' = h|A$, and let $v = rh'$; then X, A, $\widetilde{w}$, and v satisfy the assumptions of (2), and hence, by (2), v can be extended to a continuous $w: X \longrightarrow E$ such that $w(x) \in \widetilde{w}(x)$ for every x in X. Now define $u: Z \longrightarrow E$ by "$u(z) = r(z)$ if $z \in Y'$, and $u(z) = wh^{-1}(z)$ if $z \in Z - Y'$". Clearly $u(z) \in \widetilde{u}(z)$ for every z in Z, and u is continuous, since $u|Y' = r$ and $uh = w$ are both continuous. This completes the proof.

Finally, let us mention the following result of Hu [**18**, Th. 3.2].

PROPOSITION 2.5. (Hu) *If Y is a completely regular space which is an AR* (resp. *ANR*) *for completely regular spaces, then Y is an AE* (resp. *ANE*) *for normal spaces.*

Having just covered the similarities between extensors and retracts, let us end this section with some comments about their differences. These differences occur in two ways:

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[1] A function $\widetilde{u}$ from a topological space Z to the space of nonempty subsets of a topological space E is called *l.s.c.* (= *lower semi-continuous*) if, whenever U is an open subset of E, then $\{ z \in Z \mid \widetilde{u}(z) \cap U \neq \phi \}$ is open in Z.

(a) If Y is *not* a metric (resp. paracompact, etc.) space, then Y is (vacuously!) *always* an AR and an ANR for metric (resp. paracompact, etc.) spaces. But Y need by no means always be an AE or an ANE for metric (resp. paracompact, etc.) spaces, and when it is, this is a fact which cannot be restated in terms of retracts. As examples, we mention the theorems of Dugundji [**11**, Th. 4.1] and [**12**, p. 9] which we have encountered earlier in this paper.

(b) If Y is completely regular and has more than one point, then it is easy to see that Y cannot be an AE or ANE for any class of spaces which contains a nonnormal space. But such a Y may very well be an AR or ANR for completely regular spaces (see Theorem 3.1 (e) and Theorem 3.2 (e)).

3. The theorems. We will now state the theorems answering question (2) of the introduction.

THEOREM 3.1. *Let Y be a metrizable space which is an AE* (resp. *ANE*) *for metric spaces. Then:*

(a) *Y is an AE* (resp. *ANE*) *for spaces which are paracompact and perfectly normal.*

(b) *Y is an AE* (resp. *ANE*) *for paracompact spaces if and only if Y is topologically complete.*

(c) *Y is an AE* (resp. *ANE*) *for perfectly normal spaces if and only if Y is separable.*

(d) *Y is an AE* (resp. *ANE*) *for normal spaces if and only if Y is separable and topologically complete.*

(e) *Y is an AE* (resp. *ANE*) *for completely regular spaces if and only if Y has at most one point.*

THEOREM 3.2. *Let Y be a metrizable space which is an AR* (resp. *ANR*) *for metric spaces. Then:*

(a) *Y is an AR* (resp. *ANR*) *for paracompact spaces containing Y as a G_δ.*

(b) *Y is an AR* (resp. *ANR*) *for paracompact spaces if and only if Y is topologically complete.*

(c) *Y is an AR* (resp. *ANR*) *for perfectly normal spaces if and only if Y is separable.*

(d) *Y is an AR* (resp. *ANR*) *for normal spaces if and only if Y is separable and topologically complete.*

(e) *Y is an AR* (resp. *ANR*) *for completely regular spaces if and only if Y is compact* (resp. *locally compact and separable*).

The foregoing theorems make a rather formidable array of statements, but because of their interdependence we will not have to prove all of them separately. In fact, we will prove only the following assertions (whose labeling is self-explanatory):

(*) 1(a), 2(a), 1(b) "if", 2(b) "only if", 1(c), 1(d) "if", 2(e).

Let us show that these assertions imply all the others. To begin with, the assumptions on Y made at the beginning of Theorems 3.1 and 3.2 are equivalent, by Proposition 2.2. We therefore have the following implications:

1(b) "if" $\Longrightarrow$ 2(b) "if": by Remark after Definition 2.

2(b) "only if" $\Longrightarrow$ 1(b) "only if": by Remark after Definition 2.

1(c) $\Longrightarrow$ 2(c): by Proposition 2.

1(b) "only if" and 1(c) "only if" $\Longrightarrow$ 1(d) "only if": obvious.

1(d) $\Longrightarrow$ 2(d): by Proposition 2.

$\Longrightarrow$ 1(e): this follows from the definitions.

These implications, together with the assertions (*) which we are going to prove, cover all the assertions of both theorems.

Before concluding this section, let us comment on the novelty and significance of the results (*). First of all, 1(d) "if" has been proved by Hanner [**16**, Th. 4.1 and Th. 4.2], and 2(e) "if" (*AR*) has been observed by Hu [**18**]; our proofs of these results are short, and we include them for the sake of unity of approach. Result 1(b) "if" follows easily from Arens' [**2**, Th. 4.1] by means of a technique due to Dugundji [**11**]. Results 1(a), 2(a), 2(b) "only if", and 1(c) "if" are proved by minor variations of techniques due to Hanner [**16**]. This leaves 1(c) "only if", 2(e) "only if", and 2(e) "if" (ANR) as the only results with some claim to originality; among these, 1(c) "only if" solves a problem of Arens [**2**, p. 19], and the others solve some problems of Hu [**18**].

In the next section we will prove 1(a), 2(a), and the "if" parts of the other (*) assertions; in the section after that we will prove the "only if" parts. The lemmas and propositions in these sections have some independent interest, and are sometimes stated with greater generality than is needed in their application.

4. Proofs of sufficiency. Assertions 1(a), 2(a), 1(b) "if", 1(c) "if", and 1(d) "if" will be proved after Lemma 4.3. Assertion 2(e) "if" will be proved after Lemma 4.6.

In the following lemmas, $R^{\aleph_0}$ will denote a countably infinite cartesian product of real lines.

LEMMA 4.1. *Every (complete) metric space can be embedded homeomorphically as a (closed) subset in a Banach space. Every (complete) separable metric space can be embedded homeomorphically as a (closed) subset in* $R^{\aleph_0}$.

Proof. It is well known (see, for instance [**20**]) that every metric space can be embedded *isometrically* in a Banach space, and the first sentence follows from this fact. It is also well known [**19**, p. 104] that every separable metric space X can be embedded in $R^{\aleph_0}$, which proves the second sentence with parenthetical words omitted. If X is moreover complete, then it is a G_δ in $R^{\aleph_0}$ [**19**, p. 215]. By [**19**, p. 151], X is therefore homeomorphic to a closed subset of $R^{\aleph_0} \times R^{\aleph_0}$, and the latter space is homeomorphic to $R^{\aleph_0}$. This completes the proof.

The proof of the following lemma uses an idea which the author found in Hanner [**16**] who in turn ascribes it to Fox [**13**].

LEMMA 4.2. *Let X be a normal space, A a closed G_δ in X, and g a continuous function from A into a metric space E. Then there exists a metric space F containing $g(A)$ as a closed subset, and a continuous function h from X into F which agrees with g on A.*

Proof. Let $G = E \times I$, where I is the closed unit interval, and identify E with $E \times \{0\} \subset G$. Let $F = G - (E - g(A))$. Since A is a closed G_δ in the normal space X, there exists a continuous function ϕ from X into the nonnegative real numbers, which is zero exactly on A. Finally we define $h: X \longrightarrow F$ by $h(x) = (g(x), \phi(x))$, and we see that F and h satisfy all our requirements.

LEMMA 4.3. *Let X be a topological space, A a closed subset of X, M a metric space, and f a continuous function from A into M. Suppose either that M is a complete metric space, or that A is a G_δ in X. Suppose also either that X is paracompact, or that X is normal and M separable. Then there exists a metric space F containing M as a closed subset, and a continuous function from X into F which agrees with f on A.*

Proof. If X is paracompact, embed M in a Banach space E according to

Lemma 4.1. By [2, Th. 4.1] we may extend f to a continuous function g from X into E. If M is complete, then we may suppose that M is closed in E, and we are through. If A is a G_δ in X, we need only apply Lemma 4.2.

If X is normal and M separable, embed M in $R^{\aleph_0}$ according to Lemma 4.1. The proof now proceeds exactly as above, except that we use the Urysohn-Tietze extension theorem instead of [2, Th. 4.1]. This completes the proof.

Proof of 1(a), 1(b) *"if"*, 1(c) *"if"*, *and* 1(d) *"if"*. These all follow almost immediately from Lemma 4.3.

Our next two lemmas deal with locally compact spaces, and are stated without proof. The crux of Lemma 4.4 is essentially stated as an exercise in [6] and proved in [8]; the first proof which the author saw was due to J. Tits.

LEMMA 4.4. *The following properties of a Hausdorff space X are equivalent:*

a) *X is locally compact.*

b) *If X is a dense subset of a Hausdorff space Y, then X is open in Y.*

c) *If X is a subset of a Hausdorff space Y, then $X = U \cap C$, where U is open in Y, and C is closed in Y.*

LEMMA 4.5. *Let X be a locally compact space, and A a σ-compact subset of X. Then there exists an open, σ-compact subset Z of X which contains A.*

One part of the following lemma is trivial, while the other part is not; we state them together to emphasize the parallelism.

LEMMA 4.6. *Let X be a completely regular space, and A a compact* (resp. *locally compact and σ-compact*) *subset of X. Then X* (resp. *some neighborhood V of A in X*) *can be embedded in a compact* (resp. *locally compact and σ-compact*) *Hausdorff space Z such that A is closed in Z.*

Proof. The assertion where A is compact is trivial. To prove the other assertion, let Y be any compact Hausdorff space containing X. By Lemma 4.4, there exists an open subset U of Y such that A is a subset of U which is closed relative to U. Since U is open in Y, it is locally compact. Hence, by Lemma 4.5, there exists an open, σ-compact subset Z of U which contains A. Since Z is open in U, Z is locally compact. Letting $V = Z \cap X$, we see that Z and V satisfy our requirements. This completes the proof.

Proof of 2(e) *"if"*. This follows easily from Lemma 4.6 as follows: (We will prove the part about ANR; the part about AR is even easier). Let Y be a locally compact, separable metric space which is an *ANR* for metric spaces, and let Y be a closed subset of the completely regular space X. We must find a neighborhood U of Y in X, and a continuous function g from U to Y which is the identity on Y.

Since Y is a locally compact, separable metric space, it is σ-compact. Hence, by Lemma 4.6, some neighborhood V of Y in X can be embedded in a locally compact and σ-compact Hausdorff space Z such that A is closed in Z. By [**10**, Th. 3], Z is paracompact. Since Y is a locally compact metric space, it is topologically complete (for instance by [**19**, p. 200] and Lemma 4.4). Hence, by Theorem 3.2(b), there exists a continuous function g from some neighborhood W of Y in Z to Y such that g is the identity on Y. Letting $U = W \cap V$, and $f = g \mid U$, we see that all our requirements are satisfied. This completes the proof.

5. Proofs of necessity. We start this section with the proof of 2(b) "only if". We will prove 1(c) "only if" after Proposition 5.1, and 2(e) "only if" after Proposition 5.3.

Proof of 2(b) *"only if"*. If "paracompact" were replaced by "normal" in this assertion, and "metric" by "separable metric", then the assertion would be contained in [**16**, Th. 4.1 and Th. 4.2]. To prove our assertion as it stands, we need only modify the proof of [**16**, Th. 4.2]. We therefore invite the reader to look at Hanner's proof of [**16**, Th. 4.2], and we will now point out the necessary modification.

Instead of embedding X (this is the space in [**16**] which corresponds to our Y) in the Hilbert cube I_ω (which can only be done if X is separable), we embed X in an arbitrary complete metric space M, and this space M will take the place of I_ω throughout the proof. With that in mind, we now define Z just as Hanner does, and the crux of the matter is that we must show Z to be paracompact (Hanner only shows that Z is normal). Once this is accomplished, the remainder of Hanner's proof goes through unchanged (except that I_ω is replaced by M) to show that X is a G_δ in M. But this implies [**19**, p. 200] that X is topologically complete, and our proof will therefore be complete.

We will use the notation of Hanner's proof (except that M replaces I_ω). Let $\{U_\alpha\}$ be a covering of Z by open sets. Then, for each α, there exists an open set O_α in M, and a subset A_α of $Z - X'$, such that

$$U_\alpha = h^{-1}(O_\alpha) \cup A_\alpha .$$

Let $O = \bigcup_\alpha O_\alpha$. Since O is a metric space (and therefore paracompact [**22**, Cor. 1]), and since $\{O_\alpha\}$ is a covering of O by open sets, $\{O_\alpha\}$ has a locally finite refinement $\{V_\beta\}$. Since each V_β is open in O, and since O is open in M, it follows that each V_β is open in M. Now let $\mathfrak{W}$ be the covering of Z whose elements are the sets $h^{-1}(V_\beta)$ and the one-point sets corresponding to the points of $Z - h^{-1}(O)$. Let us show that $\mathfrak{W}$ is a locally finite refinement of $\{U_\alpha\}$: It is clear that $\mathfrak{W}$ is a covering of Z by open sets, and that $\mathfrak{W}$ is a refinement of $\{U_\alpha\}$, so we need only show that $\mathfrak{W}$ is locally finite. If $x \in Z - h^{-1}(O)$, then $\{x\}$ is certainly a neighborhood of x which intersects only finitely many elements of $\mathfrak{W}$. If $x \in h^{-1}(O)$, then there exists an open subset S_x of O such that $h(x) \in S_x$, and such that S_x intersects only finitely many elements of $\{V_\beta\}$. But then $h^{-1}(S_x)$ is an open subset of Z which contains x, and which intersects only finitely many elements of $\mathfrak{W}$. This completes the proof.

The following proposition is more general than 1(c) "only if".

PROPOSITION 5.1. *If Y is a topological space which is an ANE for normal spaces, then every disjoint collection of open subsets of Y is countable.*

Proof. Suppose that there exists a disjoint collection $\mathfrak{U}$ of nonempty open subsets of Y which is uncountable. Then there exists a subset B of Y which contains exactly one point from every element of $\mathfrak{U}$; clearly B is a discrete space in the relative topology. Now by [3, Ex. H] there exists a perfectly normal space X, and a discrete, closed subset A of X which is homeomorphic to B, such that no collection of open subsets of X separates[2] A. Let f be the homeomorphism from A onto B. By assumption, f can be extended to a continuous function g from some open neighborhood V of A in X into Y. But now the collection of all $V \cap g^{-1}(U)$, with $U \in \mathfrak{U}$, is a collection of open subsets of X which separates A. This is a contradiction, and thus the proof is complete.

Proof of 1(c) *"only if".* This now follows immediately from Proposition 5.1, since for metric spaces the property of Y in Proposition 5.1 is equivalent to separability [21, p. 130].

LEMMA 5.2. *Let ξ be an uncountable ordinal, and let Q be the space of ordinals $\leq \xi$, in the order topology. For each α in Q, let*

$$Q_\alpha = \{q \in Q \mid q \geq \alpha\},$$

with the relative topology induced by Q. Also let X be a subset of the cartesian

[2] If Y is a topological space, and B a subset of Y, then a collection $\mathfrak{U}$ of open subsets of Y *separates* B if $\mathfrak{U}$ is a disjoint collection, and if each $U \subset \mathfrak{U}$ contains exactly one element of B.

product of $\aleph$ *copies of the real line, where* $\aleph$ *is a cardinal which is less than the cardinality of* ξ. Then:

a) *If* $\alpha < \xi$, *and* f *is a continuous function from* Q_α *into* X, *then there exists a* β *in* Q *such that* $\alpha \leq \beta < \xi$, *and such that* $f(q) = f(\xi)$ *for all* $q \geq \beta$.

b) *If* U *is a neighborhood of* $\{\xi\} \times X$ *in* $Q \times X$, *then there exists an ordinal* $\alpha < \xi$ *such that* $Q_\alpha \times X \subset U$.

Proof. a) If $\aleph = 1$, then this is proved exactly like the assertion in the middle of page 836 of [9]. In the general case, let $X \subset \Pi_{\iota \in I} R_\iota$, where I is an index set of cardinality $\aleph$ and R_ι is the real line for every $\iota \in I$, and for every $\iota \in I$ let π_ι be the projection from X into R_ι. Letting $f_\iota = f \circ \pi_\iota$ for every $\iota \in I$, we have, by the first sentence of this proof, an indexed family $\{\beta_\iota\}_{\iota \in I}$ of ordinals in Q such that $f_\iota(q) = f_\iota(\xi)$ whenever $q \geq \beta_\iota$. Letting β be the smallest ordinal which is larger than all the β_ι, we see that β satisfies all our requirements.

b) The assumptions on X imply that X has a basis of cardinality $\leq \aleph$, and hence every covering of X by open sets has a subcovering of cardinality $\leq \aleph$. Now for each x in X, we can find an α_x in Q and an open neighborhood V_x of x in X such that $Q_{\alpha_x} \times V_x \subset U$. Thus $\{V_x\}_{x \in X}$ is a covering of X by open sets, and hence there exists a subcovering $\{V_x\}_{x \in X'}$, where X' has cardinality $\leq \aleph$. If now α is the smallest ordinal which is larger than all the α_x with $x \in X'$, then α satisfies all our requirements. This completes the proof.

PROPOSITION 5.3. *If* Y *is a completely regular space which is an* AR (resp. ANR) *for completely regular spaces, then* Y *is compact* (resp. *locally compact*).

Proof. Since Y is completely regular, it may be embedded in a cartesian product of real lines. Let $\aleph$ be the cardinality of this product, and let ξ be an ordinal whose cardinality is greater than $\aleph$ and greater than the cardinality of Y. Now let Q be the space of ordinals $\leq \xi$ in the order topology, let $\overline{X}$ be a compact Hausdorff space containing X, and let

$$Z = (Q \times \overline{X}) - (\{\xi\} \times (\overline{X} - X)).$$

Since Q and $\overline{X}$ are completely regular, so is Z. Now $\{\xi\} \times X$ is closed in Z, and $\{\xi\} \times X$ is homeomorphic to X, and therefore there exists a retraction f from Z onto $\{\xi\} \times X$. For each x in X, let

$$f_x = f \mid (Q \times \{x\}).$$

By Lemma 5.2, there exists for each x in X a β_x in Q such that

$$f_x((q, x)) = f_x((\xi, x))$$

for all $q \geq \beta_x$. Now let β be the smallest ordinal larger than all the β_x; then $\beta < \xi$, and

$$f((\beta, x)) = (\xi, x)$$

for all x in X. Hence

$$f(\{\beta\} \times \overline{X}) = \{\xi\} \times X,$$

and therefore X is compact.

Let us now consider the ANR case. Suppose, therefore, that X is an ANR for completely regular spaces. Let $\overline{\overline{X}}$, Q, and Z be as in the last paragraph. Then, by assumption, there exists a retraction f from a neighborhood U of $\{\xi\} \times X$ in Z onto $\{\xi\} \times X$. Now by Lemma 5.2, there exists an ordinal $\alpha < \xi$ such that $Q_\alpha \times X \subset U$, where $Q_\alpha = \{q \in Q \mid q \geq \alpha\}$. Proceeding just as in the last paragraph (with Q replaced by Q_α), we obtain a β in Q_α such that

$$f((\beta, x)) = (\xi, x)$$

for all x in X. If we now define the continuous function

$$h : \{\xi\} \times X \longrightarrow \{\beta\} \times X$$

by

$$h((\xi, x)) = (\beta, x),$$

then the restriction of $h \circ f$ to $(\{\beta\} \times \overline{X}) \cap U$ is a retraction of $(\{\beta\} \times \overline{X}) \cap U$ onto $\{\beta\} \times X$. Hence $\{\beta\} \times X$ is closed in $(\{\beta\} \times \overline{X}) \cap U$; but $(\{\beta\} \times \overline{X}) \cap U$ is an open subset of the compact set $\{\beta\} \times \overline{X}$, and therefore both $(\{\beta\} \times \overline{X}) \cap U$ and $\{\beta\} \times X$ are locally compact. Hence X is locally compact, which is what we had to show.

Proof of 2(e) "*only if*". This now follows immediately from Proposition 5.3 and Theorem 3.2(d).

6. An example. In [2], Arens showed indirectly that there exists a compact, convex subset of a locally convex topological linear space which, while certainly an AE for metric spaces by [**11**, Th. 4.1], is not an AE for compact Hausdorff spaces. In this section we will prove this result (and a little more) by means of a direct example, which should also indicate why we assumed the space Y in Theorems 3.1 and 3.2 to be metrizable.

The proof of Proposition 6.1 is due jointly to V. L. Klee and the author, and uses a suggestion by I. E. Segal.

PROPOSITION 6.1. *Let X be the cartesian product of continuum many closed unit intervals. Then there exists a closed, convex subset of X which is not the image under a continuous function of any open subset of X.*

Proof. Let us call a topological space *separable* if it has a countable dense subset. Since the cartesian product of at most continuum many separable spaces is separable [**21**, p. 139], it follows that X is separable. Hence any continuous image of any open subset of X is also separable. To prove the proposition, it therefore suffices to produce a closed, convex subset of X which is not separable. This we will now proceed to do.

Let H be a Hilbert space whose orthonormal dimension is the continuum. Then H has continuum many elements, and is not separable. Let us show (using a proof due to I.E. Segal) that H is not even separable in the weak topology.

In fact, if H were separable in the weak topology, there would exist a countably dimensional subspace K of H which is weakly dense in H. Since H is countably dimensional, it is separable in the strong topology. Now by the Hahn-Banach theorem, the strong closure of K is weakly closed and hence coincides with H. But this implies that H is separable in the strong topology, contrary to our assumption.

Now let S be the unit sphere of H in the weak topology. Then S is compact, since H is reflexive. Also S is not separable since, as we have just shown, H is not separable in the weak topology. To complete the proof, we must show that S is homeomorphic to a convex subset of X. Now by definition,

$$X = \prod_{f \in F} I_f,$$

where F is an index set whose cardinality is the continuum, and I_f is homeomorphic to the unit interval for every f in F. Now H^*, the dual space of H, is isomorphic to H [**15**, p. 31, Th. 3], and hence we may take F to be the unit sphere of H^*.

Define $\phi : S \longrightarrow X$ by "$(\phi(x))_f = f(x)$"; then ϕ is a homeomorphism from S onto $\phi(S)$ by definition of the weak topology, and $\phi(S)$ is clearly convex in X. This completes the proof.

COROLLARY 6.2. *There exists a compact Hausdorff space which is a convex subset of a locally convex topological linear space (and hence* [**11**, Th. 4.1] *an AE for metric space) which is not even an ANR for compact Hausdorff spaces.*

7. Simultaneous extensions. The purpose of this section is to prove the following theorem:

THEOREM 7.1. *Let X be a metric space, A a closed subset of X, and E a locally convex topological linear space. Let $C(X, E)$ denote the linear space of continuous functions from X into E, and similarly for $C(A, E)$. Then there exists a mapping*

$$\phi : C(A, E) \longrightarrow C(X, E)$$

satisfying the following conditions:

(a) *$\phi(f)$ is an extension of f for every $f \in C(A, E)$.*

(b) *The range of $\phi(f)$ is contained in the convex hull of the range of f for every $f \in C(A, E)$.*

(c) *ϕ is an isomorphism* (i.e. *a one-to-one, bi-continuous linear transformation) from $C(A, E)$ into $C(X, E)$, provided $C(A, E)$ and $C(X, E)$ both carry the same one of the following three topologies:*

(1) *Topology of simple convergence* [**7**, p. 4].

(2) *Topology of compact convergence* [**7**, p. 5][3].

(3) *Topology of uniform convergence* [**7**, p. 5].

Proof. We will show below that, in his proof of [**11**, Th. 4.1], Dugundji has already constructed a mapping ϕ satisfying all our requirements. In fact, Dugundji [**11**, Th. 5.1] and Arens [**2**, Th. 2.6] have already observed that this trivially satisfies *some* of our requirements; the only property of ϕ which will need a nontrivial proof below is that ϕ is continuous for the topology (2).

We need the following fact, which is due to Dugundji [**11**] and was more concisely stated and proved by Arens [**2**, Lem. 2.1]:

(*) There exists a locally finite covering $\mathfrak{V}$ of $X - A$ by open sets, and associated with each $V \in \mathfrak{V}$ an $a_V \in A$ and a continuous real-valued function g_V on X which vanishes outside V, such that:

(i) $0 \leq g_V(x) \leq 1$ and $\sum_V g_V(x) = 1$ for all $x \in X - A$.

[3] This topology is the same as the compact-open topology [**1**, Th. 9].

(ii) If $a \in A$, and $x \in V$, then $\rho(a, a_V) < 3\rho(a, x)$, where ρ is the metric in X.

(iii) If $f \in C(A, E)$, then the function $\tilde{f}: X \longrightarrow E$, defined by "$\tilde{f}(x) = x$ for $x \in A$, and $\tilde{f}(x) = \sum_V g_V(x) f(a_V)$" for $x \in X - A$, is continuous.

The mapping ϕ may now be defined by $\phi(f) = \tilde{f}$, where $\tilde{f}$ is as in (iii) above. It is immediately evident that ϕ is a one-to-one linear transformation which satisfies conditions (a) and (b) of our theorem. The continuity of ϕ^{-1} for any of the three topologies follows from (a) and the definition of these topologies. The continuity of ϕ for topology (3) follows from (b). The continuity of ϕ for the topologies (1) and (2), finally, will be an immediate consequence of the following lemma:

LEMMA 7.2. *If C is a finite* (resp. *compact*) *subset of X, then there exists a finite* (resp. *compact*) *subset $\tilde{C}$ of A such that $\tilde{f}(C)$ is contained in the convex hull of $f(\tilde{C})$.*

Proof of lemma. Let us define function u from X to the finite subsets of A. If $x \in A$, then we let

$$u(x) = \{x\}.$$

If $x \in X - A$, then clearly x is in the closure of only finitely many $V \in \mathfrak{V}$, say $V_1, \cdots, V_n$, and we set

$$u(x) = \{x_{V_1}, \cdots, x_{V_n}\}.$$

Having thus defined u, we set

$$\tilde{C} = \cup_{x \in C} u(x)$$

for every $C \subset X$. It is clear that $\tilde{f}(C)$ is contained in the convex hull of $f(\tilde{C})$. If C is finite, then $\tilde{C}$ is clearly also finite. It therefore remains to prove that $\tilde{C}$ is compact if C is compact. To do this, it is sufficient to show that u is upper semi-continuous[4], because then the compactness of $\tilde{C}$ for compact C will be an immediate consequence of [**14**, p. 151, 21.3.4].

[4] A function h from a topological space Y to the space of nonempty subsets of a topological space Z is called *upper semi-continuous* [**14**, p. 149] *at a point* $y \in Y$ if, for every open subset U of Z which contains $h(y)$, there exists a neighborhood W of y in Y such that $h(y') \subset U$ for every $y' \in W$; h is called *upper semi-continuous* if it is upper semi-continuous at every $y \in Y$.

Let us first show that u is upper semi-continuous at points of $X - A$. Since V is locally finite, the closures (in $X - A$) of any subcollection of $\mathcal{V}$ have a closed (in $X - A$) union. Hence if $x \in X - A$, then

$$B = \bigcup \{ \overline{V} \mid V \in \mathcal{V}, x \notin \overline{V} \}$$

is closed in $X - A$, where $\overline{V}$ denotes (and will always denote below) the closure of V in $X - A$. Let

$$U = (X - A) - B.$$

Then U is a neighborhood of x in X, and $u(x') \subset u(x)$ whenever $x' \in U$; this shows that u is continuous at x.

Before proving the upper semi-continuity of u on A, we need the following consequence of (ii):

(ii*) If $a \in A$, and $x \in \overline{V}$, then $\rho(a, a_V) < 4\rho(a, x)$.

To see this, pick a $y \in V$ such that

$$\rho(x, y) < 1/3 \ \rho(a, x),$$

and then observe that

$$\rho(a, a_V) < 3\rho(a, y) \leq 3(\rho(a, x) + \rho(x, y)) \leq 4\rho(a, x).$$

Let us now prove that u is upper semi-continuous on A. Let $a \in A$, and let U be an open subset of X containing $u(a) = \{a\}$. Pick $\epsilon > 0$ such that

$$\{y \in X \mid \rho(a, y) < \epsilon\} \subset U.$$

Now let

$$W = \{x \in X \mid \rho(a, x) < \epsilon/4\}.$$

Then W is a neighborhood of a in X. If $x \in W \cap A$, then $u(x) = \{x\}$, and thus $u(x) \subset U$. If $x \in W \cap (X - A)$, then $\rho(a, a_V) < \epsilon$ whenever $x \in \overline{V}$ by (ii*), and thus again $u(x) \subset U$. Hence u is upper semi-continuous on A.

This proves the lemma, and hence also the theorem.

REMARK. It is an easy consequence of Proposition 4.3 that Theorem 7.1 remains true if the requirement that X is metric is replaced by the following weaker requirement: A is metric, and one of the following three conditions holds: (a) X is paracompact, (b) X is normal and A is separable, (c) X is completely regular and A is compact.

8. ADDED IN PROOF. Many of our results have been obtained independently by Olof Hanner [24], who was kind enough to send the author a pre-publication reprint of his paper.

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INSTITUTE FOR ADVANCED STUDY

A NOTE ON THE HÖLDER MEAN

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1. Introduction. Of the two better-known generalizations of the simple arithmetic mean, the Hölder mean and the Cesàro mean, the latter has been the more extensively studied. This is primarily due to the equivalence of the two when used to define summability methods and to the following formulas. If we define C_n^k, the k^{th} order Cesàro mean of the terms $S_0, S_1, \cdots, S_n$, by the relation

$$C_n^k = \binom{n+k}{k}^{-1} S_n^k,$$

where

$$S_n^0 = S_n \quad \text{and} \quad S_n^k = \sum_{v=0}^{n} S_v^{k-1} \quad \text{for } n \geq 0, \quad k = 1, 2, \cdots,$$

then it follows [**1**, p. 96] that

$$S_n^{k+m} = \sum_{v=0}^{n} \binom{n-v+m-1}{m-1} S_v^k \tag{1.1}$$

and

$$S_n^k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} S_{n-v}^{k+m} \qquad (m = 1, 2, \cdots). \tag{1.2}$$

The only known analogues to these formulas for the Hölder mean that this writer has been able to find are as follows. Denoting the k^{th} order Hölder mean of the terms $S_0, S_1, \cdots, S_n$ by H_n^k, and recalling the definition that

$$H_n^0 = S_n \quad \text{and} \quad H_n^k = \frac{1}{n+1} \sum_{v=0}^{n} H_v^{k-1} \qquad \text{for } n \geq 0, \quad k = 1, 2, \cdots,$$

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it can be proved [1, p. 250] that

$$(1.3)\qquad H_n^{k+m} = \sum_{v=0}^{n} (-1)^v \binom{n}{n-v} [\Delta^v (n+1-v)^{-m}]\, H_{n-v}^k$$

and

$$(1.4)\qquad H_n^k = \sum_{v=0}^{m} (-1)^v \binom{n}{v} [\Delta^v (n+1-v)^m]\, H_{n-v}^{k+m} \qquad (m = 1, 2, \cdots),$$

where $\Delta u(n) = u(n+1) - u(n)$. These formulas follow from a more general expression for the coefficients in any Hausdorff transformation. It is easily seen that the coefficients involved in (1.3) and (1.4) in many respects are not as convenient to work with as those of (1.1) and (1.2).

In § 2 below, the coefficients of (1.4) are obtained in different form, being expressed in terms of a particular set of polynomials. A few of the properties of these polynomials are considered in § 3, while applications with respect to Hölder summability are dealt with in § 4.

2. A set of polynomials. It follows from the definition of the Hölder mean that

$$(n+1)\, H_n^{k+1} - nH_{n-1}^{k+1} = H_n^k$$

for integers $k \geq 0$ and $n \geq 0$. By iteration, it follows that there exist coefficients $A_j^m(n)$ such that

$$(2.1)\qquad H_n^k = \sum_{j=0}^{m} (-1)^j A_j^m(n)\, H_{n-j}^{k+m} \qquad (m = 0, 1, 2, \cdots)$$

if

$$(2.2)\qquad A_j^{m+1}(n) = (n-j+1)\, [A_j^m(n) + A_{j-1}^m(n)]$$

for $0 \leq j \leq m$, where

$$(2.3)\qquad A_0^0(n) = 1 \text{ and } A_j^m(n) = 0$$

for $j < 0$ or $j > m$. By virtue of the identity

$$\Delta^j (n+1-j)^{m+1} = (n+1-j)\, \Delta^j (n+1-j)^m + j\Delta^{j-1}(n+2-j)^m,$$

it follows that the coefficient of (1.4),

$$A_j^m(n) = \binom{n}{j} \Delta^j (n+1-j)^m,$$

is a solution of (2.2) satisfying the boundary condition (2.3).

Another form of this solution is obtained when we consider the following set of polynomials. For arbitrary nonnegative integers m and j, $0 \leq j \leq m$, let

$$(2.4) \qquad \begin{aligned} F_0^m(x) &= x^{m+1}, \\ F_1^m(x) &= \sum_{m+1} x^i (x-1)^j, \\ \cdots \quad &\cdots \quad \cdots \\ F_j^m(x) &= \sum_{m+1} x^p (x-1)^q \cdots (x-j)^s, \\ \cdots \quad &\cdots \quad \cdots \\ F_m^m(x) &= x(x-1) \cdots (x-m), \end{aligned}$$

the symbol

$$\sum_{m+1} x^p (x-1)^q \cdots (x-j)^s$$

denoting the sum of all possible but different such products where $p, q, \cdots, s$ are positive integers such that $p + q + \cdots + s = m + 1$. If we further let

$$(2.5) \qquad F_j^m(x) = 0$$

whenever $j < 0$ or $j > m$, it follows that

$$(2.6) \qquad F_j^{m+1}(x) = (x-j)\,[F_{j-1}^m(x) + F_j^m(x)]$$

for integers j and $m \geq 0$. To prove the latter relation, apply (2.4) to get

$$(2.7) \quad (x-j)[F_{j-1}^m(x) + F_j^m(x)] = \sum_{m+1} x^p (x-1)^q \cdots (x-j+1)^r (x-j)$$

$$+ \sum_{m+1} x^p (x-1)^q \cdots (x-j+1)^r (x-j)^{s+1}$$

for $0 < j \leq m$. In the first sum on the right, the exponents $p, q, \cdots, r$ take on all possible positive integral values such that $(p+q+\cdots+r)+1 = m+2$. In the second sum, the integers $p, q, \cdots, r, s$ take on all possible integral values such that $(p+q+\cdots+r)+(s+1) = m+2$. It follows that if we consider both sums on the right of (2.7) together, then their sum is $F_j^{m+1}(x)$, thus completing the proof of (2.6) when $0 < j \leq m$. Its truth for $j \leq 0$ or $j > m$ follows when we further consider (2.5) as well as (2.4).

Reconsidering equations (2.4), we note that each of the polynomials defined there has x as a factor. Consequently there exists a unique polynomial $G_j^m(x)$ such that

$$F_j^m(x) = x G_j^m(x) \tag{2.8}$$

for integral $m \geq 0$ and j. Substituting into (2.5) and (2.6), and noting that $G_0^0(x) = 1$ for all x, we see that $G_j^m(n+1)$ is a solution for (2.2) satisfying the boundary conditions (2.3). Consequently, we assert that

$$H_n^k = \sum_{j=0}^{m} (-1)^j \, G_j^m(n+1) \, H_{n-j}^{k+m} \tag{2.9}$$

for integers $k \geq 0$ and $m \geq 0$.[1]

3. Properties of the polynomials $G_j^m(x)$. In the work that follows, it will be more convenient to consider the polynomials $G_j^m(x)$ defined by (2.8). As might be expected, we find a considerable number of recurrence relations and other formulas involving these polynomials and their coefficients. Before proceeding to the particular applications in view, we shall list a few such relations. For integral $m \geq 0$ and j,

[1]The author is indebted to the referee for suggesting the above derivation of (2.9) which is somewhat simpler than the proof originally presented. The referee also proposed the following alternative derivation. We write

$$H^k(x) = \sum_{n=0}^{\infty} H_n^k x^n,$$

and then with $D = d/dx$,

$$(1-x) \, D\{xH^{k+1}(x)\} = H^k(x),$$

and symbolically,

$$[(1-x)\, Dx]^m \, H^{k+m}(x) = H^k(x).$$

Interpretation of the operator leads to the same results. This derivation is worth noting, for it is analogous to the classical development of equations (1.1) and (1.2).

(3.1) $$G_j^{m+1}(x) = (x-j)\,[G_{j-1}^m(x) + G_j^m(x)];$$

for integral $m \geq 1$ and j,

(3.2) $$G_j^{m+1}(x) = (x-1)\,G_{j-1}^m(x-1) + xG_j^m(x);$$

and for integral $m \geq 0$ and j,

(3.3) $$(j/2+x)\,G_j^m(j/2+x) = (-1)^{m+1}\,(j/2-x)\,G_j^m(j/2-x).$$

Equation (3.1) is obtained by substituting from (2.8) into (2.6). The proof of (3.2) is carried out by first deriving the relation

$$F_j^{m+1}(x) = x[F_{j-1}^m(x-1) + F_j^m(x)]$$

in the same manner as we derived (2.6), then substituting from (2.8). Equation (3.3) follows from the defining equation of $F_j^m(x)$ when (-1) is factored from each of the factors of the defining sum giving

$$F_j^m(x) = (-1)^{m+1}\,F_j^m(j-x)$$

for $0 \leq j \leq m$. Replacing x by $(j/2)+x$ and substituting from (2.8) yields the desired result. This relation displays the symmetric nature of the polynomials $F_j^m(x) = xG_j^m(x)$ in that they are symmetric with respect to the line $x = j/2$ when m is odd, and symmetric with respect to the point $(j/2, 0)$ when m is even.

Determine coefficients ${}_jA_{m,i}$ such that

(3.4) $$G_j^m(x) = {}_jA_{m,0}\,x^m + {}_jA_{m,1}\,x^{m-1} + \cdots + {}_jA_{m,m-1}\,x + {}_jA_{m,m}$$

for $m > 0$. It follows from the definition that

(3.5) $${}_jA_{m,i} = 0$$

for either $i < 0$, $i > m > 0$, $j < 0$, or $j > m > 0$, and in particular ${}_0A_{m,0} = 1$ while ${}_0A_{m,i} = 0$ for $i > 0$. The following is a table of the polynomials $G_j^m(x)$ when $m = 1, 2, 3,$ and 4:

$k = 1$	$k = 2$
$G_0^1(x) = x$	$G_0^2(x) = x^2$
$G_1^1(x) = x - 1$	$G_1^2(x) = 2x^2 - 3x + 1$
	$G_2^2(x) = x^2 - 3x + 2$

$k = 3$	$k = 4$
$G_0^3(x) = x^3$	$G_0^4(x) = x^4$
$G_1^3(x) = 3x^3 - 6x^2 + 4x - 1$	$G_1^4(x) = 4x^4 - 10x^3 + 10x^2 - 5x + 1$
$G_2^3(x) = 3x^3 - 12x^2 + 15x - 6$	$G_2^4(x) = 6x^4 - 30x^3 + 55x^2 - 45x + 14$
$G_3^3(x) = x^3 - 6x^2 + 11x - 6$	$G_3^4(x) = 4x^4 - 30x^3 + 80x^2 - 90x + 36$
	$G_4^4(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

Substituting from (3.4) into (3.1), collecting like terms with respect to x, replacing m by $m-1$, and equating coefficients, yields the recurrence relation

$$(3.6) \qquad {}_jA_{m,i} = ({}_jA_{m-1,i} + {}_{j-1}A_{m-1,i}) - j({}_jA_{m-1,i-1} + {}_{j-1}A_{m-1,i-1})$$

for integral $m \geq 1$ and j. Summing the latter expression with respect to j results in the relation

$$(3.7) \qquad \sum_{v=0}^{j} (-1)^v \, {}_vA_{m,i} = (-1)^j \, {}_jA_{m-1,i} - j(-1)^j \, {}_jA_{m-1,i-1} + \sum_{v=0}^{j-1} (-1)^v \, {}_vA_{m-1,i-1}$$

for $0 \leq i \leq m$. An interesting particular case of the latter formula is obtained by letting $j = m$ and considering (3.5). It follows that

$$\sum_{v=0}^{m} (-1)^v \, {}_vA_{m,i} = \sum_{v=0}^{m-1} (-1)^v \, {}_vA_{m-1,i-1}.$$

From repeated substitution, we conclude that

$$\sum_{v=0}^{m} (-1)^v \, {}_vA_{m,i} = {}_0A_{0,i-m},$$

whence

$$(3.8) \qquad \sum_{v=0}^{m} (-1)^v \, {}_vA_{m,i} = \begin{cases} 0 & \text{for } i < m \\ 1 & \text{for } i = m \end{cases}$$

when $m \geq 1$.

Recalling the factorial notation $x^{(m+1)} = x(x-1)\cdots(x-m)$, $m \geq 0$, we obtain

$$x G_m^m(x) = x^{(m+1)}.$$

But by definition, the numbers $s_{m,v}$ such that

$$x^{(m)} = s_{m,m} x^m + s_{m,m-1} x^{m-1} + \cdots + s_{m,1} x$$

are the Stirling numbers of the first kind [2, p. 143].[2] It now follows, since

$$G_m^m(x) = s_{m+1,m+1} x^m + s_{m+1,m} x^{m-1} + \cdots + s_{m+1,1},$$

that

$$(3.9) \qquad {}_mA_{m,i} = s_{m+1,m-i+1}.$$

In turn, letting $i = 0$ in (3.6), we find that

$${}_jA_{m,0} = {}_jA_{m-1,0} + {}_{j-1}A_{m-1,0}.$$

As a consequence of the initial conditions that ${}_0A_{m,0} = 1$ and ${}_jA_{m,0} = 0$ for $j > 0$, it follows [2, p. 615] that the solution of this partial difference equation is

$$(3.10) \qquad {}_jA_{m,0} = \binom{m}{j}.$$

When considering the polynomials $G_j^m(x)$ as displayed in the table, we see that, for any m, the coefficients considered by rows in light of (3.9) and (3.6) give a possible extension of the Stirling numbers. On the other hand, when the coefficients are considered by columns in light of (3.10), they present a possible extension of the binomial coefficients. This latter property is better displayed when we consider the known formula [2, p. 169]

$$\sum_{v=1}^{m} (-1)^v \binom{m}{v} v^j = (-1)^m \, m! S_{j,m} \qquad (j \geq 1),$$

where $S_{j,m}$ is the Stirling number of the second kind and thus $S_{j,m} = 0$ for $0 < j < m$. Make the definitions

$$P^m(i, j) = \sum_{v=1}^{m} (-1)^v \, {}_vA_{m,i} \, v^j \quad \text{and} \quad Q^m(i, j) = \sum_{v=0}^{m} (-1)^v \, {}_vA_{m,i} \, (v+1)^j,$$

where $m \geq 1$. It follows from a straightforward induction proof that

$$(3.11) \qquad P^m(0, 0) = -1 \quad \text{and} \quad P^m(i, j) = 0$$

[2] The notation used here for the Stirling numbers of the first and second kind is not the same as that used by Jordan in [2].

whenever $0 \leq i < m$, $0 \leq j < m - i$, and $i + j \neq 0$. The induction can be carried out by using the identity

$$P^{m+1}(i, j) = [P^m(i, j) - Q^m(i, j)] - [P^m(i-1, j+1) - Q^m(i-1, j+1)]$$

and the fact that the truth of (3.11) implies that both

$$Q^m(i, j) = 0 \tag{3.12}$$

for $0 \leq i < m$, $0 \leq j < m - i$, and

$$Q^m(i, m-i) = P^m(i, m-i)$$

for $0 \leq i \leq m$.

It is of interest that

$$\sum_{i=0}^{m} (-1)^i G_i^m(x + in) = \sum_{i=0}^{m-1} n^{m-i} P^m(i, m-i) + 1 \tag{3.13}$$

for $m \geq 1$, $n = 0, \pm 1, \pm 2, \cdots$, and all x. That is, the sum

$$\sum_{i=0}^{m} (-1)^i G_i^m(x + in)$$

is a function of n and m alone, independent of x. This follows from (3.8), (3.11), (3.12), and the identity

$$\begin{aligned}\sum_{i=0}^{m} (-1)^i G_i^m(x + in) &= \{ {}_0A_{m,0} + P^m(0, 0)\} x^m \\ &+ \sum_{j=1}^{m-1} \left[\sum_{v=0}^{j} \binom{m-v}{j-v} n^{j-v} P^m(v, j-v) \right] x^{m-j} \\ &+ \sum_{i=0}^{m-1} n^{m-i} P^m(i, m-i) + P^m(m, 0),\end{aligned}$$

where $m \geq 1$. Since the sum on the left of (3.13) is independent of x, we can write

$$\sum_{i=0}^{m} (-1)^i G_i^m(x + in) = \sum_{i=0}^{m} (-1)^i G_i^m(in)$$

for $m \geq 1$, $n = 0, \pm 1, \pm 2, \cdots$, and all x. Letting $n = 1$, recalling that $G_i^m(x)$ has $(x - i)$ as a factor for $i > 0$ and that $G_0^m(x) = x^m$, we see that

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x + i) = 0$$

for $m \geq 1$ and all x. If we let $n = 0$ in (3.13), then

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x) = 1 \tag{3.14}$$

for $m \geq 1$ and all x. It turns out that $n = 0, 1$ are the only two cases where the sum

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x + in)$$

is independent of m as well as x.

Consideration of (1.4) with (2.9) yields

$$G_j^m(n) = \binom{n-1}{j} \, \Delta^j \, (n - j)^m . \tag{3.15}$$

As might be expected, more is found concerning the nature of the coefficients of the polynomial $G_j^m(x)$ by studying the expression on the right of (3.15). Substituting into (3.15) from the identity

$$\Delta^j \, x^m = \sum_{v=1}^{m+1} v^{(j)} \, S_{m,v} \, x^{(v-j)},$$

where $S_{m,v}$ denotes the Stirling number of the second kind [2, p. 181], and simplifying, we obtain the relation

$$G_j^m(n) = \frac{(n-j)}{n} \sum_{v=j}^{m} \binom{v}{j} \, S_{m,v} \, n^{(v)} .$$

Substituting from the defining relation for the Stirling numbers of the first kind,

$$x^{(v)} = \sum_{i=1}^{v} s_{v,i} \, x^i ,$$

collecting like terms with respect to n^v, $v = 0, 1, \cdots, m$, and equating coefficients, yields the relation

$$_jA_{m,i} = \sum_{v=0}^{m-j} \binom{j+v}{j} S_{m,j+v}(s_{j+v,m-i} - j s_{j+v,m-i+1})$$

for integral $m \geq 0$, i, and j.

4. Application to Hölder summability. For the remainder of this paper $\{S_n\}$ denotes the sequence of partial sums of the arbitrary infinite series $\sum a_n$, and H_n^k denotes the k^{th} order Holder mean of the terms $S_0, S_1, \cdots, S_n$. If

$$\lim_{n \to \infty} H_n^k = S,$$

then $\sum a_n$ is said to be *summable Hölder of order k to S*, and this fact is denotéd by

$$\sum a_n = S(H, k).$$

In the same manner, the sequence $\{C_n^k\}$ defines Cesàro summability of order k. Likewise, Cesàro summability of order k is denoted by

$$\sum a_n = S(C, k).$$

The Hölder and Cesàro summability methods are equivalent in that

$$\sum a_n = S(H, k)$$

if and only if

$$\sum a_n = S(C, k).$$

At times it will be convenient to use the operator form of denoting the Hölder mean. That is, the k^{th} order Hölder mean of the terms $p_0, p_1, \cdots, p_n$ is denoted by $H^k(p_n)$. If $p_n = S_{n-k}$, $k > 0$, and $S_m = 0$ for $m < 0$, then we have

$$H^1(S_{n-k}) = \frac{1}{n+1} \sum_{v=0}^{n-k} S_v, \qquad H^k(S_{n-k}) = H^1(H^{k-1}(S_{n-k}))$$

for $k > 1$, and

$$H^0(S_{n-k}) = S_{n-k}.$$

It follows that

$$H^m(H^k(p_n)) = H^{m+k}(p_n) \tag{4.1}$$

and

$$H^m(p_n + q_n) = H^m(p_n) + H^m(q_n), \tag{4.2}$$

where m and k are nonnegative integers.

Letting $k = -m$ in (2.9), $m \geq 0$, we have the following definition for Hölder means of negative integral order.

DEFINITION 1. *For* $m \geq 0$,

$$H_n^{-m} = \sum_{i=0}^{m} (-1)^i\, G_i^m(n+1)\, S_{n-i}. \tag{4.3}$$

Referring to the defining equation for the Cesàro mean,

$$C_n^m = \binom{n+m}{m}^{-1} S_n^m,$$

we see that the first factor on the right is undefined for negative m when n is sufficiently large.

From Definition 1, it follows that (2.9) can be extended to all integral values of k. The Hölder method of summation is said to be *regular* since

$$\sum a_n = S$$

implies

$$\sum a_n = S(H, m)$$

for $m > 0$. With respect to negative order summation, the following extended sense of regularity is immediate.

(i) If $\sum a_n$ is divergent, then it is not summable $(H, -m)$ for any $m \geq 0$.

(ii) If

$$\sum a_n = S(H, -m)$$

for $m \geq 0$, then

$$\sum a_n = S(H, p)$$

for all $p \geq -m$.

Also, the right side of (4.3) can be used to define the operator H^{-m}. From this definition, it follows that properties (4.1) and (4.2) are true for all integral m and k.

Applying summation by parts to (4.3), considering (3.14), and using the operator notation, we find that

$$H^{-m}(S_n) = \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \, G_j^m(n+1) \right) a_{n-i} + S_{n-m} \tag{4.4}$$

for $m \geq 0$. Applying the operator H^{q+m}, we see that

$$H^q(S_n) = H^{q+m} \left[\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \, G_j^m(n+1) \right) a_{n-i} \right] + H^{q+m}(S_{n-m}) \tag{4.5}$$

for integers $m \geq 0$ and q. Since

$$\lim_{n \to \infty} H^q(S_n) = S$$

implies

$$\lim_{n \to \infty} H^{q+m}(S_{n-m}) = S$$

for $m \geq 0$, we have the following theorem as a formal statement of our results.

THEOREM 1. *If*

$$\sum a_n = S(H, q+m), \quad m \geq 0,$$

then

$$\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \, G_j^m(n+1) \right) a_{n-i} = 0(H, q+m)$$

is a necessary and sufficient condition that

$$\sum a_n = S(H, q).$$

Letting $q = 0$ in Theorem 1 yields a Tauberian theorem, that is, a theorem in which ordinary convergence is deduced from the fact that the series is summable and satisfies some further condition (which will vary with the method of summation).

Letting $q = -m$ in Theorem 1, we have the following corollary with respect to negative order summation.

COROLLARY 1. *If*

$$\sum a_n = S,$$

then

$$\lim_{n \to \infty} \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j G_j^m (n+1) \right) a_{n-i} = 0$$

is a necessary and sufficient condition that

$$\sum a_n = S(H, -m), \quad m \geq 0.$$

Noting that

$$\sum_{j=0}^{i} (-1)^j G_j^m (n+1)$$

is a polynomial at least of degree m, it follows that

$$\lim_{n \to \infty} n^m a_n = 0$$

implies

$$\lim_{n \to \infty} \left[\sum_{j=0}^{i} (-1)^j G_j^m (n+1) \right] a_{n-i} = 0,$$

and consequently we assert:

COROLLARY 2. *If*

$$\sum a_n = S,$$

then

$$\lim_{n\to\infty} n^m a_n = 0, \; m > 0,$$

is sufficient for

$$\sum a_n = S(H, -m).$$

Letting $m = 1$ in (4.5) we have

$$H^q(S_n) = H^{q+1}((n+1)\,a_n) + H^{q+1}(S_{n-1}),$$

or, applying the distributive property of this operator,

$$H^q(S_n) = H^{q+1}(na_n) + H^{q+1}(S_n). \tag{4.6}$$

This relation is equivalent to a well-known analogue to Kronecker's theorem [3, p. 485] which states that if $\sum a_n$ is summable (C, q), then

$$H^1(na_n) = 0\,(C, q).$$

Conversely, it follows from (4.6) that if $\sum a_n$ is summable $(H, q+1)$, then a necessary and sufficient condition that it be summable (H, q) is that

$$na = 0\,(H, q+1).$$

For integral $q \geq 0$ this is analogous to Theorem 65 of [1]. However, in the foregoing case, the statement is true for all integral q. As a further extension of the analogue to Kronecker's theorem, we have the following.

COROLLARY 3. *If*

$$\sum a_n = S(H, q),$$

then

$$\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \, G_j^m (n+1) \right) a_{n-i} = 0\,(H, q+m)$$

for integral $m > 0$.

For a special case where the condition of Corollary 2 is necessary as well as sufficient, we shall prove the following.

THEOREM 2. *If* $\sum a_n$ *is a convergent alternating series, then*

$$\lim_{n\to\infty} n^m a_n = 0, \; m \geq 0,$$

is a necessary and sufficient condition for $\sum a_n$ *to be summable* $(H, -m)$.

Proof. Letting $i = 0$ in (3.7), we conclude that there exist constants ${}_k a_{m,j}$, $j = 1, 2, \cdots, m$, such that

$$(4.7) \quad \sum_{j=0}^{k} (-1)^j G_j^m(n) = (-1)^k \, {}_k A_{m-1,0} \, n^m + {}_k a_{m,1} \, n^{m-1} + {}_k a_{m,2} \, n^{m-2} \cdots + {}_k a_{m,m}$$

for $0 \leq k < m$. We recall from the definition of $G_k^m(x)$ that ${}_k A_{m-1,0} > 0$ for $0 \leq k < m$. Consequently, for a given m, it follows that there exists an n_0 such that for all even k,

$$\sum_{i=0}^{k} (-1)^i G_i^m(n) > 0;$$

and for all odd k,

$$\sum_{i=0}^{k} (-1)^i G_i^m(n) < 0$$

whenever $n \geq n_0$. But by hypothesis, a_{n-k} is alternating in sign with respect to m, whence

$$(4.8) \quad \left| \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j G_j^m(n) \right) a_{n-i-1} \right| = \sum_{i=0}^{m-1} \left| \sum_{j=0}^{i} (-1)^j G_j^m(n) \right| |a_{n-i-1}|$$

for $n \geq n_0$. Also, it follows from (4.7) that

$$\lim_{n\to\infty} n^{-m} \left| \sum_{j=0}^{i} (-1)^j G_j^m(n) \right| = {}_i A_{m-1,0},$$

consequently there exist positive constants $n_1 > n_0$, $M(m)$, and $N(m)$ such that

$$n^m M(m) \le \left| \sum_{j=0}^{i} (-1)^j \, G_j^m(n) \right| \le n^m N(m)$$

for $0 \le i < m$ and $n \ge n_1$. Considering this with (4.8) yields

$$M(m) \sum_{i=0}^{m-1} \left(\frac{n}{n-i-1} \right)^m (n-i-1)^m \, |a_{n-i-1}| \le \left| \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j G_j^m(n) \right) a_{n-i-1} \right|$$

$$\le N(m) \sum_{i=1}^{m-1} \left(\frac{n}{n-i-1} \right)^m (n-i-1)^m \, |a_{n-i-1}|$$

for $n \ge n_1$. We conclude that

$$\lim_{n \to \infty} \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \, G_j^m(n) \right) a_{n-i-1} = 0$$

if and only if

$$\lim_{n \to \infty} n^m a_n = 0.$$

The theorem now follows from Corollary 1.

Letting $q = -1$ in (4.6), we see that any convergent series for which

$$\lim_{n \to \infty} n a_n \ne 0$$

is not summable Hölder for any negative order. On the other hand, $\sum 1/(n+1)^2$ is convergent and

$$\lim_{n \to \infty} n^2 a_n \ne 0,$$

yet it follows from direct application of Corollary 1 that this series is summable $(H, -2)$.

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ON A THEOREM OF PLANCHEREL AND PÓLYA

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1. Introduction. Paley and Wiener [6] have shown that the following classes of entire functions are equivalent:

(A) those which are $o(e^{a|z|})$ in the whole plane and belong to L^2 on the real axis;

(B) those which can be represented in the form

$$F(z) = \int_{-a}^{a} e^{izt} f(t)\, dt,$$

with $f(t) \in L^2$ on $[-a, a]$.

A simple proof was given later by Plancherel and Pólya [7], and they showed how the condition $o(e^{a|z|})$ could be weakened in the passage from (A) to (B). Their result leads at once to the following, which is the form to be used in the present discussion:

THEOREM A (Plancherel and Pólya). *Let $F(z)$ be an entire function of order 1, type a. If $F(x) \in L^2$ on $(-\infty, \infty)$ then $F(z)$ can be represented in the form*

$$F(z) = \int_{-a}^{a} e^{izt} f(t)\, dt,$$

with $f(t) \in L^2$ on $[-a, a]$.

The hypothesis concerning order and type means

$$\limsup \log |F(z)|/|z| \leq a, \quad |z| \longrightarrow \infty. \tag{1}$$

Theorem A implies a nontrivial result about entire functions; namely, if $F(z)$ satisfies (1) and is in L^2 on the real axis, then [7]

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$$(2) \qquad F(z) = o(e^{ar|\sin\theta|}), \qquad z = r\,e^{i\theta}.$$

We shall show here how Theorem A can be used to give very simple proofs of other results, some of which seem accessible only with more difficulty to purely complex-variable methods.

2. The growth of $F(z)$. The Plancherel-Pólya result determines the growth of $F(z)$ in the whole plane from the growth on the real axis:

THEOREM 1. *Let $F(z)$ be an entire function satisfying (1), such that*

$$F(x) = O(|x|^n)$$

for some positive or negative integer n, as $x \longrightarrow \infty$ on the real axis. Then

$$F(r\,e^{i\theta}) = O(r^n\,e^{ar|\sin\theta|})$$

uniformly in θ, as $r \longrightarrow \infty$.

THEOREM 2. *Let $F(z)$ be an entire function satisfying (1), such that*

$$|F(x)| \le A$$

for all real x. Then

$$|F(x+iy)| \le A\,e^{a|y|}$$

in the whole plane. If $\lambda = p + iq$ is a zero of $F(z)$, then

$$|F(z)| \le A\,e^{a|y|}\,|z - y|/|q|.$$

These results (which are probably well known) can be obtained at once by [8]; for example, applying [8] to $F(iz)\,e^{-az}/(A\,z^n + B)$ gives Theorem 1 when $n \ge 0$. Since our primary purpose here is to illustrate a method, however, we deduce them from Theorem A. Assume that $F(z)$ in Theorem 2 has a complex zero $\lambda = p + iq$, $q \ne 0$. (In the contrary case consider $F(z)(z - \lambda - iq)/(z - \lambda)$, where λ is a real zero, and let $q \longrightarrow 0$.) We have

$$(1) \qquad G(z) = [F(z)]^m/(z-\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-ma}^{ma} f(t)\,e^{izt}\,dt, \quad f(t) \in L^2,$$

where m is an integer. (A similar use of the m^{th} power of a function is made in [5] and [7].) By a short calculation, we get

(2) $$\int_{-ma}^{ma} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |G(t)|^2\, dx \le A^m\, \pi/|q|,$$

so that, by the Schwartz inequality in (1),

(3) $$|G(z)|^2 \le \frac{1}{\sqrt{2\pi}} \left(\int_{-ma}^{ma} e^{2|y|t}\, dt\right) (A^{2m}\pi/|q|), \quad z = x + iy.$$

Hence

$$|F(t)|^{2m} \le |z - \lambda|^2\, C\, A^{2m}\, e^{2ma|y|},$$

where C is constant. Taking the m^{th} root and letting $m \longrightarrow \infty$ completes the proof. The proof of Theorem 1 is similar, if we define

$$G(z) = (z - \lambda)^{-1}\, [F(z)/\mathrm{p}(z)]^m,$$

where $p(z)$ is a polynomial of degree n formed from the zeros, other than λ, of $F(z)$.

The second part of Theorem 2 results when we apply the first part to $F(z)/(z - \lambda)$; it could be sharpened by including more zeros. As it stands, however, this second part already gives the following:

COROLLARY. *Let $F(z)$ satisfy the hypothesis of Theorem 2, and suppose*

$$F(re^{i\theta}) \sim A\, e^{ar|\sin\theta|}$$

for a particular θ, as $r \longrightarrow \infty$. Then at most a finite number of zeros λ satisfy $\pi + \theta - \delta > 2 \arg \lambda > \theta + \delta$ for any positive δ.

3. Complex roots. A consequence of Theorem 1 is:

THEOREM 3. *Let $F(z)$ satisfy the hypothesis of Theorem 1, and let $n(x)$ denote the number of real roots of the equation $F(z) = 0$ which lie in the circle $|z| \le x$. If*

(4) $$\limsup_{r \to \infty} \int_1^r n(x)\, dx/x - 2ar/\pi + b \log r > -\infty,$$

then the equation $F(z) = 0$ has at most $b + n$ complex roots in the whole plane.

The proof is practically contained in a discussion of Levinson [5]. If $N(x)$ denotes the number of roots of $F(z) = 0$ in the circle $|z| \le x$, Jensen's theorem combines with the conclusion of Theorem 1 to give

$$(5)\qquad \int_1^r N(x)\,dx/x - A = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} (n \log r + ar|\sin\theta|)\,d\theta + B$$

$$= n \log r + 2ar/\pi + B,$$

where A and B are constants. Hence the number of complex zeros,

$$c(x) = N(x) - n(x),$$

satisfies

$$(6)\qquad \int_1^r c(x)\,dx/x \le (n+b)\log r + C$$

for some arbitrarily large r's, where C is constant. It follows that $c(x) \le n + b$, as was to be shown.

By means of the following result, Duffin and Schaeffer have given simple proofs, and improvements, of some theorems due to Szegö, Bernstein and Boas (see below):

THEOREM 4 (Duffin and Schaeffer). *Let $F(z)$ be an entire function such that*

$$F(z) = O(e^{a|z|}).$$

If $F(x)$ is real for all real x and satisfies $|F(x)| \le A$, then the equation

$$F(z) = A \cos(az + B)$$

has no complex roots.

Theorem 3 contains Theorem 4, and in fact gives a slight generalization of it:

THEOREM 5. *Let $F(z)$ be an entire function satisfying (1). If $F(x)$ is real for real x and satisfies*

$$|F(x)| \le |P(x)|,$$

where $P(x)$ is a real polynomial of degree n, then the equation

$$F(z) = P(z) \cos(az + B)$$

has at most $n + 1$ complex roots.

A linear change of variable enables us to assume $a = \pi$, $B = 0$. Since $F(z) - P(z) \cos \pi z$ is nonpositive when $\cos \pi z = 1$, and nonnegative when $\cos \pi z = -1$, the equation

$$F(z) = P(z) \cos \pi z$$

has a root in every interval $m \leq z \leq m + 1$, where m is an integer (cf. [3]). Any root occurring at the ends of these intervals is multiple. Hence if $n(x)$ is the number of real zeros λ satisfying $|\lambda| \leq x$, then $n(x)$ is at least equal to the function $n_1(x)$, defined as 0 for $0 \leq x \leq 1$, as 2 for $1 < n \leq 2$, and so on. A short calculation gives

$$\int_1^{n+} n(x) dx/x \geq \int_1^{n+} n_1(x) dx/x = 2 \log(n^n/n!) \sim 2n - \log n,$$

so that Theorem 5 follows from Theorem 3 with $b = 1$. Since complex zeros occur in pairs, Theorem 5 contains Theorem 4.

According to Paley and Wiener [6], a set of functions $\{e^{i\lambda_n x}\}$ has *deficiency* d on a given closed interval if it becomes complete in L^2 when d but not fewer functions $\{e^{i\lambda x}\}$ are adjoined to the set. Similarly, the set has *excess e* if it remains complete when e terms, but not more, are removed. Here we adopt the convention that a negative deficiency d means an excess $-d$. That the deficiency d is well defined follows from a theorem of Levinson [5]:

THEOREM 6 (Levinson). *If the set $\{e^{i\lambda_n x}\}$ is complete L^p on a finite interval, it remains complete when any λ_n is changed to another number.*

The result remains true even when several λ's are equal, if we agree to require a zero of the corresponding multiplicity in the entire function

$$F(z) = \int_{-a}^{a} e^{izt} f(t)\, dt, \quad f \in L^p,$$

which vanishes at the λ_n's. In this setting, the previous theorems concerning zeros appear as special cases of the following:

THEOREM 7. *Let $F(z)$ be an entire function satisfying (1), and suppose*

$$F(x) = O(|x|^n)$$

on the real axis. If $F(z) = 0$ *at a set* $\{\lambda_n\}$ *such that* $\{e^{i\lambda_n x}\}$ *has deficiency d on an interval of length* $2\pi a$*, then* $F(z)$ *has at most* $d + n$ *zeros other than the* λ_n*'s.*

The truth of the assertion is evident from

$$Q(z)\, F(z)/P(z) = \int_{-a}^{a} f(t)\, e^{izt}\, dt, \quad f(t) \in L^2,$$

where $Q(z)$ is any polynomial of degree d, and $P(x)$ is a polynomial of degree $d + n + 1$ formed from the (supposed) extra zeros of $F(z)$. That the result contains Theorem 5 and hence Theorem 6 follows from a theorem of Levinson [5] to the effect that $\{e^{i\lambda_n x}\}$ has deficiency at most d on $[0, 2\pi]$ if

$$|\lambda_n| \le |n| + d/2 + 1/4, \qquad -\infty < n < \infty$$

(cf. also [6]).

4. Completeness. Pursuing the subject of completeness in more detail, we find that some of Paley and Wiener's work can be simplified and generalized by use of Theorem A (cf. Theorems XXIX and XXX of [6]).

THEOREM 8. *Let* $\{\lambda_n\}$ *be a set of complex numbers such that the set* $\{e^{\pm i\lambda_n x}\}$ *has finite (positive, zero or negative) deficiency on some finite interval. Then the deficiency is d if and only if*

$$(7) \qquad \int_1^{\infty} x^{2d-2}\, |F(x)|^2\, dx < \infty, \quad \int_1^{\infty} x^{2d}\, |F(x)|^2\, dx = \infty,$$

where

$$F(z) = \prod (1 - z^2/\lambda_n^2).$$

We confine our attention to the case $d = 1$, since the general case is reduced to that by considering $P(z)\, F(z)$ or $F(z)/P(z)$ as heretofore. Suppose, then, that the set has deficiency $d = 1$ on an interval of length $2a$. Since the set is not complete, there is a function $G(z)$,

$$(8) \qquad G(z) = \int_{-a}^{a} f(t)\, e^{izt}\, dt, \quad f(t) \in L^2,$$

such that $G(\lambda_n) = 0$. By the Hadamard factorization theorem (cf. also [5]) we have

$$G(z) = F(z)\, e^{bz}\, P(z), \tag{9}$$

where $P(z)$ is a polynomial. Now actually $P(z)$ is constant, since otherwise $G(z)$ would have an extra zero, and the deficiency of the original set would be greater than 1. Hence (9) gives

$$F(z) = e^{-bz}\, G(z)\, C, \tag{10}$$

where C is constant. If b has positive real part, then (10) shows that $F(x)$ decreases exponentially as $x \longrightarrow \infty$. Since F is even, the same is true as $x \longrightarrow -\infty$, and hence $F(z) \equiv 0$ by a well-known result of Carlson. Similarly if b has negative real part. It follows that b is pure imaginary, so that

$$F(x) = \int_{-a}^{a} f(t)\, e^{i(x+c)t}\, dt, \quad c \text{ real}, \tag{11}$$

and hence $F(x) \in L^2$ by the Plancherel theorem.

On the other hand, if $xF(x) \in L^2$ then Theorem A yields the representation

$$zF(z) = \int_{-a}^{a} f(t)\, e^{izt}\, dt,$$

since (11) ensures (1); and hence the deficiency exceeds 1.

Suppose next that the deficiency is an unknown but finite number, and that

$$\int_{1}^{\infty} |F(x)|^2\, dx < \infty, \quad \int_{1}^{\infty} x^2\, |F(x)|^2\, dx = \infty. \tag{12}$$

With $2a$ as the interval of completeness, there is a function $G(z)$,

$$G(z) = \int_{-a}^{a} f(t)\, e^{izt}\, dt, \quad f(t) \in L^2,$$

such that $G(z) = 0$ at all but a finite number, say n, of the λ's, and has no other zeros. (Otherwise the set would have infinite negative deficiency). The Hadamard theorem gives

$$F(z) = e^{bz}\, P(z)\, G(z),$$

where $P(z)$ is a polynomial. If the imaginary part of $b = p + iq$ is positive, then

$$\limsup \log |F(iy)/y| \leq a - q \quad \text{as } y \longrightarrow \infty,$$

and hence the same is true as $y \longrightarrow -\infty$. Similarly if the imaginary part is negative. In either case, then, $F(z)$ satisfies (1). Equation (12) now combines with Theorem A to show that

$$F(z) = \int_{-a}^{a} g(t)\, e^{izt}\, dt, \quad g(t) \in L^2,$$

so that the set $\{e^{i\lambda_n x}\}$ is not complete. Thus the deficiency is at least 1.

On the other hand, if the deficiency is $n > 1$ then the Hadamard theorem, as before, gives

$$P(z)\, F(z)\, e^{bz} = \int_{-a}^{a} f(t)\, e^{izt}\, dt, \quad f(t) \in L^2$$

where $P(z)$ is a polynomial of degree $n - 1$. As before, the presence of b causes no difficulty, so that $P(x)\, F(x) \in L^2$. This contradicts (12).

Theorem 8 contains Theorem 6 for the case L^2, although Levinson's general case L^p seems somewhat deeper. We give an application:

THEOREM 9. *Let*

$$F(z) = \prod (1 - z^2/\lambda_n^2),$$

where the λ_n *are complex numbers, and let the equation* $F(x) = A$ *have roots* λ_n', *where* A *is a complex nonzero constant. If* $\{e^{i\lambda_n x}\}$ *has finite deficiency* d *and* $\{e^{i\lambda_n' x}\}$ *has finite deficiency* d', *then* $d < 0$ *implies* $d' = d$, *and* $d > 0$ *implies* $d' = 0$. *If* $d = 0$ *then* $d' \geq 0$.

It should be observed that d' is restricted to be finite in the hypothesis of the theorem, and only then can we evaluate d' more exactly. With regard to this assumption, the following may be said. First, the set $\exp(i\lambda_n' x)$ cannot have infinite excess; that is, $d' \neq -\infty$. In the other direction, the set is complete on every interval of length less than the interval for $\{\lambda_n\}$ (which does not mean, however, that d' is finite). For the case of real λ_n, an elementary but long argument shows that in fact d' is finite, so that we can then dispense with this extra hypothesis. These matters lie to one side of the present discussion, since their proof does not involve Theorem A, and we omit them.

A second remark may be in order. It is well known that all the A-points of a canonical product have the same exponent of convergence, and in Theorem 9 one can prove the stronger result that $\lim \Lambda(u)/u$ and $\lim \Lambda'(u)/u$ both exist and are equal. Even this statement is less precise than the conclusion of the theorem, however. It is easy to construct sets with equal density, such that one set has infinite excess and the other has infinite deficiency on a given interval. We conjecture, incidentally, that one can make $d = 0$, $d' = m$, where m is any positive integer, so that the nebulous case $d = 0$ cannot be improved.

To establish Theorem 9, write

$$\int_1^\infty |F(x) - A|^2 x^{2d-2} \leq \int |F|^2 x^{2d-2} + 2|A| \int |F| x^{2d-2} + |A|^2 \int x^{2d-2},$$

which is finite if $d \leq 0$, by Theorem 8 and the Schwartz inequality applied to the second integral. Hence, by Theorem 8 again,

$$d' \geq d \text{ if } d \leq 0. \tag{13}$$

Writing

$$F(z) = [F(z) - a] + a,$$

and turning the argument about, gives

$$d \geq d' \text{ if } d' \leq 0. \tag{14}$$

Suppose now $d > 0$, so that, by Theorem 8,

$$\int_1^\infty |F(x)|^2 \, dx < \infty.$$

This implies $F(x) \longrightarrow 0$, as is well known, so that $F(x) - A$ is dominated by A. Hence by Theorem 8 the zeros form an exact set:

$$d' = 0 \text{ if } d > 0. \tag{15}$$

Similarly,

$$d = 0 \text{ if } d' > 0. \tag{16}$$

Equations (13) and (16) show that $d < 0$ implies $0 > d' > d$. But then (14) gives $d \geq d'$, since $d' \leq 0$; and thus $d < 0$ implies $d' = d$.

5. An inequality for entire functions. In a series of interesting papers [2], [3], [4], Duffin and Schaeffer establish some inequalities for entire functions of exponential type bounded on the real axis. From these they obtain, sometimes in sharpened form, the classical inequalities of Bernstein and others for bounded polynomials. The main results are as follows:

THEOREM 10 (Duffin and Schaeffer). *Let $F(z)$ be an entire function, real on the real axis, which satisfies*

$$F(z) = O(e^{a|z|})$$

in the whole plane and $|F(x)| \leq 1$ for $-\infty < x < \infty$. Then, with $z = x + iy$, we have

$$|F(z)| \leq \cosh ay, \quad |F(z)|^2 + |F'(z)|^2/a^2 \leq \cosh 2ay.$$

If there is equality at any point except points on the real axis where $F(x) = \pm 1$, then $F(x) = \cos(bx + c)$.

Our Theorem 1 shows that the hypothesis $O(a^{a|z|})$ can be replaced by (1). The procedure in [2] is to deduce the result for $y = 0$ first, by means of Theorem 4. In this form the statement seems due chiefly to Boas [1]:

THEOREM 11 (Duffin, Schaeffer, and Boas). *Let $F(z)$ be an entire function satisfying (1) and real on the real axis. If $|F(x)| \leq 1$ for all real x then*

$$|F(x)|^2 + |F'(x)|^2/a^2 \leq 1$$

for all real x.

A modification of Duffin and Schaeffer's argument[1] enables us to deduce Theorem 11 from Theorem A. Suppose the hypothesis fulfilled, but let the conclusion be violated at a particular point $x = b$. By considering $\pm F(\pm z/a)$, we may assume

$$F(b) \geq 0, \quad F'(b) \leq 0, \quad \text{and} \quad a = 1,$$

besides

$$(17) \qquad |F(b)|^2 + |F'(b)|^2 > 1.$$

The equation $F(b) = \cos z$ has a root $z = r$, $0 \leq r \leq \pi/2$, since $1 \geq F(a) \geq 0$.

[1]The author regrets having presented this discussion to the American Mathematical Society without knowing of Duffin and Schaeffer's work.

Now, in fact $r > 0$. For if $r = 0$ then $F(b) = 1$, and (2) yields $F'(b) < 0$. Hence $F(x)$ is strictly decreasing at $x = b$, so that $F(x) > 1$ for some $x < b$.

If we define

$$G(z) = F(z + b - r) - \cos z, \tag{18}$$

then

$$G(r) = 0, \quad G(0) \leq 0, \quad G(\pi) \geq 0. \tag{19}$$

Moreover,

$$G'(r) = F'(b) + \sin r = F'(b) + [1 - F^2(b)]^{1/2} < 0,$$

the last inequality being a consequence of $F'(b) \leq 0$ and (17). Combined with (19), the condition $G'(r) < 0$ shows readily that $G(z) = 0$ has three roots $r_0 < r < r_1$ in the interval $0 \leq z \leq \pi$; and if $r_0 = 0$ or $r_1 = \pi$ the corresponding root is multiple, since $|F(x)| \leq 1$. Besides these roots, $G(z) = 0$ has roots r_n in each interval $[n\pi, (n+1)\pi]$, $n = \pm 1, \pm 2, \cdots$. Thus, the function

$$H(z) = G(z)/(z - r) = \int_{-1}^{1} e^{izt}\, h(t)\, dt, \quad h(t) \in L^2, \tag{20}$$

has roots at r_0 and r_n, $n = \pm 1, \pm 2, \cdots$, where the enumeration can be so managed that

$$|r_n| \leq |n|\,\pi. \tag{21}$$

By Levinson's theorem cited above, the set $\{e^{ir_n x}\}$ is complete L^2 on $[-1, 1]$, and therefore $h(t) = 0$ almost everywhere. If the inequality of Theorem 11 becomes an equality at a point where $F(x) \neq \pm 1$, then the corresponding root of $G(z)$ is easily seen to be triple, so that the same discussion holds

6. Differences and derivatives. We conclude with a theorem of different type, concerning classes of functions:

THEOREM 12. *Let C denote the class of entire functions which satisfy* (1) *and belong to L^2 on the real axis. Let h be any complex or real nonzero number, except that $|h| < 2\pi/a$ if h is real. Then the class of functions $F'(z)$, where F ranges over C, is identical with the class of functions $G(z+h) - G(z)$, where G ranges over C. But if h is real and $|h| \geq 2\pi/a$, the latter class is always a proper subset of the former.*

Results of the same sort without L^2 condition are well known; for example, Carmichael has shown that the equation

$$F(z+1) - F(z) = G(z)$$

has a unique solution of type a on the real axis and c on the imaginary axis, if $G(z)$ is of this same type, $G(0) = 1$, and $c < \pi$. To prove Theorem 12, let $F(z)$ be in C, so that by Theorem A we have

$$\begin{aligned} F(z+h) - F(z) &= \int_{-a}^{a} e^{izt}\,(e^{iht}-1)\, f(t)\, dt, \quad f \in L^2 \\ &= \int_{-a}^{a} e^{izt}\,[(e^{iht}-1)/it]\, f(t)\, it\, dt \\ &= \int_{-a}^{a} e^{izt}\, it\, g(t)\, dt, \quad g \in L^2. \end{aligned}$$

Hence, every function of the form $F(z+h) - F(z)$, with $F \in C$, is representable as $G'(t)$ with $G \in C$. Similarly, let $G \in C$, so that

$$\begin{aligned} G'(z) &= \int_{-a}^{a} it\, e^{izt}\, g(t)\, dt, \quad g \in L^2 \\ &= \int_{-a}^{a} e^{izt} \frac{it}{e^{iht}-1}\,(e^{iht}-1)\, g(t)\, dt \ (ht \neq 2n\pi) \\ &= \int_{-a}^{a} e^{izt}\,(e^{iht}-1)\, f(t)\, dt, \quad f(t) \in L^2. \end{aligned}$$

Thus $G'(z)$ is representable as $F(z+h) - F(z)$ with $F \in C$, provided $ht \neq 2n\pi$ for $-a \leq t \leq a$. The latter condition is fulfilled unless h is real, and $h = 0$ or $|h| \geq 2\pi/a$.

Suppose now that h is real and $|h| \geq 2\pi/a$. If

$$\int_{-a}^{a} ite^{izt}\, dt = \int_{-a}^{a} e^{izt}(e^{iht}-1)\, f(t)\, dt \tag{22}$$

for $f(t) \in L^2$, then uniqueness of Fourier transforms in L^2 ensures that

$$f(t) = it/(e^{iht}-1)$$

almost everywhere; but $f(t)$ is not in L^2 with the assumed condition on h. Thus the function on the left of (22) is representable as $G'(z)$ but not as $F(z+h) - F(z)$.

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ON THE COMPLEX ZEROS OF FUNCTIONS OF STURM-LIOUVILLE TYPE

CHOY-TAK TAAM

1. Let $Q(z)$ **be an analytic function** of the complex variable z in a region D. In the present paper only those solutions of

$$(1.1) \qquad W'' + Q(z)W = 0$$

which are distinct from the trivial solution ($\equiv 0$) shall be considered.

In this paper the following results shall be established.

THEOREM 1. *Suppose that the following conditions are satisfied:*

(a) *the circle* $|z| \leq R$ *is contained in* D,

(b) $W(z)$ *is a solution of* (1.1), $W(0) \neq 0$,

(c) $n(r)$ *is the number of zeros of* $W(z)$ *in* $|z| \leq r$, $r < R$.

Then $n(r)$ *satisfies the inequality*

$$(1.2) \qquad n(r) \leq (\log(Rr^{-1}))^{-1}[\log(1 + R|W'(0)|\,|W(0)|^{-1}) + (2\pi)^{-1}\int_0^{2\pi}\int_0^R (R-t)\,|Q(t\,e^{i\theta})|\,dtd\theta].$$

COROLLARY 1.1. *Suppose that the following conditions are satisfied:*

(a) $Q(z)$ *is a polynomial of degree* k,

(b) *conditions* (b) *and* (c) *of Theorem* 1 *hold.*

Then $W(z)$ *is an integral function of order at most* $k+2$. *Furthermore, as* $r \longrightarrow \infty$,

$$(1.3) \qquad n(r) = O(r^{k+2}).$$

Obviously the result of Theorem 1 is not good if r is close to R. Also it

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does not apply to a solution which vanishes at the origin. The following theorem is free of these restrictions.

THEOREM 2. *Suppose that the following conditions are satisfied:*

(a) *S is a closed region contained in D,*

(b) *the boundary C of S is a closed contour,*

(c) *the maximum value of $|Q(z)|$ on C is M,*

(d) *S can be divided into n subregions such that each subregion has a diameter not greater than $\pi M^{-1/2}$; and for any two points z_1 and z_2 of a subregion, the linear segment $z_1 z_2$ lies in S (we agree that the common boundary of two subregions belongs to both subregions).*

Then

(e) *if $Q(z)$ is not a constant, the number of zeros of any solution $W(z)$ of (1.1) in S is not greater than n,*

(f) *more accurately, if $Q(z)$ is not a constant, each solution $W(z)$ of (1.1) has at most one zero in each subregion, and when it is known that $W(z)$ has some zero z_i which belongs to n_i ($n_i > 1$) different subregions, $i = 1, 2, \cdots, k$, its total number of zeros in S is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$,*

(g) *if some solution of (1.1) has more than one zero in some subregion, $Q(z)$ must be a constant and $|Q(z)| = M > 0$ in D.*

We may observe that if $Q(z)$ is not a constant, M must be positive, according to the principle of the maximum modulus. If $Q(z)$ is a constant, the problem is trivial as the distribution of the zeros is known.

2. To prove Theorem 1, we need the following known results.

LEMMA 1. *Suppose that the following conditions are satisfied:*

(a) *$f(x)$ and $g(x)$ are real-valued functions, continuous and nonnegative for $x \geq 0$,*

(b) *M is a positive constant,*

(c) $f(x) \leq M + \int_0^x f(t)g(t)\,dt, \qquad x \geq 0.$

Then we have

$$f(x) \le M\, e^{\int_0^x g(t)dt} \qquad x \ge 0.$$

This lemma is due to R. Bellman. For a proof of it see [1] or [5].

LEMMA 2. *Suppose that the following conditions are satisfied:*

(a) *$f(z)$ is analytic for $|z| \le R$, $f(0) \ne 0$,*

(b) *the moduli of the zeros of $f(z)$ in the circle $|z| \le R$ are $r_1, r_2, \cdots, r_k$ arranged as a nondecreasing sequence (a zero of order p is counted p times).*

Then we have

$$\log [R^k (r_1 r_2 \cdots r_k)^{-1}] = (2\pi)^{-1} \int_0^{2\pi} \log |f(R\, e^{i\theta})|\, d\theta - \log |f(0)|.$$

Lemma 2 is known as Jensen's theorem (see [4]).

3. Now we shall prove Theorem 1. Along a fixed ray radiating out from the origin, $z = r \exp(i\theta)$, equation (1.1) becomes

$$\frac{d^2 W}{dr^2} + e^{2i\theta} Q(re^{i\theta}) W = 0. \tag{3.1}$$

Integrating (3.1) twice from 0 to r, we obtain

$$W(re^{i\theta}) = W(0) + W'(0) e^{i\theta} r - e^{2i\theta} \int_0^r \int_0^h Q(te^{i\theta}) W(te^{i\theta})\, dt dh, \tag{3.2}$$

where $W'(0) \exp(i\theta)$ is the value of dW/dr at the origin. Integration by parts of the integral in (3.2) gives

$$W(re^{i\theta}) = W(0) + W'(0) e^{i\theta} r - e^{2i\theta} \int_0^r (r-t) Q(te^{i\theta}) W(te^{i\theta})\, dt. \tag{3.3}$$

For $r \le R$, (3.3) yields

$$|W(re^{i\theta})| \le |W(0)| + |W'(0)|\, R + \int_0^r (R-t) |Q(te^{i\theta}) W(te^{i\theta})|\, dt. \tag{3.4}$$

Applying Lemma 1 to (3.4), we have

$$|W(Re^{i\theta})| \le (|W(0)| + |W'(0)| R)\, e^{\int_0^R (R-t) |Q(te^{i\theta})| dt}. \tag{3.5}$$

Let the moduli of the zeros of $W(z)$ in the circle $|z| \leq r < R$ be $r_1, r_2, \cdots, r_k$, arranged as a nondecreasing sequence. Then an appeal to Lemma 2 gives

$$(3.6) \quad \log [R^k(r_1 r_2 \cdots r_k)^{-1}] \leq (2\pi)^{-1} \int_0^{2\pi} \log |W(Re^{i\theta})| \, d\theta - \log |W(0)|.$$

Clearly

$$(3.7) \quad \log [R^k(r_1 r_2 \cdots r_k)^{-1}] \geq \log [R^{n(r)} r^{-n(r)}]$$

$$= n(r) \log (Rr^{-1}), \qquad r < R,$$

where $n(r)$ is the number of zeros of $W(z)$ in $|z| \leq r$. On the other hand, (3.5) gives

$$(3.8) \quad \int_0^{2\pi} \log |W(Re^{i\theta})| \, d\theta \leq 2\pi \log [|W(0)| + |W'(0)| R]$$

$$+ \int_0^{2\pi} \int_0^R (R-t) \, |Q(te^{i\theta})| \, dt \, d\theta .$$

Combining (3.6), (3.7), and (3.8), we have

$$(3.9) \quad n(r) \log (Rr^{-1}) \leq \log [|W(0)| + |W'(0)| R] - \log |W(0)|$$

$$+ (2\pi)^{-1} \int_0^{2\pi} \int_0^R (R-t) \, |Q(te^{i\theta})| \, dt \, d\theta$$

for $r < R$. But (3.9) is equivalent to (1.2), so that this completes the proof of Theorem 1.

If $Q(z)$ is a polynomial of degree k, then $W(z)$ is analytic except at infinity and, from (3.5),

$$|W(Re^{i\theta})| = O\left(e^{A \cdot R^{k+2}}\right), \qquad R \longrightarrow \infty ,$$

where A is a constant. Hence $W(z)$ is an integral function of order at most $k+2$. Finally if we set $R = 2r$ in (3.9), it is clear that

$$n(r) = O(r^{k+2}).$$

This proves Corollary 1.1.

4. To prove Theorem 2, we need the following known result. On the real axis, equation (1.1) becomes

$$\frac{d^2W}{dx^2} + Q(x)W = 0, \tag{4.1}$$

where x is the real part of the complex variable z. Denote by $q_1(x)$ the real part of $Q(x)$.

LEMMA 3. *Let $W(x)$ be a solution of (4.1), $W(0) = 0$. Suppose that one of the following conditions is satisfied.*

(a) max $q_1(x) = m > 0$ *in* $[0, a]$, $0 < a \leq \pi m^{-1/2}$, *and* $Q(x) \neq m$ *in* $[0, a]$,

(b) $q_1(x) \leq 0$ *in* $[0, a]$.

Then $W(x) \neq 0$ *in* $(0, a]$.

This lemma was proved in [3; Theorems 5.1, 5.2]. Part (b) is also covered by a theorem of Hille [2, p. 512 ff.]. Its proof remains valid even if $Q(x)$ is assumed only to be a continuous (complex-valued) function of a real variable x; consequently the lemma remains true under such an assumption on $Q(x)$.

We first prove (f) of Theorem 2.

Let S_i be one of the subregions of S with a diameter not greater than $\pi M^{-1/2}$. Suppose that $W(z)$ is a solution of (1.1) which vanishes at a point z_0, say, of S_i. Consider a fixed ray radiating out from z_0, $z - z_0 = r \exp(i\theta)$. Along this ray, equation (1.1) becomes

$$\frac{d^2W}{dr^2} + e^{2i\theta} Q(z_0 + re^{i\theta})W = 0. \tag{4.2}$$

By virtue of the principle of the maximum modulus, we have

$$|e^{2i\theta}Q(z)| = |Q(z)| \leq M$$

for any point z of S on this ray. Hence on a segment of this ray between z_0 and any other point of S_i (by assumption, this segment lies in S) the maximum value m, say, of the real part of exp $(2i\theta)Q(z)$ is not greater than M. If m is positive, then $\pi m^{-1/2} \geq \pi M^{-1/2}$. Since $Q(z)$ is not a constant, $\exp(2i\theta)Q(z) \neq m$ on this segment. By virtue of the fact that the diameter of S_i is not greater than $\pi M^{-1/2}$ and Lemma 3, it is clear that $W(z)$ does not vanish again on that part of the ray in S_i, regardless of the sign of m. Repeating this process for each ray radiating out from z_0, we see clearly that $W(z)$ cannot vanish again in S_i. Since S_i is an arbitrary subregion, $W(z)$ can vanish at most at one point of each subregion.

On the other hand, if $W(z)$ has a zero z_i which belongs to n_i $(n_i > 1)$ different subregions, then $W(z)$ cannot vanish again in any of these n_i subregions, as the foregoing proof shows. If it is known that there are k such zeros z_i, each z_i belonging to n_i subregions, $i = 1, 2, \cdots, k$, it is clear that the total number of zeros of $W(z)$ in S is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$.

To prove (g), let $W(z)$ be a solution of (1.1) having two zeros, say z_0 and z_1, in some subregion S_i. Let the argument of $z_1 - z_0$ be θ. Then along the linear segment $z_0 z_1$, equation (1.1) becomes (4.2). According to Lemma 3, the maximum value m of the real part of $\exp(2i\theta)Q(z)$ on the linear segment $z_0 z_1$ must be positive. Further, since

$$|z_1 - z_0| \leq \pi M^{-1/2} \leq \pi m^{-1/2}, \tag{4.3}$$

z_0 and z_1 can both be the zeros of $W(z)$ only if

$$e^{2i\theta}Q(z) \equiv m \tag{4.4}$$

on the linear segment $z_0 z_1$, by Lemma 3 again. But if (4.4) is true, the general solution of (4.2) is $A \sin(m^{1/2} r + B)$, A and B being constants. If a solution of (4.2) has two zeros, the distance between them must not be less than $\pi m^{-1/2}$. In other words, the equality signs in (4.3) must hold. That is, $M = m$. From (4.4), we have $\exp(2i\theta)Q(z) \equiv M$ on the linear segment $z_0 z_1$. Since $Q(z)$ is an analytic function and constant on the linear segment $z_0 z_1$, $Q(z)$ is a constant in D. Obviously $|Q(z)| = M$; and since m is positive, so is M. This proves (g).

Clearly (e) follows from (f), and this completes the proof of Theorem 2.

5. Added in proof. The author is indebted to a referee for calling his attention to the fact that, in connection with Corollary 1.1, an entire function which satisfies a linear differential equation with coefficients which are rational functions of z is always of finite rational order and of perfectly regular growth. (See G. Valiron, *Lectures on the theory of integral functions*, Toulouse, 1923, p. 106 ff.)

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