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**ON UNIFORM DISTRIBUTION MODULO A SUBDIVISION**

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1. Let  $\Delta$  be a subdivision of the interval  $(0, \infty)$ :  $\Delta = (z_0, z_1, \dots)$ , where

$$0 = z_0 < z_1 < \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} z_n = \infty.$$

For  $z_{n-1} \leq x < z_n$ , put

$$[x]_{\Delta} = z_{n-1}, \quad \delta(x) = z_n - z_{n-1}, \quad \langle x \rangle_{\Delta} = \frac{x - [x]_{\Delta}}{\delta(x)}, \quad \phi(x) = n + \langle x \rangle_{\Delta},$$

so that  $0 \leq \langle x \rangle_{\Delta} < 1$ . Let  $\{x_k\}$  be an increasing sequence of positive numbers. If the sequence  $\{\langle x_k \rangle_{\Delta}\}$  is uniformly distributed over  $[0, 1]$ , in the sense that the proportion of the numbers  $\langle x_1 \rangle_{\Delta}, \dots, \langle x_k \rangle_{\Delta}$  which lie in  $[0, \alpha)$  approaches  $\alpha$  as  $k \rightarrow \infty$ , for each  $\alpha \in [0, 1)$ , then we shall say that the sequence  $\{x_k\}$  is *uniformly distributed modulo  $\Delta$* . If  $\Delta$  is the subdivision  $\Delta_0$  for which  $z_n = n$ , this reduces to the ordinary concept of uniform distribution (mod 1), since then  $[x]_{\Delta} = [x]$ ,  $\delta(x) = 1$  for all  $x$ , and  $\langle x \rangle_{\Delta} = x - [x]$  is the fractional part of  $x$ . Even in other cases, the generalization is more apparent than real, since the uniform distribution of one sequence (mod  $\Delta$ ) is equivalent to the uniform distribution of another sequence (mod 1). But most of the known theorems concerning uniform distribution (mod 1) are not applicable to the sequences  $\{\langle x_k \rangle_{\Delta}\}$ , if  $\Delta$  is not  $\Delta_0$ , for in such theorems  $x_k$  is ordinarily taken to be the value  $f(k)$  of a function whose derivative exists and is monotonic for positive  $x$ . Here, on the other hand,  $\langle x_k \rangle_{\Delta} \equiv \phi(x_k) \pmod{1}$ , and  $\phi$ , although a continuous polygonal function, is not necessarily everywhere differentiable; and unless  $\delta(x)$  is assumed monotonic,  $\phi'$  is not monotonic even over the set on which it exists. This lack of monotonicity introduces serious difficulties; it is the object of the present work to show how they can be dealt with in certain cases.

For brevity, "uniformly distributed" will be abbreviated to "u.d.". The symbols " $\nearrow$ ", " $\nearrow$ ", " $\searrow$ " and " $\searrow$ " indicate monotonic approach: increasing, non-decreasing, decreasing, and non-increasing, respectively.

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## 2. Put

$$N(\alpha, x) = \sum_{\substack{x_k \leq x \\ \langle x_k \rangle_\Delta < \alpha}} 1, \quad N(x) = N(1, x);$$

then  $\{x_k\}$  is u.d. (mod  $\Delta$ ) if and only if, for each  $\alpha \in [0, 1)$ ,

$$\lim_{x \rightarrow \infty} \frac{N(\alpha, x)}{N(x)} = \alpha.$$

THEOREM 1. A necessary condition that  $\{x_k\}$  be u.d. (mod  $\Delta$ ) is that

$$N(z_{n+1}) \sim N(z_n)$$

as  $n \rightarrow \infty$ .

For suppose that  $\{x_k\}$  is u.d. (mod  $\Delta$ ). Then since

$$N\left(\frac{1}{2}, \frac{z_n + z_{n+1}}{2}\right) - N(1/2, z_n) = N\left(\frac{z_n + z_{n+1}}{2}\right) - N(z_n),$$

we have

$$\begin{aligned} \frac{1}{2} &\sim \frac{N(1/2, (z_n + z_{n+1})/2)}{N((z_n + z_{n+1})/2)} = \frac{N(1/2, z_n)}{N((z_n + z_{n+1})/2)} + \frac{N((z_n + z_{n+1})/2) - N(z_n)}{N((z_n + z_{n+1})/2)} \\ &= \frac{N(1/2, z_n)}{N(z_n)} \cdot \frac{N(z_n)}{N((z_n + z_{n+1})/2)} + 1 - \frac{N(z_n)}{N((z_n + z_{n+1})/2)} \\ &= 1 + \frac{N(z_n)}{N((z_n + z_{n+1})/2)} \left( \frac{N(1/2, z_n)}{N(z_n)} - 1 \right) \sim 1 - \frac{1}{2} \frac{N(z_n)}{N((z_n + z_{n+1})/2)} \end{aligned}$$

as  $n \rightarrow \infty$ , and so

$$N(z_n) \sim N\left(\frac{z_n + z_{n+1}}{2}\right).$$

In the same way it can be shown that

$$N\left(\frac{z_n + z_{n+1}}{2}\right) \sim N(z_{n+1}),$$

and consequently  $N(z_n) \sim N(z_{n+1})$ .

3. The following theorem, due in a slightly different form to Fejér (see [1, p.88-89]), expresses the fact that if  $f$  is sufficiently smooth and  $[f(x)]$  is constant over increasingly long intervals as  $x$  increases, such that the length of the  $n$ -th interval is of smaller order of magnitude than the total length of all preceding intervals, then  $f(k)$  is u.d. (mod 1):

Suppose that  $f(x)$  has the following properties:

- (i)  $f$  is continuously differentiable for  $x > x_0$ ,
- (ii)  $f(x) \uparrow \infty$  as  $x \uparrow \infty$ ,
- (iii)  $f'(x) \searrow 0$  as  $x \uparrow \infty$ ,
- (iv)  $xf'(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Then  $f(k)$  is u.d. (mod 1).

The following theorem uses the same general idea:

**THEOREM 2.** Suppose that, for a given subdivision  $\Delta$  and a sequence  $\{x_k\}$ ,  $N(z_n) - N(z_{n-1}) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then  $\{x_k\}$  is u.d. (mod  $\Delta$ ) if the following conditions are satisfied:

- (i)  $N(z_{n-1}) \sim N(z_n)$  as  $n \rightarrow \infty$ ,
- (ii) except possibly on a sequence of intervals  $[z_{n_t-1}, z_{n_t})$  such that

$$(1) \quad \sum_{t=1}^m (N(z_{n_t}) - N(z_{n_t-1})) = o(N(z_{n_m})),$$

the relation

$$\max(x_k - x_{k-1}) \sim \min(x_k - x_{k-1})$$

holds as  $n \rightarrow \infty$ , the maximum and minimum being taken independently, for given  $n \neq n_1, n_2, \dots$ , over all  $k$  for which at least one of  $x_{k-1}$  and  $x_k$  is in  $[z_{n-1}, z_n]$ .

Give the name  $\delta_n$  to the interval  $[z_{n-1}, z_n]$ , and put

$$N(\alpha, \delta_n) = N(z_{n-1} + \alpha(z_n - z_{n-1})) - N(z_{n-1}),$$

$$N(\delta_n) = N(1, \delta_n) = N(z_n) - N(z_{n-1}).$$

It will be shown that

$$\lim_{\substack{n \rightarrow \infty \\ n \neq n_1, n_2, \dots}} \frac{N(\alpha, \delta_n)}{N(\delta_n)} = \alpha;$$

in other words, that in the limit the  $x_k$ 's which lie in  $\delta_n \neq \delta_{n_t}$  are u.d. there. This implies the theorem, for using it, (1), and (i) we have, for  $x \in \delta_n$ ,

$$\begin{aligned} \frac{N(\alpha, x)}{N(x)} &= \frac{1}{N(x)} \left\{ \sum_{\nu=1}^{n-1} N(\alpha, \delta_n) + N(\min(x, z_{n-1} + \alpha(z_n - z_{n-1}))) - N(z_{n-1}) \right\} \\ &= \frac{1}{N(x)} \sum^{\circ} N(\alpha, \delta_{\nu}) + o(1) \\ &= \frac{\sum^{\circ} (\alpha + o(1)) N(\delta_{\nu})}{\sum^{\circ} N(\delta_{\nu}) + o\left(\sum^{\circ} N(\delta_{\nu})\right) + N(x) - N(z_{n-1})} + o(1) \\ &= \frac{\alpha}{1 + o(1)} + o(1) = \alpha + o(1), \end{aligned}$$

where  $\sum^{\circ}$  denotes summation from  $\nu = 1$  to  $\nu = n - 1$ ,  $\nu \neq n_1, n_2, \dots$ .

To prove (2), suppose that  $n \neq n_1, n_2, \dots$ , that  $z_{n-1} \in (x_{k_n}, x_{k_{n+1}}]$ , and that

$$\min_{k_n \leq k \leq k_{n+1}} (x_k - x_{k-1}) = X_n.$$

Then for  $k_n \leq k \leq k_{n+1}$ , we have  $x_k - x_{k-1} = (1 + \epsilon_{kn}) X_n$ , where  $\epsilon_{kn}$  is a positive quantity tending to zero as  $n \rightarrow \infty$ . Put

$$\epsilon_n = \max_{k_n \leq k \leq k_{n+1}} \epsilon_{kn},$$

and put  $\Delta x_k = x_k - x_{k-1}$ . Now if

$$x_{k_n+t} \leq z_{n-1} + \alpha(z_n - z_{n-1}) < x_{k_n+t+1},$$

then

$$\begin{aligned} \alpha(z_n - z_{n-1}) &= (x_{k_{n+1}} - z_{n-1}) + \sum_{k=k_n+2}^{k_n+t} \Delta x_k + (z_{n-1} + \alpha(z_n - z_{n-1}) - x_{k_n+t}) \\ &= \sum_{s=1}^t \Delta x_{k_n+s} + \epsilon'_n X_n, \end{aligned}$$

where  $\epsilon'_n = O(1)$  as  $n \rightarrow \infty$ . But

$$tX_n \leq \sum_{s=1}^t \Delta x_{k_n+s} \leq tX_n + t\epsilon_n X_n \leq tX_n + u\epsilon_n X_n,$$

where  $u = N(z_n) - N(z_{n-1})$ . Hence

$$\alpha \frac{z_n - z_{n-1}}{X_n} - \epsilon'_n - u\epsilon_n \leq t \leq \alpha \frac{z_n - z_{n-1}}{X_n} - \epsilon'_n.$$

Similarly,

$$\frac{z_n - z_{n-1}}{X_n} - \epsilon'_n - u\epsilon_n \leq u \leq \frac{z_n - z_{n-1}}{X_n} - \epsilon'_n,$$

so that

$$\frac{\alpha(z_n - z_{n-1})/X_n - \epsilon'_n - u\epsilon_n}{(z_n - z_{n-1})/X_n - \epsilon'_n} \leq \frac{t}{u} \leq \frac{(z_n - z_{n-1})/X_n - \epsilon'_n}{(z_n - z_{n-1})/X_n - \epsilon'_n - u\epsilon_n}.$$

Since  $N(z_n) - N(z_{n-1}) \rightarrow \infty$  as  $n \rightarrow \infty$ , also  $(z_n - z_{n-1})/X_n \rightarrow \infty$ , and so

$$\frac{\alpha + o(1) - u\epsilon_n X_n / (z_n - z_{n-1})}{1 + o(1)} \leq \frac{t}{u} \leq \frac{\alpha + o(1)}{1 + o(1) - u\epsilon_n X_n / (z_n - z_{n-1})}.$$

But since

$$uX_n \leq \sum_{k=k_n+1}^{k_{n+1}} \Delta x_k \leq z_n - z_{n-1},$$

$uX_n = O(z_n - z_{n-1})$ ; thus

$$\frac{\alpha + o(1)}{1 + o(1)} \leq \frac{t}{u} \leq \frac{\alpha + o(1)}{1 + o(1)},$$

and therefore

$$\frac{N(\alpha, \delta_n)}{N(\delta_n)} = \frac{t}{u} \sim \alpha.$$

This completes the proof.

In case  $\Delta = \Delta_0$  and  $x_k = f(k)$ , it is easily seen that the hypotheses of Fejér's theorem imply two of the hypotheses of Theorem 2, namely that  $N(z_n) -$

$N(z_{n-1}) \uparrow \infty$  and  $N(z_{n-1}) \sim N(z_n)$  as  $n \rightarrow \infty$ . But I do not know whether Theorem 2 includes Fejér's theorem; the most that I can show is that the exceptional sequence  $\{z_{n_t}\} = \{n_t\}$  mentioned in (ii) of Theorem 2 is in this case of density zero, which does not imply (1) for all functions  $f$  satisfying the hypotheses of Fejér's theorem. Certainly, however, Theorem 2 deals with cases not covered by the following direct extension of Fejér's theorem, since it does not require the monotonicity of either  $z_n - z_{n-1}$  or  $\Delta x_k$ .

**THEOREM 3.** *The sequence  $\{x_k\}$  is u.d. (mod  $\Delta$ ) if the following conditions are satisfied:*

- (i)  $z_n - z_{n-1} \geq z_{n-1} - z_{n-2}$  for  $n = 2, 3, \dots$ ,
- (ii)  $\Delta x_k \downarrow 0$  as  $k \uparrow \infty$ ,
- (iii)  $N(z_{n-1}) \sim N(z_n)$  as  $n \rightarrow \infty$ .

We sketch the proof. Let  $\psi$  be the continuous polygonal function such that  $\psi(x_k) = k$ ; then  $0 \leq \psi(x) - N(x) < 1$ . Let  $\{\epsilon_k\}$  be such that  $\epsilon_k = o(\Delta x_k)$  and  $0 < \epsilon_k < \Delta x_k/2$  for  $k = 1, 2, \dots$ . Define  $\psi_1$  as follows:

$$\psi_1(x) = \frac{1}{2\epsilon_k} \int_{x-\epsilon_k}^{x+\epsilon_k} \psi(t) dt \quad \text{for } x \in \left[ x - \frac{1}{2} \Delta x_k, x_k + \frac{1}{2} \Delta x_{k+1} \right]$$

$$(k = 2, 3, \dots).$$

Then  $\psi_1$  is continuously differentiable, and is identical with  $\psi$  except at the corners of  $\psi$ , where it is smooth. For  $0 \leq \alpha \leq 1$ ,  $n = 1, 2, 3, \dots$ , put

$$\rho(n + \alpha) = \psi_1(z_{n-1} + \alpha(z_n - z_{n-1}));$$

$\rho$  is continuously differentiable except at  $x = 1, 2, \dots$ . A function  $\rho_1$  can now be defined in terms of  $\rho$ , just as  $\psi_1$  was determined from  $\psi$ , so that  $\rho_1$  is everywhere continuously differentiable, and  $\rho_1$  differs from  $\rho$  only on an interval about  $x = n$  ( $n = 1, 2, \dots$ ) whose length  $\epsilon'_n$  is of lower order of magnitude than  $\Delta x_{k_n}$  if  $z_n \in [x_{k_{n-1}}, x_{k_n}]$ . If  $x = n + \alpha$  is such that

$$\rho_1(x) = \rho(x), \quad \psi_1(z_{n-1} + \alpha(z_n - z_{n-1})) = \psi(z_{n-1} + \alpha(z_n - z_{n-1})),$$

and

$$z_{n-1} + \alpha(z_n - z_{n-1}) \in (x_{k_{n-1}}, x_k),$$

then

$$\rho_1'(x) = \frac{z_n - z_{n-1}}{\Delta x_k};$$

it follows that  $\rho_1'(x) \nearrow \infty$ . Moreover, since

$$\frac{\rho_1(n+1)}{\rho_1(n)} \sim \frac{\psi(z_n)}{\psi(z_{n-1})} \sim \frac{N(z_n)}{N(z_{n-1})} \rightarrow 1,$$

it follows that  $\rho_1'(x)/\rho_1(x) \rightarrow 0$  as  $x \rightarrow \infty$ . But if  $f$  is the function inverse to  $\rho_1$ , these facts imply that  $f(x) \uparrow \infty$ ,  $f'(x) \searrow 0$ , and  $xf'(x) \rightarrow \infty$  as  $x \uparrow \infty$ . Since  $f(k) \rightarrow x_k$  as the arbitrary numbers  $\epsilon_k$  and  $\epsilon'_n$  approach zero, the conclusion follows from Fejér's theorem.

A trivial variation of Theorem 3 has, instead of (i) and (ii), the hypotheses

(i')  $z_n - z_{n-1} \uparrow \infty$ ,

(ii')  $\Delta x_{k-1} \geq \Delta x_k$  for  $k = 2, 3, \dots$ .

For then it will still be true that  $\rho_1'(x) \nearrow \infty$  as  $x \uparrow \infty$ .

4. It follows from Theorem 2 (and also from the variation of Theorem 3 just mentioned) that if  $z_n - z_{n-1} \nearrow \infty$  in such a way that  $z_{n-1} \sim z_n$ , the sequence  $\{k\theta\}$  is u.d. (mod  $\Delta$ ) for each  $\theta > 0$ . In this section we examine the distribution of  $\{k\theta\}$  (mod  $\Delta$ ) when  $\delta(x) \searrow 0$ . This is a problem of a very different kind from the earlier one; the result is expressed in the following metric theorem:

**THEOREM 4.** *If  $\delta(x) \searrow 0$  and  $\delta(x) = O(x^{-1})$  then  $\{k\theta\}$  is u.d. (mod  $\Delta$ ) for almost all  $\theta > 0$ .*

The proof depends on a principle used in an earlier paper [2]:

*If  $C$  and  $\epsilon$  are positive constants and  $\{f_k\}$  is a sequence of real-valued functions such that*

$$(3) \quad \left| \int_a^b e^{i(f_j(x) - f_k(x))} dx \right| \leq \frac{C}{\max(1, |j - k|^\epsilon)}, \quad (j, k = 1, 2, \dots),$$

*then  $\{f_k(x)\}$  is u.d. (mod 1) for almost all  $x \in (a, b)$ .*

This will be applied with  $f_k(x) = \phi(kx)$ , where  $\phi$  is the function defined in § 1; it was noted there that the u.d. (mod  $\Delta$ ) of  $\{x_k\}$  is equivalent to the u.d. (mod 1) of  $\{\phi(x_k)\}$ . Let  $a$  and  $b$  be arbitrary positive numbers with  $a < b$ , and put



$$J_{jk} = \int_a^b e^{i(f_j(x) - f_k(x))} dx;$$

since  $J_{kj}$  and  $J_{jk}$  are complex conjugates, it suffices to consider the case  $j > k$ . For fixed  $j$  and  $k$ , denote by  $\xi_0, \dots, \xi_r$  all the numbers of the form  $z_m/j$  or  $z_m/k$  in the interval  $(a, b)$ , so named that  $\xi_0 < \dots < \xi_r$ . Then the function

$$f_j(x) - f_k(x) = \left( \frac{j}{\delta(jx)} - \frac{k}{\delta(kx)} \right) x - \left( \frac{[jx]_{\Delta}}{\delta(jx)} - \frac{[kx]_{\Delta}}{\delta(kx)} \right) = xA(x) + B(x)$$

is linear in each interval  $[\xi_{l-1}, \xi_l]$ ,  $A(x)$  and  $B(x)$  being certain constants  $A_l$  and  $B_l$  there. Hence

$$J_{jk} = \sum_{l=1}^r \int_{\xi_{l-1}}^{\xi_l} e^{i(A_l x + B_l)} dx = \sum_{l=1}^r \frac{e^{i(A_l \xi_l + B_l)} - e^{i(A_l \xi_{l-1} + B_l)}}{iA_l}.$$

Since  $f$  is continuous,

$$A_l \xi_l + B_l = A_{l+1} \xi_l + B_{l+1},$$

and so for  $1 \leq t \leq r$ ,

$$\sum_{l=1}^t [e^{i(A_l \xi_l + B_l)} - e^{i(A_l \xi_{l-1} + B_l)}] = e^{i(A_t \xi_t + B_t)} - e^{i(A_1 \xi_0 + B_1)}$$

Thus, using the relation

$$\sum_{m=1}^n a_m b_m = \sum_{m=1}^{n-1} \left( \sum_{\mu=1}^m a_{\mu} \right) (b_m - b_{m+1}) + b_n \sum_{\mu=1}^n a_{\mu},$$

we have

$$J_{jk} = \frac{1}{i} \sum_{t=1}^{r-1} (e^{i(A_t \xi_t + B_t)} - e^{i(A_1 \xi_0 + B_1)}) \left( \frac{1}{A_t} - \frac{1}{A_{t+1}} \right) + (e^{i(A_r \xi_r + B_r)} - e^{i(A_1 \xi_0 + B_1)}) \frac{1}{iA_r},$$

and so

$$(4) \quad |J_{jk}| \leq 2 \sum_{t=1}^{r-1} \left| \frac{1}{A_t} - \frac{1}{A_{t+1}} \right| + \frac{2}{|A_r|}.$$

By the facts that  $\xi_t \geq a > 0$ ,  $\delta(x) \searrow 0$  as  $x \rightarrow \infty$ , and

$$A_t = \frac{j}{\delta(j\xi_{t-1})} - \frac{k}{\delta(k\xi_{t-1})},$$

it is clear that

$$A_t > C(j-k) > 0$$

for  $t = 1, 2, \dots, r$ , so that (3) will follow from (4) if it can be shown that for some  $c, \epsilon > 0$ , the inequality

$$\sum_{t=1}^{r-1} \left| \frac{1}{A_t} - \frac{1}{A_{t+1}} \right| < \frac{c}{(j-k)^\epsilon}$$

holds. Moreover, writing

$$C_t = \frac{1}{A_t} - \frac{1}{A_{t+1}}$$

and

$$\sum_{t=1}^{r-1} |C_t| = \sum_{t=1}^r C_t - 2 \sum' C_t = \frac{1}{A_1} - \frac{1}{A_r} - 2 \sum' C_t,$$

where  $\sum'$  is the sum over those  $t$  for which  $C_t < 0$ , we see that it suffices to show that

$$\sum' |C_t| < \frac{c}{(j-k)^\epsilon}.$$

We consider three cases. Suppose first that  $t$  is such that  $\xi_{t+1} = z_m/j$  for some  $m$ , but that for no  $l$  is  $\xi_{t+1} = z_l/k$ . Then

$$A_t = \frac{j}{\delta(z_{m-1})} - \frac{k}{\delta(k\xi_t)}, \quad A_{t+1} = \frac{j}{\delta(z_m)} - \frac{k}{\delta(k\xi_t)},$$

so that  $A_{t+1} \geq A_t$ , and the term  $C_t$  does not occur in  $\sum'$ . If

$$\xi_{t+1} = z_m/j = z_l/k,$$

then  $z_m > z_l$  and

$$\begin{aligned} C_t &= \frac{1}{j/\delta(z_{m-1}) - k/\delta(z_{l-1})} - \frac{1}{j/\delta(z_m) - k/\delta(z_l)} \\ &\geq \frac{-k(1/\delta(z_l) - 1/\delta(z_{l-1}))}{(j/\delta(z_{m-1}) - k/\delta(z_{l-1})) (j/\delta(z_m) - k/\delta(z_l))}. \end{aligned}$$

Finally, if  $\xi_{t+1} = z_l/k$  for some  $l$ , but  $\xi_{t+1} \neq z_m/j$  for every  $m$ , then

$$C_t = \frac{-k(1/\delta(z_l) - 1/\delta(z_{l-1}))}{(j/\delta(j\xi_{t+1}) - k/\delta(z_{l-1})) (j/\delta(j\xi_{t+1}) - k/\delta(z_l))}.$$

Thus, writing  $\delta(x^+)$  and  $\delta(x^-)$  for  $\lim_{\xi \rightarrow x^+} \delta(\xi)$  and  $\lim_{\xi \rightarrow x^-} \delta(\xi)$ , we have

$$\begin{aligned} \sum' |C_t| &\leq k \sum'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j/\delta(j\xi_{t+1}^-) - k/\delta(z_{l-1})) (j/\delta(j\xi_{t+1}^+) - k/\delta(z_l))} \\ &= \sum'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j/\delta(jz_l^-/k) - k/\delta(z_{l-1})) (j/\delta(jz_l^+/k) - k/\delta(z_l))}, \end{aligned}$$

where  $\sum''$  denotes summation with respect to  $l$  with  $z_l/k \in (a, b)$ . But

$$\delta(jz_l^-/k) \leq \delta(z_{l-1})$$

and

$$\delta(jz_l^+/k) \leq \delta(z_l),$$

and so

$$\begin{aligned} \sum' |C_t| &\leq k \sum'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j-k)^2/\delta(z_{l-1})\delta(z_l)} \\ &= \frac{k}{(j-k)^2} \sum'' \{\delta(z_{l-1}) - \delta(z_l)\} \leq \frac{2k \delta(ka)}{(j-k)^2}. \end{aligned}$$

If now  $\delta(x) = O(1/x)$ , then

$$\sum' |C_t| = O\left(\frac{1}{(j-k)^2}\right),$$

and the proof is complete.

5. The preceding result can be generalized considerably by using the following transfer theorem:

**THEOREM 5.** *Suppose that  $\{x_k\}$  is u.d. (mod  $\Delta$ ), where  $\Delta = \{z_n\}$ , and that  $f$  is a function which is differentiable except possibly at the points  $z_1, z_2, \dots$ , such that  $f(x) \uparrow \infty$  as  $x \uparrow \infty$  and*

$$(5) \quad \inf_{x \in (z_{n-1}, z_n)} f'(x) \sim \sup_{x \in (z_{n-1}, z_n)} f'(x).$$

Then the sequence  $\{x_k^*\} = \{f(x_k)\}$  is u.d. (mod  $\Delta^*$ ), where  $\Delta^* = \{f(z_n)\}$ .

Put

$$N(\alpha, x) = \sum 1, N(1, x) = N(x), N^*(\alpha, x) = \sum^* 1, N^*(1, x) = N^*(x),$$

where  $\sum$  denotes summation with  $x_k \leq x$  and  $\langle x_k \rangle_\Delta < \alpha$  and  $\sum^*$  denotes summation with  $x_k^* \leq x$ ,  $\langle x_k^* \rangle_{\Delta^*} < \alpha$ . Since  $f$  is an increasing function,

$$N^*(f(x)) = \sum_{f(x_k) \leq f(x)} 1 = \sum_{x_k \leq x} 1 = N(x).$$

By assumption, the relation

$$\lim_{x \rightarrow \infty} \frac{N(\alpha, x)}{N(x)} = \alpha$$

holds for  $\alpha \in [0, 1]$ . So we need only show that  $N^*(\alpha, f(x)) \sim N(\alpha, x)$  as  $x \rightarrow \infty$ , and by Theorem 1 it suffices to prove this as  $x$  runs through the sequence  $\{z_n\}$ . But

$$N(\alpha, z_n) = \sum_{m=1}^n \{N(z_{m-1} + \alpha(z_m - z_{m-1})) - N(z_{m-1})\},$$

and so

$$\begin{aligned} N^*(\alpha, f(z_n)) &= \sum_{m=1}^n \{N^*(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) - N^*(z_{m-1}^*)\} \\ &= N(\alpha, z_n) + \sum_{m=1}^n \{N^*(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\}. \end{aligned}$$

Thus the problem reduces to showing that

$$\sum_{m=1}^n \{N^*(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\} = o(N(\alpha, z_n)),$$

or what is the same thing, that

$$(6) \sum_{m=1}^n \{N(f^{-1}(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*))) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\} = o(N(z_n)).$$

Put

$$\begin{aligned} f^{-1}(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) &= u_m(\alpha), \\ z_{m-1} + \alpha(z_m - z_{m-1}) &= v_m(\alpha). \end{aligned}$$

If it can be shown that

$$(7) \quad |u_m(\alpha) - v_m(\alpha)| < \epsilon_m(z_m - z_{m-1}),$$

where  $\epsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ , then for every  $\epsilon > 0$ ,

$$\begin{aligned} &\sum_{m=1}^n \{N(u_m(\alpha)) - N(v_m(\alpha))\} \\ &= O\left(\sum_{m=1}^n \{N(v_m(\alpha) + \epsilon(z_m - z_{m-1})) - N(v_m(\alpha))\}\right) \\ &= O(N(\epsilon, z_n)) = O(\epsilon N(z_n)), \end{aligned}$$

which implies (6).

Now

$$u_m(0) = v_m(0), u_m(1) = v_m(1),$$

and

$$\begin{aligned} u_m(\alpha) - v_m(\alpha) &= f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1}))) \\ &\quad - (z_{m-1} + \alpha(z_m - z_{m-1})); \end{aligned}$$

hence

$$u'_m(\alpha) - v'_m(\alpha) = \frac{f(z_m) - f(z_{m-1})}{f'\{f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1})))\}} - (z_m - z_{m-1}).$$

To maximize  $u_m(\alpha) - v_m(\alpha)$ , we must have

$$f(z_m) - f(z_{m-1}) - (z_m - z_{m-1}) f'\{f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1})))\} = 0.$$

There is a  $Z_0 \in (z_{m-1}, z_m)$  such that

$$\frac{f(z_m) - f(z_{m-1})}{z_m - z_{m-1}} = f'(Z_0),$$

and a corresponding  $\alpha_0 \in (0, 1)$  such that

$$f(z_{m-1}) + \alpha_0(f(z_m) - f(z_{m-1})) = f(Z_0),$$

(so that  $u'_m(\alpha_0) - v'_m(\alpha_0) = 0$ ) for which

$$|u_m(\alpha) - v_m(\alpha)| \leq |u_m(\alpha_0) - v_m(\alpha_0)| = |Z_0 - v_m(\alpha_0)|$$

for all  $\alpha \in (0, 1)$ . But

$$\begin{aligned} v_m(\alpha_0) &= z_{m-1} + \frac{f(Z_0) - f(z_{m-1})}{f(z_m) - f(z_{m-1})} (z_m - z_{m-1}) \\ &= z_{m-1} + \frac{f(Z_0) - f(z_{m-1})}{f'(Z_0)}, \end{aligned}$$

so that

$$Z_0 - v_m(\alpha_0) = Z_0 - z_{m-1} - \frac{f(Z_0) - f(z_{m-1})}{f'(Z_0)}$$

and

$$|u_m(\alpha) - v_m(\alpha)| \leq \sup_{Z \in \delta_m} \left( |Z - z_{m-1}| \left| 1 - \frac{f(Z) - f(z_{m-1})}{(Z - z_{m-1}) f'(Z)} \right| \right),$$

whence

$$\left| \frac{u_m(\alpha) - v_m(\alpha)}{z_m - z_{m-1}} \right| \leq \sup_{\substack{Z \in \delta_m \\ W \in \delta_m}} \left| 1 - \frac{f'(W)}{f'(Z)} \right|,$$

and this last upper bound is  $o(1)$  as  $m \rightarrow \infty$ . Thus (7) holds, and the proof is complete.

If the  $f$  of Theorem 5 is taken to be an arbitrary increasing polygonal function, with vertices on the abscissas  $x = z_1, z_2, \dots$ , then the condition (5) on the derivative is trivially satisfied. Such a transformation merely represents a change of scale inside each interval  $\delta_n$ , and the distribution modulo  $\Delta$  of any sequence  $\{x_k\}$  is identical with the distribution of  $\{f(x_k)\}$  modulo  $\Delta^*$ .

In case  $f'$  is monotone, (5) can be replaced by the simpler condition

$$(5') \quad f'(z_{n-1}) \sim f'(z_n) \quad \text{as } n \rightarrow \infty.$$

Combining this version of Theorem 5 with Theorem 4, we have:

**THEOREM 6.** *The sequence  $\{f(k\theta)\}$  is u.d. (mod  $\Delta$ ) for almost all  $\theta > 0$  if  $f(x) \uparrow \infty$ ,  $f'$  is monotonic, and*

$$\begin{aligned} f^{-1}(z_n) - f^{-1}(z_{n-1}) &\searrow 0, \\ f^{-1}(z_n) - f^{-1}(z_{n-1}) &= O\left(\frac{1}{f^{-1}(z_n)}\right), \\ f'(f^{-1}(z_n)) &\sim f'(f^{-1}(z_{n-1})), \end{aligned}$$

where  $f^{-1}$  is the function inverse to  $f$ .

**COROLLARY.** *The sequence  $\{\alpha^k\}$  is u.d. (mod  $\Delta$ ) for almost all  $\alpha > 1$  if  $z_n = g(n)$ , where  $g$  is an increasing function with monotonic logarithmic derivative such that*

$$(8) \quad \frac{g'(x)}{g(x)} = O(x^{-1/2}).$$

For writing  $\alpha^k$  as  $e^{k \log \alpha}$ , we see that we can take the  $f$  of Theorem 6 to be the exponential function, and the conditions displayed there become

$$\begin{aligned} \log z_n - \log z_{n-1} &\searrow 0, \\ \log z_n - \log z_{n-1} &= O\left(\frac{1}{\log z_n}\right), \end{aligned}$$

$$z_n \sim z_{n-1}.$$

Of these, the third is implied by the first. Since

$$\frac{d}{dx} \log g(x) \searrow 0,$$

it is clear that  $\log g(n) - \log g(n-1) \searrow 0$ . From the extended law of the mean,

$$\frac{G(x) - G(x-1)}{H(x) - H(x-1)} = \frac{G'(X)}{H'(X)}, \quad X \in (x-1, x),$$

it follows that if  $G'(x) = O(H'(x))$ , then

$$G(x) - G(x-1) = O(H(x) - H(x-1)).$$

Taking

$$G(x) = \log g(x), \quad H(x) = \log e^{\sqrt{x}} = \sqrt{x},$$

we have by (8) that

$$\log g(n) - \log g(n-1) = O(n^{-1/2}).$$

But it also follows from the relation  $G'(x) = O(H'(x))$  that  $G(x) = O(H(x))$ ; hence

$$\log g(x) = O(x^{1/2}), \quad n^{-1/2} = O((\log g(n))^{-1}),$$

and the proof is complete.

For sufficiently smooth  $g$ , (8) can be replaced by the condition  $g(x) = O(\exp \sqrt{x})$ .

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