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1. Introduction. In a recent paper, J. Dugundji proved [**11**, Th. 4.1] that every convex subset Y of a locally convex topological linear space has the following property:

(1) If X is a metric space, A a closed subset of X, and f a continuous function from A into Y, then f can be extended to a continuous function from X into Y.

Let us call a topological space Y which has property (1) an *absolute extensor for metric spaces*, and let *absolute extensor for normal* (*or paracompact, etc.*) *spaces* be defined analogously. According to Dugundji's theorem above, the supply of spaces which are absolute extensors for metric spaces is quite substantial, and it becomes reasonable to ask the following question:

(2) Suppose that Y is an absolute extensor for metric spaces. Under what conditions is it also an absolute extensor for normal (or paracompact, etc.) spaces?

Most of this paper (§§2-6) will be devoted to answering this question and related questions. The related questions arise in connection with the concepts of absolute retract, absolute neighborhood retract, and absolute neighborhood extensor (in §2 these are all defined and their interrelations and significance explained), and it is both convenient and natural to answer all the questions simultaneously. Assuming that the space Y of (2) is metrizable, we are able to answer these questions completely (thereby solving some heretofore unsolved problems of Arens [2, p. 19] and Hu [**18**]) in Theorems 3.1 and 3.2 of §3; §§4 and 5 are devoted to proving these theorems. In §6 we show by an example that things can go completely awry if Y is not assumed to be metrizable.

Our final section (§7), which is also based on Dugundji's [**11**, Th. 4.1], deals with *simultaneous* extensions of continuous functions. It is entirely independent of §§2-6, and is the only part of this paper which might interest those readers who are interested only in metric spaces.

We conclude this introduction with a summary of some of the less familiar

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or possibly ambiguous terms used in this paper. All our *normal* spaces are assumed to be Hausdorff. A *perfectly normal* space is a normal space in which every closed subset is a G_δ (i.e., the intersection of countably many open sets). A covering $\mathfrak{V}$ of *a* topological space X is called *locally finite* [**10**, p. 66] if every x in X has a neighborhood which intersects only finitely many $V \in \mathfrak{V}$. A topological space X is *paracompact* [**10**, p. 66] if it is Hausdorff, and if to every open covering $\mathfrak{U}$ of X there corresponds a locally finite open covering $\mathfrak{V}$ of X such that every $V \in \mathfrak{V}$ is a subset of some $U \in \mathfrak{U}$. (Every paracompact space is normal [**10**, Th. 1], every metric space is paracompact [**22**, Cor. 1], and a Hausdorff space is paracompact if and only if it is fully normal [**22**, Th. 1 and Th. 2].) A metrizable space is *topologically complete* if it can be given a complete metric which agrees with the topology. A topological space is *σ-compact* if it is the union of countably many compact subsets.

2. Definitions and interrelations. Let us begin this section by formally defining the concepts which were mentioned in the introduction, and which will be the objects of investigation of most of this paper. For convenience, we will use the following abbreviations:

$$\begin{aligned} AE &= \text{absolute extensor} \\ ANE &= \text{absolute neighborhood extensor} \\ AR &= \text{absolute retract} \\ ANR &= \text{absolute neighborhood retract} \end{aligned}$$

DEFINITION 2.1. A topological space Y is called an AE (resp. ANE) *for metric spaces* if, whenever X is a metric space and A is a closed subset of X, then any continuous function from A into Y can be extended to a continuous function from X (resp. some neighborhood of A in X) into Y. Similarly if "metric" is replaced by the name of some other kind of space in the above.

DEFINITION 2.2. A topological space Y is called an AR (resp. ANR) *for metric spaces* if, whenever Y is a closed subset of a metric space X, there exists a continuous function from X (resp. some neighborhood of Y in X) onto Y which keeps Y pointwise fixed. Similarly if "metric" is replaced by the name of some other kind of space in the above.

REMARK. Observe that if Y is an AE (resp. ANE) for a certain class of spaces, then Y is *a fortiori* an AR (resp. ANR) for this class of spaces.

The concepts defined in Definition 2.2 are essentially due to Borsuk [**4** and **5**], who proved [**5**, p. 227] that every finite simplicial complex is an *ANR* for compact metric spaces; this was, in fact, Borsuk's motive for introducing *ANR*'s. More recently, Hanner [**17**] generalized that result by showing that every locally finite simplicial complex is an *ANR* for separable metric spaces. Finally this result was generalized still further by Dugundji [**12**, Th. 5.2], who proved that every simplicial complex with J.H.C. Whitehead's *CW* topology is an *ANE* for metric spaces.

The following propositions summarize the known relations between the various concepts defined above. Propositions 1 and 3 are due to Hu [**18**], and parts of Proposition 2 are essentially due to Dugundji [**11**] and Hanner [**16**].

PROPOSITION 2.3 (Hu). *Let Y be a separable metric space. Then Y is an AR* (resp. *ANR*) *for metric spaces if and only if Y is an AR* (resp. *ANR*) *for separable metric spaces.*

Proof. This follows at once from [18, Th. 3.1].

PROPOSITION 2.4. *Let Y be a metric space. Then Y is an AR* (resp. *ARN*) *for metric spaces if and only if Y is an AE* (resp. *ANE*) *for metric spaces. This assertion remains true if "metric" is everywhere replaced by "paracompact", or "normal", or "perfectly normal".*

Proof. The "if" assertions are clear (see the Remark after Definition 2.2), so let us turn to the "only if" assertions. Here the metric case was proved by Dugundji [**11**, Th. 7.1]; to prove the results in the other cases, we shall use the method employed by Hanner in his proof of the normal case [**16**, Th. 3.1 and Th. 3.2].

Let X and Y be topological spaces, A a closed subset of X, and $f: A \longrightarrow Y$ a continuous function. Let $X \cup Y$ denote the disjoint union of X and Y, and let Z be the identification space which we get from $X \cup Y$ by identifying $x \in A$ with $f(x) \in Y$. To prove our results, it is sufficient, as in Hanner's proof of the normal case, to show that if X and Y are both paracompact (resp. normal, perfectly normal), then so is Z. For normal spaces this was proved by Hanner [**16**, Lem. 3.3], and for perfectly normal spaces the proof is almost the same as that for normal spaces; this leaves paracompact spaces, where our proof depends on the following two facts. The first of these is a characterization of paracompact spaces which the author will prove in another paper, and the second is an immediate consequence of the first.

(1) *If Z is a T_1-space, then Z is paracompact if and only if it has the following property: If E is a Banach space, and if $\widetilde{u}$ is a l.s.c.*[1] *function from Z to the space $\mathcal{C}(E)$ of nonempty, closed, convex subsets of E, then there exists a continuous $u : Z \longrightarrow E$ such that $u(z) \in \widetilde{u}(z)$ for every z in Z.*

(2) *Let X be a paracompact space, E a Banach space, $\widetilde{w} : X \longrightarrow C(E)$ a l.s.c. function, and A a closed subset of X. Then any continuous $v : A \longrightarrow E$ such that $v(x) \in \widetilde{w}(x)$ for every x in A can be extended to a continuous $w : X \longrightarrow E$ such that $w(x) \in \widetilde{w}(x)$ for every x in X.*

We shall also need the following elementary facts about Z. Let g be the natural mapping from $X \cup Y$ onto Z, and denote $g|X$ by h and $g|Y$ by k; also denote $k(Y)$ by Y'. As observed by Hanner, k is a homeomorphism onto Y', and $h|X - A$ is a homeomorphism onto $Z - Y'$. It follows that a function u with domain Z is continuous if and only if $u|Y'$ and uh are both continuous.

Suppose now that X and Y are paracompact, and let us prove that Z is also paracompact. Since Z is certainly T_1, we need only show that Z has the property in (1). Suppose, therefore, that E is a Banach space, and $\widetilde{u} : Z \longrightarrow \mathcal{C}(E)$ a l.s.c. function; we must find a continuous $u : Z \longrightarrow E$ such that $u(z) \in \widetilde{u}(z)$ for every z in Z. Now Y' is paracompact, and $\widetilde{u}|Y'$ is l.s.c.; hence, by (1), there exists a continuous $r : Y' \longrightarrow E$ such that $r(z) \in \widetilde{u}(z)$ for every z in Y'. Let $\widetilde{w} = \widetilde{u}h$, let $h' = h|A$, and let $v = rh'$; then X, A, $\widetilde{w}$, and v satisfy the assumptions of (2), and hence, by (2), v can be extended to a continuous $w : X \longrightarrow E$ such that $w(x) \in \widetilde{w}(x)$ for every x in X. Now define $u : Z \longrightarrow E$ by "$u(z) = r(z)$ if $z \in Y'$, and $u(z) = wh^{-1}(z)$ if $z \in Z - Y'$". Clearly $u(z) \in \widetilde{u}(z)$ for every z in Z, and u is continuous, since $u|Y' = r$ and $uh = w$ are both continuous. This completes the proof.

Finally, let us mention the following result of Hu [**18**, Th. 3.2].

PROPOSITION 2.5. (Hu) *If Y is a completely regular space which is an AR* (resp. *ANR*) *for completely regular spaces, then Y is an AE* (resp. *ANE*) *for normal spaces.*

Having just covered the similarities between extensors and retracts, let us end this section with some comments about their differences. These differences occur in two ways:

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[1] A function $\widetilde{u}$ from a topological space Z to the space of nonempty subsets of a topological space E is called *l.s.c.* (= *lower semi-continuous*) if, whenever U is an open subset of E, then $\{ z \in Z \mid \widetilde{u}(z) \cap U \neq \phi \}$ is open in Z.

(a) If Y is *not* a metric (resp. paracompact, etc.) space, then Y is (vacuously!) *always* an AR and an ANR for metric (resp. paracompact, etc.) spaces. But Y need by no means always be an AE or an ANE for metric (resp. paracompact, etc.) spaces, and when it is, this is a fact which cannot be restated in terms of retracts. As examples, we mention the theorems of Dugundji [**11**, Th. 4.1] and [**12**, p. 9] which we have encountered earlier in this paper.

(b) If Y is completely regular and has more than one point, then it is easy to see that Y cannot be an AE or ANE for any class of spaces which contains a nonnormal space. But such a Y may very well be an AR or ANR for completely regular spaces (see Theorem 3.1 (e) and Theorem 3.2 (e)).

3. The theorems. We will now state the theorems answering question (2) of the introduction.

THEOREM 3.1. *Let Y be a metrizable space which is an AE* (resp. ANE) *for metric spaces. Then:*

(a) *Y is an AE* (resp. ANE) *for spaces which are paracompact and perfectly normal.*

(b) *Y is an AE* (resp. ANE) *for paracompact spaces if and only if Y is topologically complete.*

(c) *Y is an AE* (resp. ANE) *for perfectly normal spaces if and only if Y is separable.*

(d) *Y is an AE* (resp. ANE) *for normal spaces if and only if Y is separable and topologically complete.*

(e) *Y is an AE* (resp. ANE) *for completely regular spaces if and only if Y has at most one point.*

THEOREM 3.2. *Let Y be a metrizable space which is an AR* (resp. ANR) *for metric spaces. Then:*

(a) *Y is an AR* (resp. ANR) *for paracompact spaces containing Y as a G_δ.*

(b) *Y is an AR* (resp. ANR) *for paracompact spaces if and only if Y is topologically complete.*

(c) *Y is an AR* (resp. ANR) *for perfectly normal spaces if and only if Y is separable.*

(d) *Y is an AR* (resp. ANR) *for normal spaces if and only if Y is separable and topologically complete.*

(e) *Y is an AR* (resp. *ANR*) *for completely regular spaces if and only if Y is compact* (resp. *locally compact and separable*).

The foregoing theorems make a rather formidable array of statements, but because of their interdependence we will not have to prove all of them separately. In fact, we will prove only the following assertions (whose labeling is self-explanatory):

(*) 1(a), 2(a), 1(b) "if", 2(b) "only if", 1(c), 1(d) "if", 2(e).

Let us show that these assertions imply all the others. To begin with, the assumptions on Y made at the beginning of Theorems 3.1 and 3.2 are equivalent, by Proposition 2.2. We therefore have the following implications:

1(b) "if" $\Longrightarrow$ 2(b) "if": by Remark after Definition 2.

2(b) "only if" $\Longrightarrow$ 1(b) "only if": by Remark after Definition 2.

1(c) $\Longrightarrow$ 2(c): by Proposition 2.

1(b) "only if" and 1(c) "only if" $\Longrightarrow$ 1(d) "only if": obvious.

1(d) $\Longrightarrow$ 2(d): by Proposition 2.

$\Longrightarrow$ 1(e): this follows from the definitions.

These implications, together with the assertions (*) which we are going to prove, cover all the assertions of both theorems.

Before concluding this section, let us comment on the novelty and significance of the results (*). First of all, 1(d) "if" has been proved by Hanner [**16**, Th. 4.1 and Th. 4.2], and 2(e) "if" (*AR*) has been observed by Hu [**18**]; our proofs of these results are short, and we include them for the sake of unity of approach. Result 1(b) "if" follows easily from Arens' [**2**, Th. 4.1] by means of a technique due to Dugundji [**11**]. Results 1(a), 2(a), 2(b) "only if", and 1(c) "if" are proved by minor variations of techniques due to Hanner [**16**]. This leaves 1(c) "only if", 2(e) "only if", and 2(e) "if" (ANR) as the only results with some claim to originality; among these, 1(c) "only if" solves a problem of Arens [**2**, p. 19], and the others solve some problems of Hu [**18**].

In the next section we will prove 1(a), 2(a), and the "if" parts of the other (*) assertions; in the section after that we will prove the "only if" parts. The lemmas and propositions in these sections have some independent interest, and are sometimes stated with greater generality than is needed in their application.

4. Proofs of sufficiency. Assertions 1(a), 2(a), 1(b) "if", 1(c) "if", and 1(d) "if" will be proved after Lemma 4.3. Assertion 2(e) "if" will be proved after Lemma 4.6.

In the following lemmas, $R^{\aleph_0}$ will denote a countably infinite cartesian product of real lines.

LEMMA 4.1. *Every (complete) metric space can be embedded homeomorphically as a (closed) subset in a Banach space. Every (complete) separable metric space can be embedded homeomorphically as a (closed) subset in* $R^{\aleph_0}$.

Proof. It is well known (see, for instance [20]) that every metric space can be embedded *isometrically* in a Banach space, and the first sentence follows from this fact. It is also well known [19, p. 104] that every separable metric space X can be embedded in $R^{\aleph_0}$, which proves the second sentence with parenthetical words omitted. If X is moreover complete, then it is a G_δ in $R^{\aleph_0}$ [19, p. 215]. By [19, p. 151], X is therefore homeomorphic to a closed subset of $R^{\aleph_0} \times R^{\aleph_0}$, and the latter space is homeomorphic to $R^{\aleph_0}$. This completes the proof.

The proof of the following lemma uses an idea which the author found in Hanner [16] who in turn ascribes it to Fox [13].

LEMMA 4.2. *Let X be a normal space, A a closed G_δ in X, and g a continuous function from A into a metric space E. Then there exists a metric space F containing $g(A)$ as a closed subset, and a continuous function h from X into F which agrees with g on A.*

Proof. Let $G = E \times I$, where I is the closed unit interval, and identify E with $E \times \{0\} \subset G$. Let $F = G - (E - g(A))$. Since A is a closed G_δ in the normal space X, there exists a continuous function ϕ from X into the nonnegative real numbers, which is zero exactly on A. Finally we define $h: X \longrightarrow F$ by $h(x) = (g(x), \phi(x))$, and we see that F and h satisfy all our requirements.

LEMMA 4.3. *Let X be a topological space, A a closed subset of X, M a metric space, and f a continuous function from A into M. Suppose either that M is a complete metric space, or that A is a G_δ in X. Suppose also either that X is paracompact, or that X is normal and M separable. Then there exists a metric space F containing M as a closed subset, and a continuous function from X into F which agrees with f on A.*

Proof. If X is paracompact, embed M in a Banach space E according to

Lemma 4.1. By [2, Th. 4.1] we may extend f to a continuous function g from X into E. If M is complete, then we may suppose that M is closed in E, and we are through. If A is a G_δ in X, we need only apply Lemma 4.2.

If X is normal and M separable, embed M in $R^{\aleph_0}$ according to Lemma 4.1. The proof now proceeds exactly as above, except that we use the Urysohn-Tietze extension theorem instead of [2, Th. 4.1]. This completes the proof.

Proof of 1(a), 1(b) *"if"*, 1(c) *"if"*, *and* 1(d) *"if"*. These all follow almost immediately from Lemma 4.3.

Our next two lemmas deal with locally compact spaces, and are stated without proof. The crux of Lemma 4.4 is essentially stated as an exercise in [6] and proved in [8]; the first proof which the author saw was due to J. Tits.

LEMMA 4.4. *The following properties of a Hausdorff space X are equivalent:*

a) *X is locally compact.*

b) *If X is a dense subset of a Hausdorff space Y, then X is open in Y.*

c) *If X is a subset of a Hausdorff space Y, then $X = U \cap C$, where U is open in Y, and C is closed in Y.*

LEMMA 4.5. *Let X be a locally compact space, and A a σ-compact subset of X. Then there exists an open, σ-compact subset Z of X which contains A.*

One part of the following lemma is trivial, while the other part is not; we state them together to emphasize the parallelism.

LEMMA 4.6. *Let X be a completely regular space, and A a compact* (resp. *locally compact and σ-compact*) *subset of X. Then X* (resp. *some neighborhood V of A in X*) *can be embedded in a compact* (resp. *locally compact and σ-compact*) *Hausdorff space Z such that A is closed in Z.*

Proof. The assertion where A is compact is trivial. To prove the other assertion, let Y be any compact Hausdorff space containing X. By Lemma 4.4, there exists an open subset U of Y such that A is a subset of U which is closed relative to U. Since U is open in Y, it is locally compact. Hence, by Lemma 4.5, there exists an open, σ-compact subset Z of U which contains A. Since Z is open in U, Z is locally compact. Letting $V = Z \cap X$, we see that Z and V satisfy our requirements. This completes the proof.

Proof of 2(e) *"if"*. This follows easily from Lemma 4.6 as follows: (We will prove the part about *ANR*; the part about *AR* is even easier). Let Y be a locally compact, separable metric space which is an *ANR* for metric spaces, and let Y be a closed subset of the completely regular space X. We must find a neighborhood U of Y in X, and a continuous function g from U to Y which is the identity on Y.

Since Y is a locally compact, separable metric space, it is σ-compact. Hence, by Lemma 4.6, some neighborhood V of Y in X can be embedded in a locally compact and σ-compact Hausdorff space Z such that A is closed in Z. By [**10**, Th. 3], Z is paracompact. Since Y is a locally compact metric space, it is topologically complete (for instance by [**19**, p. 200] and Lemma 4.4). Hence, by Theorem 3.2(b), there exists a continuous function g from some neighborhood W of Y in Z to Y such that g is the identity on Y. Letting $U = W \cap V$, and $f = g \mid U$, we see that all our requirements are satisfied. This completes the proof.

5. Proofs of necessity. We start this section with the proof of 2(b) "only if". We will prove 1(c) "only if" after Proposition 5.1, and 2(e) "only if" after Proposition 5.3.

Proof of 2(b) *"only if"*. If "paracompact" were replaced by "normal" in this assertion, and "metric" by "separable metric", then the assertion would be contained in [**16**, Th. 4.1 and Th. 4.2]. To prove our assertion as it stands, we need only modify the proof of [**16**, Th. 4.2]. We therefore invite the reader to look at Hanner's proof of [**16**, Th. 4.2], and we will now point out the necessary modification.

Instead of embedding X (this is the space in [**16**] which corresponds to our Y) in the Hilbert cube I_ω (which can only be done if X is separable), we embed X in an arbitrary complete metric space M, and this space M will take the place of I_ω throughout the proof. With that in mind, we now define Z just as Hanner does, and the crux of the matter is that we must show Z to be paracompact (Hanner only shows that Z is normal). Once this is accomplished, the remainder of Hanner's proof goes through unchanged (except that I_ω is replaced by M) to show that X is a G_δ in M. But this implies [**19**, p. 200] that X is topologically complete, and our proof will therefore be complete.

We will use the notation of Hanner's proof (except that M replaces I_ω). Let $\{U_\alpha\}$ be a covering of Z by open sets. Then, for each α, there exists an open set O_α in M, and a subset A_α of $Z - X'$, such that

$$U_\alpha = h^{-1}(O_\alpha) \cup A_\alpha.$$

Let $O = \bigcup_\alpha O_\alpha$. Since O is a metric space (and therefore paracompact [**22**, Cor. 1]), and since $\{O_\alpha\}$ is a covering of O by open sets, $\{O_\alpha\}$ has a locally finite refinement $\{V_\beta\}$. Since each V_β is open in O, and since O is open in M, it follows that each V_β is open in M. Now let $\mathfrak{W}$ be the covering of Z whose elements are the sets $h^{-1}(V_\beta)$ and the one-point sets corresponding to the points of $Z - h^{-1}(O)$. Let us show that $\mathfrak{W}$ is a locally finite refinement of $\{U_\alpha\}$: It is clear that $\mathfrak{W}$ is a covering of Z by open sets, and that $\mathfrak{W}$ is a refinement of $\{U_\alpha\}$, so we need only show that $\mathfrak{W}$ is locally finite. If $x \in Z - h^{-1}(O)$, then $\{x\}$ is certainly a neighborhood of x which intersects only finitely many elements of $\mathfrak{W}$. If $x \in h^{-1}(O)$, then there exists an open subset S_x of O such that $h(x) \in S_x$, and such that S_x intersects only finitely many elements of $\{V_\beta\}$. But then $h^{-1}(S_x)$ is an open subset of Z which contains x, and which intersects only finitely many elements of $\mathfrak{W}$. This completes the proof.

The following proposition is more general than 1(c) "only if".

PROPOSITION 5.1. *If Y is a topological space which is an ANE for normal spaces, then every disjoint collection of open subsets of Y is countable.*

Proof. Suppose that there exists a disjoint collection $\mathfrak{U}$ of nonempty open subsets of Y which is uncountable. Then there exists a subset B of Y which contains exactly one point from every element of $\mathfrak{U}$; clearly B is a discrete space in the relative topology. Now by [3, Ex. H] there exists a perfectly normal space X, and a discrete, closed subset A of X which is homeomorphic to B, such that no collection of open subsets of X separates[2] A. Let f be the homeomorphism from A onto B. By assumption, f can be extended to a continuous function g from some open neighborhood V of A in X into Y. But now the collection of all $V \cap g^{-1}(U)$, with $U \in \mathfrak{U}$, is a collection of open subsets of X which separates A. This is a contradiction, and thus the proof is complete.

Proof of 1(c) *"only if"*. This now follows immediately from Proposition 5.1, since for metric spaces the property of Y in Proposition 5.1 is equivalent to separability [21, p. 130].

LEMMA 5.2. *Let ξ be an uncountable ordinal, and let Q be the space of ordinals $\leq \xi$, in the order topology. For each α in Q, let*

$$Q_\alpha = \{q \in Q \mid q \geq \alpha\},$$

with the relative topology induced by Q. Also let X be a subset of the cartesian

[2] If Y is a topological space, and B a subset of Y, then a collection $\mathfrak{U}$ of open subsets of Y *separates* B if $\mathfrak{U}$ is a disjoint collection, and if each $U \subset \mathfrak{U}$ contains exactly one element of B.

product of $\aleph$ *copies of the real line, where* $\aleph$ *is a cardinal which is less than the cardinality of* ξ. Then:

a) *If* $\alpha < \xi$, *and* f *is a continuous function from* Q_α *into* X, *then there exists a* β *in* Q *such that* $\alpha \leq \beta < \xi$, *and such that* $f(q) = f(\xi)$ *for all* $q \geq \beta$.

b) *If* U *is a neighborhood of* $\{\xi\} \times X$ *in* $Q \times X$, *then there exists an ordinal* $\alpha < \xi$ *such that* $Q_\alpha \times X \subset U$.

Proof. a) If $\aleph = 1$, then this is proved exactly like the assertion in the middle of page 836 of [9]. In the general case, let $X \subset \Pi_{\iota \in I} R_\iota$, where I is an index set of cardinality $\aleph$ and R_ι is the real line for every $\iota \in I$, and for every $\iota \in I$ let π_ι be the projection from X into R_ι. Letting $f_\iota = f \circ \pi_\iota$ for every $\iota \in I$, we have, by the first sentence of this proof, an indexed family $\{\beta_\iota\}_{\iota \in I}$ of ordinals in Q such that $f_\iota(q) = f_\iota(\xi)$ whenever $q \geq \beta_\iota$. Letting β be the smallest ordinal which is larger than all the β_ι, we see that β satisfies all our requirements.

b) The assumptions on X imply that X has a basis of cardinality $\leq \aleph$, and hence every covering of X by open sets has a subcovering of cardinality $\leq \aleph$. Now for each x in X, we can find an α_x in Q and an open neighborhood V_x of x in X such that $Q_{\alpha_x} \times V_x \subset U$. Thus $\{V_x\}_{x \in X}$ is a covering of X by open sets, and hence there exists a subcovering $\{V_x\}_{x \in X'}$, where X' has cardinality $\leq \aleph$. If now α is the smallest ordinal which is larger than all the α_x with $x \in X'$, then α satisfies all our requirements. This completes the proof.

PROPOSITION 5.3. *If* Y *is a completely regular space which is an* AR (resp. ANR) *for completely regular spaces, then* Y *is compact* (resp. *locally compact*).

Proof. Since Y is completely regular, it may be embedded in a cartesian product of real lines. Let $\aleph$ be the cardinality of this product, and let ξ be an ordinal whose cardinality is greater than $\aleph$ and greater than the cardinality of Y. Now let Q be the space of ordinals $\leq \xi$ in the order topology, let $\overline{X}$ be a compact Hausdorff space containing X, and let

$$Z = (Q \times \overline{X}) - (\{\xi\} \times (\overline{X} - X)).$$

Since Q and $\overline{X}$ are completely regular, so is Z. Now $\{\xi\} \times X$ is closed in Z, and $\{\xi\} \times X$ is homeomorphic to X, and therefore there exists a retraction f from Z onto $\{\xi\} \times X$. For each x in X, let

$$f_x = f \mid (Q \times \{x\}).$$

By Lemma 5.2, there exists for each x in X a β_x in Q such that

$$f_x((q, x)) = f_x((\xi, x))$$

for all $q \geq \beta_x$. Now let β be the smallest ordinal larger than all the β_x; then $\beta < \xi$, and

$$f((\beta, x)) = (\xi, x)$$

for all x in X. Hence

$$f(\{\beta\} \times \overline{X}) = \{\xi\} \times X,$$

and therefore X is compact.

Let us now consider the ANR case. Suppose, therefore, that X is an ANR for completely regular spaces. Let $\overline{X}$, Q, and Z be as in the last paragraph. Then, by assumption, there exists a retraction f from a neighborhood U of $\{\xi\} \times X$ in Z onto $\{\xi\} \times X$. Now by Lemma 5.2, there exists an ordinal $\alpha < \xi$ such that $Q_\alpha \times X \subset U$, where $Q_\alpha = \{q \in Q \mid q \geq \alpha\}$. Proceeding just as in the last paragraph (with Q replaced by Q_α), we obtain a β in Q_α such that

$$f((\beta, x)) = (\xi, x)$$

for all x in X. If we now define the continuous function

$$h : \{\xi\} \times X \longrightarrow \{\beta\} \times X$$

by

$$h((\xi, x)) = (\beta, x),$$

then the restriction of $h \circ f$ to $(\{\beta\} \times \overline{X}) \cap U$ is a retraction of $(\{\beta\} \times \overline{X}) \cap U$ onto $\{\beta\} \times X$. Hence $\{\beta\} \times X$ is closed in $(\{\beta\} \times \overline{X}) \cap U$; but $(\{\beta\} \times \overline{X}) \cap U$ is an open subset of the compact set $\{\beta\} \times \overline{X}$, and therefore both $(\{\beta\} \times \overline{X}) \cap U$ and $\{\beta\} \times X$ are locally compact. Hence X is locally compact, which is what we had to show.

Proof of 2(e) *"only if"*. This now follows immediately from Proposition 5.3 and Theorem 3.2(d).

6. An example. In [2], Arens showed indirectly that there exists a compact, convex subset of a locally convex topological linear space which, while certainly an AE for metric spaces by [**11**, Th. 4.1], is not an AE for compact Hausdorff spaces. In this section we will prove this result (and a little more) by means of a direct example, which should also indicate why we assumed the space Y in Theorems 3.1 and 3.2 to be metrizable.

The proof of Proposition 6.1 is due jointly to V. L. Klee and the author, and uses a suggestion by I. E. Segal.

PROPOSITION 6.1. *Let X be the cartesian product of continuum many closed unit intervals. Then there exists a closed, convex subset of X which is not the image under a continuous function of any open subset of X.*

Proof. Let us call a topological space *separable* if it has a countable dense subset. Since the cartesian product of at most continuum many separable spaces is separable [**21**, p. 139], it follows that X is separable. Hence any continuous image of any open subset of X is also separable. To prove the proposition, it therefore suffices to produce a closed, convex subset of X which is not separable. This we will now proceed to do.

Let H be a Hilbert space whose orthonormal dimension is the continuum. Then H has continuum many elements, and is not separable. Let us show (using a proof due to I. E. Segal) that H is not even separable in the weak topology.

In fact, if H were separable in the weak topology, there would exist a countably dimensional subspace K of H which is weakly dense in H. Since H is countably dimensional, it is separable in the strong topology. Now by the Hahn-Banach theorem, the strong closure of K is weakly closed and hence coincides with H. But this implies that H is separable in the strong topology, contrary to our assumption.

Now let S be the unit sphere of H in the weak topology. Then S is compact, since H is reflexive. Also S is not separable since, as we have just shown, H is not separable in the weak topology. To complete the proof, we must show that S is homeomorphic to a convex subset of X. Now by definition,

$$X = \prod_{f \in F} I_f,$$

where F is an index set whose cardinality is the continuum, and I_f is homeomorphic to the unit interval for every f in F. Now H^*, the dual space of H, is isomorphic to H [**15**, p. 31, Th. 3], and hence we may take F to be the unit sphere of H^*.

Define $\phi : S \longrightarrow X$ by "$(\phi(x))_f = f(x)$"; then ϕ is a homeomorphism from S onto $\phi(S)$ by definition of the weak topology, and $\phi(S)$ is clearly convex in X. This completes the proof.

COROLLARY 6.2. *There exists a compact Hausdorff space which is a convex subset of a locally convex topological linear space (and hence* [**11**, Th. 4.1] *an AE for metric space) which is not even an ANR for compact Hausdorff spaces.*

7. Simultaneous extensions. The purpose of this section is to prove the following theorem:

THEOREM 7.1. *Let X be a metric space, A a closed subset of X, and E a locally convex topological linear space. Let $C(X, E)$ denote the linear space of continuous functions from X into E, and similarly for $C(A, E)$. Then there exists a mapping*

$$\phi : C(A, E) \longrightarrow C(X, E)$$

satisfying the following conditions:

(a) *$\phi(f)$ is an extension of f for every $f \in C(A, E)$.*

(b) *The range of $\phi(f)$ is contained in the convex hull of the range of f for every $f \in C(A, E)$.*

(c) *ϕ is an isomorphism* (i.e. *a one-to-one, bi-continuous linear transformation*) *from $C(A, E)$ into $C(X, E)$, provided $C(A, E)$ and $C(X, E)$ both carry the same one of the following three topologies:*

(1) *Topology of simple convergence* [**7**, p. 4].

(2) *Topology of compact convergence* [**7**, p. 5][3].

(3) *Topology of uniform convergence* [**7**, p. 5].

Proof. We will show below that, in his proof of [**11**, Th. 4.1], Dugundji has already constructed a mapping ϕ satisfying all our requirements. In fact, Dugundji [**11**, Th. 5.1] and Arens [**2**, Th. 2.6] have already observed that this trivially satisfies *some* of our requirements; the only property of ϕ which will need a nontrivial proof below is that ϕ is continuous for the topology (2).

We need the following fact, which is due to Dugundji [**11**] and was more concisely stated and proved by Arens [**2**, Lem. 2.1]:

(*) There exists a locally finite covering $\mathcal{V}$ of $X - A$ by open sets, and associated with each $V \in \mathcal{V}$ an $a_V \in A$ and a continuous real-valued function g_V on X which vanishes outside V, such that:

(i) $0 \leq g_V(x) \leq 1$ and $\sum_V g_V(x) = 1$ for all $x \in X - A$.

[3] This topology is the same as the compact-open topology [**1**, Th. 9].

(ii) If $a \in A$, and $x \in V$, then $\rho(a, a_V) < 3\rho(a, x)$, where ρ is the metric in X.

(iii) If $f \in C(A, E)$, then the function $\widetilde{f}: X \longrightarrow E$, defined by "$\widetilde{f}(x) = x$ for $x \in A$, and $\widetilde{f}(x) = \sum_V g_V(x) f(a_V)$" for $x \in X - A$, is continuous.

The mapping ϕ may now be defined by $\phi(f) = \widetilde{f}$, where $\widetilde{f}$ is as in (iii) above. It is immediately evident that ϕ is a one-to-one linear transformation which satisfies conditions (a) and (b) of our theorem. The continuity of ϕ^{-1} for any of the three topologies follows from (a) and the definition of these topologies. The continuity of ϕ for topology (3) follows from (b). The continuity of ϕ for the topologies (1) and (2), finally, will be an immediate consequence of the following lemma:

LEMMA 7.2. *If C is a finite* (resp. *compact*) *subset of X, then there exists a finite* (resp. *compact*) *subset $\widetilde{C}$ of A such that $\widetilde{f}(C)$ is contained in the convex hull of $f(\widetilde{C})$.*

Proof of lemma. Let us define function u from X to the finite subsets of A. If $x \in A$, then we let

$$u(x) = \{x\}.$$

If $x \in X - A$, then clearly x is in the closure of only finitely many $V \in \mathfrak{V}$, say $V_1, \cdots, V_n$, and we set

$$u(x) = \{x_{V_1}, \cdots, x_{V_n}\}.$$

Having thus defined u, we set

$$\widetilde{C} = \bigcup_{x \in C} u(x)$$

for every $C \subset X$. It is clear that $\widetilde{f}(C)$ is contained in the convex hull of $f(\widetilde{C})$. If C is finite, then $\widetilde{C}$ is clearly also finite. It therefore remains to prove that $\widetilde{C}$ is compact if C is compact. To do this, it is sufficient to show that u is upper semi-continuous[4], because then the compactness of $\widetilde{C}$ for compact C will be an immediate consequence of [**14**, p. 151, 21.3.4].

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[4]A function h from a topological space Y to the space of nonempty subsets of a topological space Z is called *upper semi-continuous* [**14**, p. 149] *at a point* $y \in Y$ if, for every open subset U of Z which contains $h(y)$, there exists a neighborhood W of y in Y such that $h(y') \subset U$ for every $y' \in W$; h is called *upper semi-continuous* if it is upper semi-continuous at every $y \in Y$.

Let us first show that u is upper semi-continuous at points of $X - A$. Since V is locally finite, the closures (in $X - A$) of any subcollection of $\mathcal{V}$ have a closed (in $X - A$) union. Hence if $x \in X - A$, then

$$B = \bigcup\{\overline{V} \mid V \in \mathcal{V}, x \notin \overline{V}\}$$

is closed in $X - A$, where $\overline{V}$ denotes (and will always denote below) the closure of V in $X - A$. Let

$$U = (X - A) - B.$$

Then U is a neighborhood of x in X, and $u(x') \subset u(x)$ whenever $x' \in U$; this shows that u is continuous at x.

Before proving the upper semi-continuity of u on A, we need the following consequence of (ii):

(ii*) If $a \in A$, and $x \in \overline{V}$, then $\rho(a, a_V) < 4\rho(a, x)$.

To see this, pick a $y \in V$ such that

$$\rho(x, y) < 1/3\ \rho(a, x),$$

and then observe that

$$\rho(a, a_V) < 3\rho(a, y) \leq 3(\rho(a, x) + \rho(x, y)) \leq 4\rho(a, x).$$

Let us now prove that u is upper semi-continuous on A. Let $a \in A$, and let U be an open subset of X containing $u(a) = \{a\}$. Pick $\epsilon > 0$ such that

$$\{y \in X \mid \rho(a, y) < \epsilon\} \subset U.$$

Now let

$$W = \{x \in X \mid \rho(a, x) < \epsilon/4\}.$$

Then W is a neighborhood of a in X. If $x \in W \cap A$, then $u(x) = \{x\}$, and thus $u(x) \subset U$. If $x \in W \cap (X - A)$, then $\rho(a, a_V) < \epsilon$ whenever $x \in \overline{V}$ by (ii*), and thus again $u(x) \subset U$. Hence u is upper semi-continuous on A.

This proves the lemma, and hence also the theorem.

REMARK. It is an easy consequence of Proposition 4.3 that Theorem 7.1 remains true if the requirement that X is metric is replaced by the following weaker requirement: A is metric, and one of the following three conditions holds: (a) X is paracompact, (b) X is normal and A is separable, (c) X is completely regular and A is compact.

8. ADDED IN PROOF. Many of our results have been obtained independently by Olof Hanner [24], who was kind enough to send the author a pre-publication reprint of his paper.

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