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ON THE COMPLEX ZEROS OF FUNCTIONS OF STURM-LIOUVILLE TYPE

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1. Let $Q(z)$ be an analytic function of the complex variable z in a region D. In the present paper only those solutions of

$$(1.1) \qquad W'' + Q(z)W = 0$$

which are distinct from the trivial solution ($\equiv 0$) shall be considered.

In this paper the following results shall be established.

THEOREM 1. *Suppose that the following conditions are satisfied:*

(a) *the circle $|z| \le R$ is contained in D,*

(b) *$W(z)$ is a solution of (1.1), $W(0) \neq 0$,*

(c) *$n(r)$ is the number of zeros of $W(z)$ in $|z| \le r$, $r < R$.*

Then $n(r)$ satisfies the inequality

$$(1.2) \qquad n(r) \le (\log(Rr^{-1}))^{-1}[\log(1 + R|W'(0)|\,|W(0)|^{-1}) + (2\pi)^{-1}\int_0^{2\pi}\int_0^R (R-t)\,|Q(t\,e^{i\theta})|\,dtd\theta].$$

COROLLARY 1.1. *Suppose that the following conditions are satisfied:*

(a) *$Q(z)$ is a polynomial of degree k,*

(b) *conditions (b) and (c) of Theorem 1 hold.*

Then $W(z)$ is an integral function of order at most $k+2$. Furthermore, as $r \longrightarrow \infty$,

$$(1.3) \qquad n(r) = O(r^{k+2}).$$

Obviously the result of Theorem 1 is not good if r is close to R. Also it

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does not apply to a solution which vanishes at the origin. The following theorem is free of these restrictions.

THEOREM 2. *Suppose that the following conditions are satisfied:*

(a) *S is a closed region contained in D,*

(b) *the boundary C of S is a closed contour,*

(c) *the maximum value of $|Q(z)|$ on C is M,*

(d) *S can be divided into n subregions such that each subregion has a diameter not greater than $\pi M^{-1/2}$; and for any two points z_1 and z_2 of a subregion, the linear segment $z_1 z_2$ lies in S (we agree that the common boundary of two subregions belongs to both subregions).*

Then

(e) *if $Q(z)$ is not a constant, the number of zeros of any solution $W(z)$ of (1.1) in S is not greater than n,*

(f) *more accurately, if $Q(z)$ is not a constant, each solution $W(z)$ of (1.1) has at most one zero in each subregion, and when it is known that $W(z)$ has some zero z_i which belongs to n_i $(n_i > 1)$ different subregions, $i = 1, 2, \cdots, k$, its total number of zeros in S is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$,*

(g) *if some solution of (1.1) has more than one zero in some subregion, $Q(z)$ must be a constant and $|Q(z)| = M > 0$ in D.*

We may observe that if $Q(z)$ is not a constant, M must be positive, according to the principle of the maximum modulus. If $Q(z)$ is a constant, the problem is trivial as the distribution of the zeros is known.

2. To prove Theorem 1, we need the following known results.

LEMMA 1. *Suppose that the following conditions are satisfied:*

(a) *$f(x)$ and $g(x)$ are real-valued functions, continuous and nonnegative for $x \geq 0$,*

(b) *M is a positive constant,*

(c) $$f(x) \leq M + \int_0^x f(t) g(t)\, dt, \qquad x \geq 0.$$

Then we have

$$f(x) \leq M\, e^{\int_0^x g(t)dt} \qquad x \geq 0.$$

This lemma is due to R. Bellman. For a proof of it see [1] or [5].

LEMMA 2. *Suppose that the following conditions are satisfied:*

(a) *$f(z)$ is analytic for $|z| \leq R$, $f(0) \neq 0$,*

(b) *the moduli of the zeros of $f(z)$ in the circle $|z| \leq R$ are $r_1, r_2, \cdots, r_k$ arranged as a nondecreasing sequence (a zero of order p is counted p times).*

Then we have

$$\log[R^k(r_1 r_2 \cdots r_k)^{-1}] = (2\pi)^{-1}\int_0^{2\pi} \log|f(R\,e^{i\theta})|\,d\theta - \log|f(0)|.$$

Lemma 2 is known as Jensen's theorem (see [4]).

3. Now we shall prove Theorem 1. Along a fixed ray radiating out from the origin, $z = r\exp(i\theta)$, equation (1.1) becomes

$$\frac{d^2W}{dr^2} + e^{2i\theta}Q(re^{i\theta})W = 0. \tag{3.1}$$

Integrating (3.1) twice from 0 to r, we obtain

$$W(re^{i\theta}) = W(0) + W'(0)e^{i\theta}r - e^{2i\theta}\int_0^r\int_0^h Q(te^{i\theta})W(te^{i\theta})\,dtdh, \tag{3.2}$$

where $W'(0)\exp(i\theta)$ is the value of dW/dr at the origin. Integration by parts of the integral in (3.2) gives

$$W(re^{i\theta}) = W(0) + W'(0)e^{i\theta}r - e^{2i\theta}\int_0^r (r-t)Q(te^{i\theta})W(te^{i\theta})\,dt. \tag{3.3}$$

For $r \leq R$, (3.3) yields

$$|W(re^{i\theta})| \leq |W(0)| + |W'(0)|\,R + \int_0^r (R-t)\,|Q(te^{i\theta})W(te^{i\theta})|\,dt. \tag{3.4}$$

Applying Lemma 1 to (3.4), we have

$$|W(Re^{i\theta})| \leq (|W(0)| + |W'(0)|R)\,e^{\int_0^R (R-t)|Q(te^{i\theta})|dt}. \tag{3.5}$$

Let the moduli of the zeros of $W(z)$ in the circle $|z| \leq r < R$ be $r_1, r_2, \cdots, r_k$, arranged as a nondecreasing sequence. Then an appeal to Lemma 2 gives

$$(3.6)\quad \log [R^k(r_1 r_2 \cdots r_k)^{-1}] \leq (2\pi)^{-1} \int_0^{2\pi} \log |W(Re^{i\theta})|\, d\theta - \log |W(0)|.$$

Clearly

$$(3.7)\quad \log [R^k(r_1\, r_2 \cdots r_k)^{-1}] \geq \log [R^{n(r)} r^{-n(r)}]$$

$$= n(r) \log (Rr^{-1}), \qquad r < R,$$

where $n(r)$ is the number of zeros of $W(z)$ in $|z| \leq r$. On the other hand, (3.5) gives

$$(3.8)\quad \int_0^{2\pi} \log |W(Re^{i\theta})|\, d\theta \leq 2\pi \log [|W(0)| + |W'(0)| R]$$

$$+ \int_0^{2\pi} \int_0^R (R-t)\, |Q(te^{i\theta})|\, dt\, d\theta\,.$$

Combining (3.6), (3.7), and (3.8), we have

$$(3.9)\quad n(r) \log (Rr^{-1}) \leq \log [|W(0)| + |W'(0)| R] - \log |W(0)|$$

$$+ (2\pi)^{-1} \int_0^{2\pi} \int_0^R (R-t)\, |Q(te^{i\theta})|\, dt\, d\theta$$

for $r < R$. But (3.9) is equivalent to (1.2), so that this completes the proof of Theorem 1.

If $Q(z)$ is a polynomial of degree k, then $W(z)$ is analytic except at infinity and, from (3.5),

$$|W(Re^{i\theta})| = O\left(e^{A \cdot R^{k+2}}\right), \qquad R \longrightarrow \infty\,,$$

where A is a constant. Hence $W(z)$ is an integral function of order at most $k+2$. Finally if we set $R = 2r$ in (3.9), it is clear that

$$n(r) = O(r^{k+2}).$$

This proves Corollary 1.1.

4. To prove Theorem 2, we need the following known result. On the real axis, equation (1.1) becomes

$$\frac{d^2W}{dx^2} + Q(x)W = 0, \tag{4.1}$$

where x is the real part of the complex variable z. Denote by $q_1(x)$ the real part of $Q(x)$.

LEMMA 3. *Let $W(x)$ be a solution of (4.1), $W(0) = 0$. Suppose that one of the following conditions is satisfied.*

(a) $\max\ q_1(x) = m > 0$ *in* $[0, a]$, $0 < a \leq \pi m^{-1/2}$, *and* $Q(x) \not\equiv m$ *in* $[0, a]$,

(b) $q_1(x) \leq 0$ *in* $[0, a]$.

Then $W(x) \neq 0$ *in* $(0, a]$.

This lemma was proved in [3; Theorems 5.1, 5.2]. Part (b) is also covered by a theorem of Hille [2, p. 512 ff.]. Its proof remains valid even if $Q(x)$ is assumed only to be a continuous (complex-valued) function of a real variable x; consequently the lemma remains true under such an assumption on $Q(x)$.

We first prove (f) of Theorem 2.

Let S_i be one of the subregions of S with a diameter not greater than $\pi M^{-1/2}$. Suppose that $W(z)$ is a solution of (1.1) which vanishes at a point z_0, say, of S_i. Consider a fixed ray radiating out from z_0, $z - z_0 = r\exp(i\theta)$. Along this ray, equation (1.1) becomes

$$\frac{d^2W}{dr^2} + e^{2i\theta}Q(z_0 + re^{i\theta})W = 0. \tag{4.2}$$

By virtue of the principle of the maximum modulus, we have

$$|e^{2i\theta}Q(z)| = |Q(z)| \leq M$$

for any point z of S on this ray. Hence on a segment of this ray between z_0 and any other point of S_i (by assumption, this segment lies in S) the maximum value m, say, of the real part of $\exp(2i\theta)Q(z)$ is not greater than M. If m is positive, then $\pi m^{-1/2} \geq \pi M^{-1/2}$. Since $Q(z)$ is not a constant, $\exp(2i\theta)Q(z) \not\equiv m$ on this segment. By virtue of the fact that the diameter of S_i is not greater than $\pi M^{-1/2}$ and Lemma 3, it is clear that $W(z)$ does not vanish again on that part of the ray in S_i, regardless of the sign of m. Repeating this process for each ray radiating out from z_0, we see clearly that $W(z)$ cannot vanish again in S_i. Since S_i is an arbitrary subregion, $W(z)$ can vanish at most at one point of each subregion.

On the other hand, if $W(z)$ has a zero z_i which belongs to n_i $(n_i > 1)$ different subregions, then $W(z)$ cannot vanish again in any of these n_i subregions, as the foregoing proof shows. If it is known that there are k such zeros z_i, each z_i belonging to n_i subregions, $i = 1, 2, \cdots, k$, it is clear that the total number of zeros of $W(z)$ in S is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$.

To prove (g), let $W(z)$ be a solution of (1.1) having two zeros, say z_0 and z_1, in some subregion S_i. Let the argument of $z_1 - z_0$ be θ. Then along the linear segment $z_0 z_1$, equation (1.1) becomes (4.2). According to Lemma 3, the maximum value m of the real part of $\exp(2i\theta)Q(z)$ on the linear segment $z_0 z_1$ must be positive. Further, since

$$(4.3) \qquad |z_1 - z_0| \leq \pi M^{-1/2} \leq \pi m^{-1/2},$$

z_0 and z_1 can both be the zeros of $W(z)$ only if

$$(4.4) \qquad e^{2i\theta}Q(z) \equiv m$$

on the linear segment $z_0 z_1$, by Lemma 3 again. But if (4.4) is true, the general solution of (4.2) is $A \sin(m^{1/2}r + B)$, A and B being constants. If a solution of (4.2) has two zeros, the distance between them must not be less than $\pi m^{-1/2}$. In other words, the equality signs in (4.3) must hold. That is, $M = m$. From (4.4), we have $\exp(2i\theta)Q(z) \equiv M$ on the linear segment $z_0 z_1$. Since $Q(z)$ is an analytic function and constant on the linear segment $z_0 z_1$, $Q(z)$ is a constant in D. Obviously $|Q(z)| = M$; and since m is positive, so is M. This proves (g).

Clearly (e) follows from (f), and this completes the proof of Theorem 2.

5. Added in proof. The author is indebted to a referee for calling his attention to the fact that, in connection with Corollary 1.1, an entire function which satisfies a linear differential equation with coefficients which are rational functions of z is always of finite rational order and of perfectly regular growth. (See G. Valiron, *Lectures on the theory of integral functions*, Toulouse, 1923, p. 106 ff.)

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