ON THE COMPLEX ZEROS OF FUNCTIONS OF STURM-LIOUVILLE TYPE

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1. Let $Q(z)$ be an analytic function of the complex variable $z$ in a region $D$. In the present paper only those solutions of

$$W'' + Q(z)W = 0$$

which are distinct from the trivial solution ($\equiv 0$) shall be considered.

In this paper the following results shall be established.

**Theorem 1.** Suppose that the following conditions are satisfied:

(a) the circle $|z| < R$ is contained in $D$,

(b) $W(z)$ is a solution of (1.1), $W(0) \neq 0$,

(c) $n(r)$ is the number of zeros of $W(z)$ in $|z| \leq r$, $r < R$.

Then $n(r)$ satisfies the inequality

$$n(r) \leq (\log R)^{-1} \left[ \log (1 + R |W'(0)| |W(0)|^{-1}) + (2\pi)^{-1} \int_0^{2\pi} \int_0^R (R - t) |Q(te^{i\theta})| \, dt \, d\theta \right].$$

**Corollary 1.1.** Suppose that the following conditions are satisfied:

(a) $Q(z)$ is a polynomial of degree $k$,

(b) conditions (b) and (c) of Theorem 1 hold.

Then $W(z)$ is an integral function of order at most $k + 2$. Furthermore, as $r \to \infty$

$$n(r) = O(r^{k+2}).$$

Obviously the result of Theorem 1 is not good if $r$ is close to $R$. Also it

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does not apply to a solution which vanishes at the origin. The following theorem is free of these restrictions.

**Theorem 2.** Suppose that the following conditions are satisfied:

(a) $S$ is a closed region contained in $D$,

(b) the boundary $C$ of $S$ is a closed contour,

(c) the maximum value of $|Q(z)|$ on $C$ is $M$,

(d) $S$ can be divided into $n$ subregions such that each subregion has a diameter not greater than $\pi M^{-1/3}$; and for any two points $z_1$ and $z_2$ of a subregion, the linear segment $z_1z_2$ lies in $S$ (we agree that the common boundary of two subregions belongs to both subregions).

Then

(e) if $Q(z)$ is not a constant, the number of zeros of any solution $W(z)$ of (1.1) in $S$ is not greater than $n$,

(f) more accurately, if $Q(z)$ is not a constant, each solution $W(z)$ of (1.1) has at most one zero in each subregion, and when it is known that $W(z)$ has some zero $z_i$ which belongs to $n_i$ ($n_i > 1$) different subregions, $i = 1, 2, \cdots, k$, its total number of zeros in $S$ is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$,

(g) if some solution of (1.1) has more than one zero in some subregion, $Q(z)$ must be a constant and $|Q(z)| = M > 0$ in $D$.

We may observe that if $Q(z)$ is not a constant, $M$ must be positive, according to the principle of the maximum modulus. If $Q(z)$ is a constant, the problem is trivial as the distribution of the zeros is known.

2. To prove Theorem 1, we need the following known results.

**Lemma 1.** Suppose that the following conditions are satisfied:

(a) $f(x)$ and $g(x)$ are real-valued functions, continuous and nonnegative for $x \geq 0$,

(b) $M$ is a positive constant,

(c) $f(x) \leq M + \int_0^x f(t)g(t)\,dt, \quad x \geq 0$.

Then we have
This lemma is due to R. Bellman. For a proof of it see [1] or [5].

**Lemma 2.** Suppose that the following conditions are satisfied:

(a) \( f(z) \) is analytic for \( |z| < R, f(0) \neq 0, \)

(b) the moduli of the zeros of \( f(z) \) in the circle \( |z| \leq R \) are \( r_1, r_2, \ldots, r_k \) arranged as a nondecreasing sequence (a zero of order \( p \) is counted \( p \) times).

Then we have

\[
\log \left[ \frac{R^k (r_1 r_2 \cdots r_k)^{-1}}{1} \right] = (2\pi)^{-1} \int_0^{2\pi} \log |f(R e^{i\theta})| \, d\theta - \log |f(0)|.
\]

Lemma 2 is known as Jensen's theorem (see [4]).

3. **Now we shall prove Theorem 1.** Along a fixed ray radiating out from the origin, \( z = r e^{i\theta} \), equation (1.1) becomes

\[
\frac{d^2 W}{dr^2} + e^{2i\theta} Q(re^{i\theta}) W = 0.
\]

Integrating (3.1) twice from 0 to \( r \), we obtain

\[
W(re^{i\theta}) = W(0) + W'(0)e^{i\theta}r - e^{2i\theta} \int_0^r \int_0^t Q(te^{i\theta}) W(te^{i\theta}) \, dt \, dh,
\]

where \( W'(0) \exp(i\theta) \) is the value of \( dW/dr \) at the origin. Integration by parts of the integral in (3.2) gives

\[
W(re^{i\theta}) = W(0) + W'(0)e^{i\theta}r - e^{2i\theta} \int_0^r (r-t) Q(te^{i\theta}) W(te^{i\theta}) \, dt.
\]

For \( r \leq R \), (3.3) yields

\[
|W(re^{i\theta})| \leq |W(0)| + |W'(0)| \cdot R + \int_0^r (R-t) |Q(te^{i\theta}) W(te^{i\theta})| \, dt.
\]

Applying Lemma 1 to (3.4), we have

\[
|W(Re^{i\theta})| \leq (|W(0)| + |W'(0)| \cdot R) e^{\int_0^R (R-t) |Q(te^{i\theta})| \, dt}.
\]
Let the moduli of the zeros of $W(z)$ in the circle $|z| \leq r < R$ be $r_1, r_2, \ldots, r_k$, arranged as a nondecreasing sequence. Then an appeal to Lemma 2 gives

\[(3.6) \quad \log [R^k(r_1 r_2 \cdots r_k)^{-1}] \leq (2\pi)^{-1} \int_0^{2\pi} \log |W(Re^{i\theta})| d\theta - \log |W(0)|.\]

Clearly

\[(3.7) \quad \log [R^k(r_1 r_2 \cdots r_k)^{-1}] \geq \log [R^{n(r)}_r r^{-n(r)}] = n(r) \log (Rr^{-1}), \quad r < R,
\]

where $n(r)$ is the number of zeros of $W(z)$ in $|z| \leq r$. On the other hand, (3.5) gives

\[(3.8) \quad \int_0^{2\pi} \log |W(Re^{i\theta})| d\theta \leq 2\pi \log [|W(0)| + |W'(0)| R] + \int_0^{2\pi} \int_0^R (R-t) |Q(te^{i\theta})| dt d\theta.
\]

Combining (3.6), (3.7), and (3.8), we have

\[(3.9) \quad n(r) \log (Rr^{-1}) \leq \log [|W(0)| + |W'(0)| R] - \log |W(0)| + (2\pi)^{-1} \int_0^{2\pi} \int_0^R (R-t) |Q(te^{i\theta})| dt d\theta
\]

for $r < R$. But (3.9) is equivalent to (1.2), so that this completes the proof of Theorem 1.

If $Q(z)$ is a polynomial of degree $k$, then $W(z)$ is analytic except at infinity and, from (3.5),

\[|W(Re^{i\theta})| = O(e^{A \cdot R^{k+2}}), \quad R \to \infty,
\]

where $A$ is a constant. Hence $W(z)$ is an integral function of order at most $k + 2$. Finally if we set $R = 2r$ in (3.9), it is clear that

\[n(r) = O(r^{k+2}).
\]

This proves Corollary 1.1.
4. To prove Theorem 2, we need the following known result. On the real axis, equation (1.1) becomes

\[ \frac{d^2W}{dx^2} + Q(x)W = 0, \]

where \( x \) is the real part of the complex variable \( z \). Denote by \( q_1(x) \) the real part of \( Q(x) \).

**Lemma 3.** Let \( W(x) \) be a solution of (4.1), \( W(0) = 0 \). Suppose that one of the following conditions is satisfied.

(a) \( \max q_1(x) = m > 0 \) in \( [0, a] \), \( 0 < a < \pi m^{-1/2} \), and \( Q(x) \neq m \) in \( [0, a] \).

(b) \( q_1(x) \leq 0 \) in \( [0, a] \).

Then \( W(x) \neq 0 \) in \( (0, a) \).

This lemma was proved in [3; Theorems 5.1, 5.2]. Part (b) is also covered by a theorem of Ille [2, p. 512 ff.]. Its proof remains valid even if \( Q(x) \) is assumed only to be a continuous (complex-valued) function of a real variable \( x \); consequently the lemma remains true under such an assumption on \( Q(x) \).

We first prove (f) of Theorem 2.

Let \( S_i \) be one of the subregions of \( S \) with a diameter not greater than \( \pi M^{-1/2} \). Suppose that \( W(z) \) is a solution of (1.1) which vanishes at a point \( z_0 \), say, of \( S_i \). Consider a fixed ray radiating out from \( z_0 \), \( z - z_0 = r \exp(i\theta) \). Along this ray, equation (1.1) becomes

\[ \frac{d^2W}{dr^2} + e^{2i\theta} Q(z_0 + re^{i\theta})W = 0. \]

By virtue of the principle of the maximum modulus, we have

\[ |e^{2i\theta} Q(z)| = |Q(z)| \leq M \]

for any point \( z \) of \( S \) on this ray. Hence on a segment of this ray between \( z_0 \) and any other point of \( S_i \) (by assumption, this segment lies in \( S \)) the maximum value \( m \), say, of the real part of \( \exp(2i\theta)Q(z) \) is not greater than \( M \). If \( m \) is positive, then \( \pi m^{-1/2} \geq \pi M^{-1/2} \). Since \( Q(z) \) is not a constant, \( \exp(2i\theta)Q(z) \neq m \) on this segment. By virtue of the fact that the diameter of \( S_i \) is not greater
than $\pi M^{1/2}$ and Lemma 3, it is clear that $W(z)$ does not vanish again on that part of the ray in $S_i$, regardless of the sign of $m$. Repeating this process for each ray radiating out from $z_0$, we see clearly that $W(z)$ cannot vanish again in $S_i$. Since $S_i$ is an arbitrary subregion, $W(z)$ can vanish at most at one point of each subregion.

On the other hand, if $W(z)$ has a zero $z_i$ which belongs to $n_i$ ($n_i > 1$) different subregions, then $W(z)$ cannot vanish again in any of these $n_i$ subregions, as the foregoing proof shows. If it is known that there are $k$ such zeros $z_i$, each $z_i$ belonging to $n_i$ subregions, $i = 1, 2, \cdots, k$, it is clear that the total number of zeros of $W(z)$ in $S$ is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$.

To prove (g), let $W(z)$ be a solution of (1.1) having two zeros, say $z_0$ and $z_1$, in some subregion $S_i$. Let the argument of $z_1 - z_0$ be $\theta$. Then along the linear segment $z_0 z_1$, equation (1.1) becomes (4.2). According to Lemma 3, the maximum value $M$ of the real part of $\exp (2i\theta) Q(z)$ on the linear segment $z_0 z_1$ must be positive. Further, since

$$\tag{4.3} |z_1 - z_0| \leq \pi M^{1/2} \leq \pi m^{-1/2},$$
$z_0$ and $z_1$ can both be the zeros of $W(z)$ only if

$$\tag{4.4} e^{2i\theta} Q(z) = M$$
on the linear segment $z_0 z_1$, by Lemma 3 again. But if (4.4) is true, the general solution of (4.2) is $A \sin (m^{1/2} r + B)$, $A$ and $B$ being constants. If a solution of (4.2) has two zeros, the distance between them must not be less than $\pi m^{-1/2}$. In other words, the equality signs in (4.3) must hold. That is, $M = m$. From (4.4), we have $\exp (2i\theta) Q(z) = M$ on the linear segment $z_0 z_1$. Since $Q(z)$ is an analytic function and constant on the linear segment $z_0 z_1$, $Q(z)$ is a constant in $D$. Obviously $|Q(z)| = M$; and since $m$ is positive, so is $M$. This proves (g).

Clearly (e) follows from (f), and this completes the proof of Theorem 2.

5. Added in proof. The author is indebted to a referee for calling his attention to the fact that, in connection with Corollary 1.1, an entire function which satisfies a linear differential equation with coefficients which are rational functions of $z$ is always of finite rational order and of perfectly regular.

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