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**ON THE COMPLEX ZEROS OF FUNCTIONS OF
STURM-LIOUVILLE TYPE**

CHOY-TAK TAAM

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1. Let $Q(z)$ be an analytic function of the complex variable z in a region D . In the present paper only those solutions of

$$(1.1) \quad \mathbb{W}'' + Q(z)\mathbb{W} = 0$$

which are distinct from the trivial solution ($\equiv 0$) shall be considered.

In this paper the following results shall be established.

THEOREM 1. *Suppose that the following conditions are satisfied:*

- (a) *the circle $|z| \leq R$ is contained in D ,*
- (b) *$\mathbb{W}(z)$ is a solution of (1.1), $\mathbb{W}(0) \neq 0$,*
- (c) *$n(r)$ is the number of zeros of $\mathbb{W}(z)$ in $|z| \leq r, r < R$.*

Then $n(r)$ satisfies the inequality

$$(1.2) \quad n(r) \leq (\log(Rr^{-1}))^{-1} [\log(1 + R|\mathbb{W}'(0)| |\mathbb{W}(0)|^{-1}) \\ + (2\pi)^{-1} \int_0^{2\pi} \int_0^R (R-t) |Q(te^{i\theta})| dt d\theta].$$

COROLLARY 1.1. *Suppose that the following conditions are satisfied:*

- (a) *$Q(z)$ is a polynomial of degree k ,*
- (b) *conditions (b) and (c) of Theorem 1 hold.*

Then $\mathbb{W}(z)$ is an integral function of order at most $k+2$. Furthermore, as $r \rightarrow \infty$,

$$(1.3) \quad n(r) = O(r^{k+2}).$$

Obviously the result of Theorem 1 is not good if r is close to R . Also it

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does not apply to a solution which vanishes at the origin. The following theorem is free of these restrictions.

THEOREM 2. *Suppose that the following conditions are satisfied:*

- (a) *S is a closed region contained in D,*
- (b) *the boundary C of S is a closed contour,*
- (c) *the maximum value of $|Q(z)|$ on C is M,*
- (d) *S can be divided into n subregions such that each subregion has a diameter not greater than $\pi M^{-1/2}$; and for any two points z_1 and z_2 of a subregion, the linear segment $z_1 z_2$ lies in S (we agree that the common boundary of two subregions belongs to both subregions).*

Then

- (e) *if $Q(z)$ is not a constant, the number of zeros of any solution $W(z)$ of (1.1) in S is not greater than n,*
- (f) *more accurately, if $Q(z)$ is not a constant, each solution $W(z)$ of (1.1) has at most one zero in each subregion, and when it is known that $W(z)$ has some zero z_i which belongs to n_i ($n_i > 1$) different subregions, $i = 1, 2, \dots, k$, its total number of zeros in S is not greater than $n + k - (n_1 + n_2 + \dots + n_k)$,*
- (g) *if some solution of (1.1) has more than one zero in some subregion, $Q(z)$ must be a constant and $|Q(z)| = M > 0$ in D.*

We may observe that if $Q(z)$ is not a constant, M must be positive, according to the principle of the maximum modulus. If $Q(z)$ is a constant, the problem is trivial as the distribution of the zeros is known.

2. To prove Theorem 1, we need the following known results.

LEMMA 1. *Suppose that the following conditions are satisfied:*

- (a) *$f(x)$ and $g(x)$ are real-valued functions, continuous and nonnegative for $x \geq 0$,*
- (b) *M is a positive constant,*
- (c) $f(x) \leq M + \int_0^x f(t)g(t)dt, \quad x \geq 0.$

Then we have

$$f(x) \leq M e^{\int_0^x g(t)dt} \quad x \geq 0.$$

This lemma is due to R. Bellman. For a proof of it see [1] or [5].

LEMMA 2. *Suppose that the following conditions are satisfied:*

(a) *$f(z)$ is analytic for $|z| \leq R$, $f(0) \neq 0$,*

(b) *the moduli of the zeros of $f(z)$ in the circle $|z| \leq R$ are r_1, r_2, \dots, r_k arranged as a nondecreasing sequence (a zero of order p is counted p times).*

Then we have

$$\log [R^k (r_1 r_2 \dots r_k)^{-1}] = (2\pi)^{-1} \int_0^{2\pi} \log |f(R e^{i\theta})| d\theta - \log |f(0)|.$$

Lemma 2 is known as Jensen's theorem (see [4]).

3. Now we shall prove Theorem 1. Along a fixed ray radiating out from the origin, $z = r \exp(i\theta)$, equation (1.1) becomes

$$(3.1) \quad \frac{d^2W}{dr^2} + e^{2i\theta} Q(re^{i\theta})W = 0.$$

Integrating (3.1) twice from 0 to r , we obtain

$$(3.2) \quad W(re^{i\theta}) = W(0) + W'(0)e^{i\theta}r - e^{2i\theta} \int_0^r \int_0^h Q(te^{i\theta})W(te^{i\theta}) dt dh,$$

where $W'(0) \exp(i\theta)$ is the value of dW/dr at the origin. Integration by parts of the integral in (3.2) gives

$$(3.3) \quad W(re^{i\theta}) = W(0) + W'(0)e^{i\theta}r - e^{2i\theta} \int_0^r (r-t)Q(te^{i\theta})W(te^{i\theta}) dt.$$

For $r \leq R$, (3.3) yields

$$(3.4) \quad |W(re^{i\theta})| \leq |W(0)| + |W'(0)|R + \int_0^r (R-t)|Q(te^{i\theta})W(te^{i\theta})| dt.$$

Applying Lemma 1 to (3.4), we have

$$(3.5) \quad |W(Re^{i\theta})| \leq (|W(0)| + |W'(0)|R) e^{\int_0^R (R-t)|Q(te^{i\theta})| dt}.$$

Let the moduli of the zeros of $W(z)$ in the circle $|z| \leq r < R$ be r_1, r_2, \dots, r_k , arranged as a nondecreasing sequence. Then an appeal to Lemma 2 gives

$$(3.6) \quad \log [R^k (r_1 r_2 \cdots r_k)^{-1}] \leq (2\pi)^{-1} \int_0^{2\pi} \log |W(Re^{i\theta})| d\theta - \log |W(0)|.$$

Clearly

$$(3.7) \quad \begin{aligned} \log [R^k (r_1 r_2 \cdots r_k)^{-1}] &\geq \log [R^{n(r)} r^{-n(r)}] \\ &= n(r) \log (Rr^{-1}), \end{aligned} \quad r < R,$$

where $n(r)$ is the number of zeros of $W(z)$ in $|z| \leq r$. On the other hand, (3.5) gives

$$(3.8) \quad \begin{aligned} \int_0^{2\pi} \log |W(Re^{i\theta})| d\theta &\leq 2\pi \log [|W(0)| + |W'(0)| R] \\ &\quad + \int_0^{2\pi} \int_0^R (R-t) |Q(te^{i\theta})| dt d\theta. \end{aligned}$$

Combining (3.6), (3.7), and (3.8), we have

$$(3.9) \quad \begin{aligned} n(r) \log (Rr^{-1}) &\leq \log [|W(0)| + |W'(0)| R] - \log |W(0)| \\ &\quad + (2\pi)^{-1} \int_0^{2\pi} \int_0^R (R-t) |Q(te^{i\theta})| dt d\theta \end{aligned}$$

for $r < R$. But (3.9) is equivalent to (1.2), so that this completes the proof of Theorem 1.

If $Q(z)$ is a polynomial of degree k , then $W(z)$ is analytic except at infinity and, from (3.5),

$$|W(Re^{i\theta})| = O(e^{A \cdot R^{k+2}}), \quad R \rightarrow \infty,$$

where A is a constant. Hence $W(z)$ is an integral function of order at most $k+2$. Finally if we set $R=2r$ in (3.9), it is clear that

$$n(r) = O(r^{k+2}).$$

This proves Corollary 1.1.

4. To prove Theorem 2, we need the following known result. On the real axis, equation (1.1) becomes

$$(4.1) \quad \frac{d^2W}{dx^2} + Q(x)W = 0,$$

where x is the real part of the complex variable z . Denote by $q_1(x)$ the real part of $Q(x)$.

LEMMA 3. *Let $W(x)$ be a solution of (4.1), $W(0) = 0$. Suppose that one of the following conditions is satisfied.*

(a) $\max q_1(x) = m > 0$ in $[0, a]$, $0 < a \leq \pi m^{-1/2}$, and $Q(x) \neq m$ in $[0, a]$,

(b) $q_1(x) \leq 0$ in $[0, a]$.

Then $W(x) \neq 0$ in $(0, a)$.

This lemma was proved in [3; Theorems 5.1, 5.2]. Part (b) is also covered by a theorem of Hille [2, p.512 ff.]. Its proof remains valid even if $Q(x)$ is assumed only to be a continuous (complex-valued) function of a real variable x ; consequently the lemma remains true under such an assumption on $Q(x)$.

We first prove (f) of Theorem 2.

Let S_i be one of the subregions of S with a diameter not greater than $\pi M^{-1/2}$. Suppose that $W(z)$ is a solution of (1.1) which vanishes at a point z_0 , say, of S_i . Consider a fixed ray radiating out from z_0 , $z - z_0 = r \exp(i\theta)$. Along this ray, equation (1.1) becomes

$$(4.2) \quad \frac{d^2W}{dr^2} + e^{2i\theta} Q(z_0 + re^{i\theta})W = 0.$$

By virtue of the principle of the maximum modulus, we have

$$|e^{2i\theta} Q(z)| = |Q(z)| \leq M$$

for any point z of S on this ray. Hence on a segment of this ray between z_0 and any other point of S_i (by assumption, this segment lies in S) the maximum value m , say, of the real part of $\exp(2i\theta)Q(z)$ is not greater than M . If m is positive, then $\pi m^{-1/2} \geq \pi M^{-1/2}$. Since $Q(z)$ is not a constant, $\exp(2i\theta)Q(z) \neq m$ on this segment. By virtue of the fact that the diameter of S_i is not greater

than $\pi M^{-1/2}$ and Lemma 3, it is clear that $W(z)$ does not vanish again on that part of the ray in S_i , regardless of the sign of m . Repeating this process for each ray radiating out from z_0 , we see clearly that $W(z)$ cannot vanish again in S_i . Since S_i is an arbitrary subregion, $W(z)$ can vanish at most at one point of each subregion.

On the other hand, if $W(z)$ has a zero z_i which belongs to n_i ($n_i > 1$) different subregions, then $W(z)$ cannot vanish again in any of these n_i subregions, as the foregoing proof shows. If it is known that there are k such zeros z_i , each z_i belonging to n_i subregions, $i = 1, 2, \dots, k$, it is clear that the total number of zeros of $W(z)$ in S is not greater than $n + k - (n_1 + n_2 + \dots + n_k)$.

To prove (g), let $W(z)$ be a solution of (1.1) having two zeros, say z_0 and z_1 , in some subregion S_i . Let the argument of $z_1 - z_0$ be θ . Then along the linear segment $z_0 z_1$, equation (1.1) becomes (4.2). According to Lemma 3, the maximum value m of the real part of $\exp(2i\theta)Q(z)$ on the linear segment $z_0 z_1$ must be positive. Further, since

$$(4.3) \quad |z_1 - z_0| \leq \pi M^{-1/2} \leq \pi m^{-1/2},$$

z_0 and z_1 can both be the zeros of $W(z)$ only if

$$(4.4) \quad e^{2i\theta}Q(z) \equiv m$$

on the linear segment $z_0 z_1$, by Lemma 3 again. But if (4.4) is true, the general solution of (4.2) is $A \sin(m^{1/2}r + B)$, A and B being constants. If a solution of (4.2) has two zeros, the distance between them must not be less than $\pi m^{-1/2}$. In other words, the equality signs in (4.3) must hold. That is, $M = m$. From (4.4), we have $\exp(2i\theta)Q(z) \equiv M$ on the linear segment $z_0 z_1$. Since $Q(z)$ is an analytic function and constant on the linear segment $z_0 z_1$, $Q(z)$ is a constant in D . Obviously $|Q(z)| = M$; and since m is positive, so is M . This proves (g).

Clearly (e) follows from (f), and this completes the proof of Theorem 2.

5. Added in proof. The author is indebted to a referee for calling his attention to the fact that, in connection with Corollary 1.1, an entire function which satisfies a linear differential equation with coefficients which are rational functions of z is always of finite rational order and of perfectly regular

growth. (See G. Valiron, *Lectures on the theory of integral functions*, Toulouse, 1923, p. 106 ff.)

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