METHODS OF SUMMATION

GORDON MARSHALL PETERSEN
1. Methods of Rogosinski and Bernstein. In this note we shall discuss certain matrix methods of summation, though otherwise § 1 and § 2 are unrelated. In this section we wish to consider some of the properties of the method \( (B^h) \), where we say that a series \( \sum_{\nu=0}^{\infty} u_\nu \) is summable \( (B^h) \) when

\[
B_n^h = \sum_{\nu=0}^{n} u_\nu \cos \frac{\pi}{2} \left( \frac{\nu}{n + h} \right) \to S, \quad n \to \infty.
\]

The method \( (B^h) \) has been the subject of recent papers by Agnew [1], Karamata [5, 6], and Petersen [7]. It has been shown in the papers by Agnew and Petersen that for \( h > 1/2 \) the method \( (B^h) \) is equivalent to the arithmetic means of Cesaro \((C)\), and in the paper by Agnew that for \( 0 < h < 1/2 \) the method is equivalent to methods stronger than \((C)\).

We shall now construct examples after a method of Hurwitz [4], to show that for \( h < 0 \) the method \( (B^h) \) sums a series not summable \((C)\). Hence, since all series summable \((C)\) are summable \( (B^h) \), we shall have proved that \( (B^h) \) is stronger than \((C)\).

We shall first consider \(-1 < h < 0\), so that all the coefficients in any row are positive except the \( n \)th coefficient \( \cos \{\pi n/[2(n + h)]\} \). We choose \( u_0 > 1 \) and assume that the first \( m - 1 \) terms of the series \( \sum_{\nu=0}^{\infty} u_\nu \) are known. Then we select \( u_m \) so that

\[
B_m^h = \sum_{\nu=0}^{m} u_\nu \cos \frac{\pi}{2} \left( \frac{\nu}{m + h} \right) = 0,
\]

or

\[
- u_m \cos \frac{\pi}{2} \left( \frac{m}{m + h} \right) = \sum_{\nu=0}^{m-1} u_\nu \cos \frac{\pi}{2} \left( \frac{\nu}{m + h} \right).
\]
All of the $u_\nu$ are positive; and since
\[
\frac{u_m}{u_{m-2}} \geq \frac{\sin \frac{\pi}{2} \left( \frac{2 + h}{m + h} \right)}{- \sin \frac{\pi}{2} \left( \frac{h}{m + h} \right)} \approx - \left( \frac{2}{h} + 1 \right)
\]
for $-1 < h < 0$, the $u_\nu$ do not satisfy $u_n = o(n)$, and hence $\sum_{\nu=0}^{\infty} u_\nu$ is not summable (C); see [3].

If $h \leq -1$, we consider
\[
B^h_m = \sum_{\nu=0}^{m-1} \left[ \cos \frac{\pi}{2} \left( \frac{\nu}{m + h} \right) - \cos \frac{\pi}{2} \left( \frac{\nu + 1}{m + h} \right) \right] S_\nu + \cos \frac{\pi}{2} \left( \frac{m}{m + h} \right) S_m.
\]

Here again we select positive increasing $S_\nu$ so that $B^h_\nu = 0$ for $\nu \leq m - 1$. Under the assumption that $S_\nu \geq \nu$, $\nu \leq m - 1$, we shall show that $S_m \geq m$. Observing that the first $m - 1$ coefficients of the $S_\nu$ are positive, we have (setting $\pi/[2(m + h)] = \theta$):
\[
- \cos m \theta \geq \sum_{\nu=0}^{m-1} \left[ \cos \nu \theta - \cos (\nu + 1) \theta \right] \nu
\]
\[
= \sum_{\nu=0}^{m-1} \cos \nu \theta - (m - 1) \cos m \theta
\]
\[
= \Re \sum_{\nu=0}^{m-1} e^{i\nu \theta} - (m - 1) \cos m \theta
\]
\[
= \Re \frac{1 - e^{im \theta}}{1 - e^{i \theta}} - (m - 1) \cos m \theta
\]
\[
= \Re \frac{i \left( e^{i \frac{\pi}{2}} - e^{i(m - 1)/2} \theta \right)}{2 \sin \frac{\theta}{2}} - (m - 1) \cos m \theta
\]
\[
\geq \left( \frac{1}{2} - \frac{\pi}{2} \right);
\]
therefore,
METHODS OF SUMMATION

\[ S_m \geq \left( \frac{1}{2} - \frac{\pi}{2} h \right) \frac{m + h}{-h} \times \frac{2}{\pi} \geq qm, \quad q > 1. \]

Hence the series constructed does not satisfy the condition \( S_n = o(n) \), and is not summable (C).

2. A Nörlund method. The method defined by

\[ \sigma_n = \left( 1 - \frac{1}{n + 3} \right) S_n + \frac{1}{n + 3} S_{n+1} \]

has been used as an example in a recent paper by Agnew [2]. We shall treat this method in a manner similar to that in which the method

\[ t_n = (1 - a)S_{n-1} + aS_n \]

is treated in [7].

**Theorem.** If

\[ \sigma_n = \left[ \left( 1 - \frac{1}{n + 3} \right) S_n + \frac{1}{n + 3} S_{n+1} \right] \rightarrow \sigma, \]

then

\[ S_n = C \cdot (-1)^{n-1}(n + 1)! + \sigma_n', \]

where \( \sigma_n' \) is convergent to \( \sigma \) and \( C \) is a constant.

**Proof.** Since (we may assume \( S_0 = 0 \))

\[
(n + 2) \sigma_{n-1} = (n + 1) S_{n-1} + S_n \\
(n + 1) \sigma_{n-2} = n S_{n-2} + S_{n-1} \\
\vdots \quad \quad \vdots \\
\vdots \quad \quad \vdots \\
3 \sigma_0 = 2 S_0 + S_1,
\]

we have
\[ S_n = (n + 2) \sigma_{n-1} - (n + 1)^2 \sigma_{n-2} + n^2 (n + 1) \sigma_{n-3} \]

\[ - (n - 1)^2 n (n + 1) \sigma_{n-4} + \ldots + (-1)^{n-2} 3^2 \cdot 4 \cdot 5 \cdot 6 \ldots (n + 1) \sigma_0, \]

or

\[ S_n = (-1)^{n-1} (n + 1)! \left[ (-1)^{n-1} \frac{n + 2}{(n + 1)!} \sigma_{n-1} - (-1)^{n-2} \frac{n + 1}{n!} \sigma_{n-2} + \ldots \right. \]

\[ + (-1)^\nu \frac{\nu + 3}{(\nu + 2)!} \sigma_\nu + \ldots + \frac{3}{2} \sigma_0 \right]. \]

Let

\[ (-1)^\nu \frac{\nu + 3}{(\nu + 2)!} \sigma_\nu = t_\nu; \]

since \( \sum_{\nu=0}^{\infty} t_\nu \) is absolutely convergent (\( \sigma_\nu \rightarrow \sigma \)), we may write

\[ t_0 + t_1 + \ldots + t_{n-1} = C - (t_n + t_{n+1} + \ldots) \]

\[ = C - \frac{1}{(n + 1)!} \left[ \frac{n + 3}{n + 2} \frac{(n + 2)!}{n + 3} t_n + \frac{n + 4}{(n + 2)(n + 3)} \frac{(n + 3)!}{n + 4} t_{n+1} + \ldots \right] \]

\[ = C - \frac{(-1)^n}{(n + 1)!} \left[ \frac{n + 3}{n + 2} \sigma_n - \frac{n + 4}{(n + 2)(n + 3)} \sigma_{n+1} + \ldots \right]. \]

Then

\[ S_n = (-1)^{n-1} (n + 1)! \left[ t_0 + t_1 + \ldots + t_{n-1} \right] \]

\[ = (-1)^{n-1} \cdot C \cdot (n + 1) + \left[ \frac{n + 3}{n + 2} \sigma_n - \frac{n + 4}{(n + 2)(n + 3)} \sigma_{n+1} + \ldots \right] \]

\[ = (-1)^{n-1} \cdot C \cdot (n + 1)! + \frac{n + 3}{n + 2} \sigma_n \]

\[ - \frac{1}{n + 2} \left[ \frac{n + 4}{n + 3} \sigma_{n+1} - \frac{n + 5}{(n + 3)(n + 4)} \sigma_{n+2} + \ldots \right] \]

\[ = (-1)^{n-1} \cdot C \cdot (n + 1)! + \frac{n + 3}{n + 2} \sigma_n - \frac{1}{n + 2} O(1) \]
This proves our assertion.

Obvious extensions can be made to the methods

$$\sigma_n = \left(1 - \frac{1}{n+k}\right) S_n + \frac{1}{n+k} S_{n+1},$$

or to iterations of these methods.

**References**


Pacific Journal of Mathematics  
Vol. 4, No. 1  
May, 1954

<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hugh D. Brunk</td>
<td>On the growth of functions having poles or zeros on the positive real axis</td>
<td>1</td>
</tr>
<tr>
<td>J. Copping</td>
<td>Application of a theorem of Pólya to the solution of an infinite matrix equation</td>
<td>21</td>
</tr>
<tr>
<td>James Richard Jackson</td>
<td>On the existence problem of linear programming</td>
<td>29</td>
</tr>
<tr>
<td>Victor Klee</td>
<td>Invariant extension of linear functionals</td>
<td>37</td>
</tr>
<tr>
<td>Shu-Teh Chen Moy</td>
<td>Characterizations of conditional expectation as a transformation on function spaces</td>
<td>47</td>
</tr>
<tr>
<td>Hukukane Nikaidô</td>
<td>On von Neumann’s minimax theorem</td>
<td>65</td>
</tr>
<tr>
<td>Gordon Marshall Petersen</td>
<td>Methods of summation</td>
<td>73</td>
</tr>
<tr>
<td>G. Power</td>
<td>Some perturbed electrostatic fields</td>
<td>79</td>
</tr>
<tr>
<td>Murray Harold Protter</td>
<td>The two noncharacteristic problem with data partly on the parabolic line</td>
<td>99</td>
</tr>
<tr>
<td>S. E. Rauch</td>
<td>Mapping properties of Cesàro sums of order two of the geometric series</td>
<td>109</td>
</tr>
<tr>
<td>Gerson B. Robison</td>
<td>Invariant integrals over a class of Banach spaces</td>
<td>123</td>
</tr>
<tr>
<td>Richard Steven Varga</td>
<td>Eigenvalues of circulant matrices</td>
<td>151</td>
</tr>
</tbody>
</table>