

Pacific Journal of Mathematics

METHODS OF SUMMATION

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METHODS OF SUMMATION

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1. Methods of Rogosinski and Bernstein. In this note we shall discuss certain matrix methods of summation, though otherwise § 1 and § 2 are unrelated. In this section we wish to consider some of the properties of the method (B^h), where we say that a series $\sum_{\nu=0}^{\infty} u_{\nu}$ is summable (B^h) when

$$B_n^h = \sum_{\nu=0}^n u_{\nu} \cos \frac{\pi}{2} \left(\frac{\nu}{n+h} \right) \rightarrow S, \quad n \rightarrow \infty.$$

The method (B^h) has been the subject of recent papers by Agnew [1], Karamata [5, 6], and Petersen [7]. It has been shown in the papers by Agnew and Petersen that for $h > 1/2$ the method (B^h) is equivalent to the arithmetic means of Cesaro (C), and in the paper by Agnew that for $0 < h < 1/2$ the method is equivalent to methods stronger than (C).

We shall now construct examples after a method of Hurwitz [4], to show that for $h < 0$ the method (B^h) sums a series not summable (C). Hence, since all series summable (C) are summable (B^h), we shall have proved that (B^h) is stronger than (C).

We shall first consider $-1 < h < 0$, so that all the coefficients in any row are positive except the n th coefficient $\cos \{ \pi n / [2(n+h)] \}$. We choose $u_0 > 1$ and assume that the first $m-1$ terms of the series $\sum_{\nu=0}^{\infty} u_{\nu}$ are known. Then we select u_m so that

$$B_m^h = \sum_{\nu=0}^m u_{\nu} \cos \frac{\pi}{2} \left(\frac{\nu}{m+h} \right) = 0,$$

or

$$-u_m \cos \frac{\pi}{2} \left(\frac{m}{m+h} \right) = \sum_{\nu=0}^{m-1} u_{\nu} \cos \frac{\pi}{2} \left(\frac{\nu}{m+h} \right).$$

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All of the u_ν are positive; and since

$$\frac{u_m}{u_{m-2}} \geq \frac{\sin \frac{\pi}{2} \left(\frac{2+h}{m+h} \right)}{-\sin \frac{\pi}{2} \left(\frac{h}{m+h} \right)} \simeq - \left(\frac{2}{h} + 1 \right)$$

for $-1 < h < 0$, the u_ν do not satisfy $u_n = o(n)$, and hence $\sum_{\nu=0}^{\infty} u_\nu$ is not summable (C); see [3].

If $h \leq -1$, we consider

$$B_m^h = \sum_{\nu=0}^{m-1} \left[\cos \frac{\pi}{2} \left(\frac{\nu}{m+h} \right) - \cos \frac{\pi}{2} \left(\frac{\nu+1}{m+h} \right) \right] S_\nu + \cos \frac{\pi}{2} \left(\frac{m}{m+h} \right) S_m.$$

Here again we select positive increasing S_ν so that $B_\nu^h = 0$ for $\nu \leq m-1$. Under the assumption that $S_\nu \geq \nu$, $\nu \leq m-1$, we shall show that $S_m \geq m$. Observing that the first $m-1$ coefficients of the S_ν are positive, we have (setting $\pi/[2(m+h)] = \theta$):

$$\begin{aligned} -\cos m\theta &\geq \sum_{\nu=0}^{m-1} [\cos \nu\theta - \cos(\nu+1)\theta] \nu \\ &= \sum_{\nu=0}^{m-1} \cos \nu\theta - (m-1) \cos m\theta \\ &= \Re \sum_{\nu=0}^{m-1} e^{i\nu\theta} - (m-1) \cos m\theta \\ &= \Re \frac{1 - e^{im\theta}}{1 - e^{i\theta}} - (m-1) \cos m\theta \\ &= \Re \frac{i(e^{-i\theta/2} - e^{i(m-1/2)\theta})}{2 \sin \theta/2} - (m-1) \cos m\theta \\ &\geq \left(\frac{1}{2} - \frac{\pi}{2} h \right); \end{aligned}$$

therefore,

$$S_m \geq \left(\frac{1}{2} - \frac{\pi}{2}h\right) \frac{m+h}{-h} \times \frac{2}{\pi} \geq qm, \quad q > 1.$$

Hence the series constructed does not satisfy the condition $S_n = o(n)$, and is not summable (C).

2. A Nörlund method. The method defined by

$$\sigma_n = \left(1 - \frac{1}{n+3}\right) S_n + \frac{1}{n+3} S_{n+1}$$

has been used as an example in a recent paper by Agnew [2]. We shall treat this method in a manner similar to that in which the method

$$t_n = (1-a)S_{n-1} + aS_n$$

is treated in [7].

THEOREM. *If*

$$\sigma_n = \left[\left(1 - \frac{1}{n+3}\right) S_n + \frac{1}{n+3} S_{n+1} \right] \rightarrow \sigma,$$

then

$$S_n = C \cdot (-1)^{n-1} (n+1)! + \sigma'_n,$$

where σ'_n is convergent to σ and C is a constant.

Proof. Since (we may assume $S_0 = 0$)

$$\begin{aligned} (n+2) \sigma_{n-1} &= (n+1) S_{n-1} + S_n \\ (n+1) \sigma_{n-2} &= \quad n S_{n-2} + S_{n-1} \\ &\cdot \qquad \qquad \cdot \qquad \cdot \\ &\cdot \qquad \qquad \cdot \qquad \cdot \\ &\cdot \qquad \qquad \cdot \qquad \cdot \\ 3 \sigma_0 &= \quad 2 S_0 + S_1, \end{aligned}$$

we have

$$S_n = (n+2)\sigma_{n-1} - (n+1)^2\sigma_{n-2} + n^2(n+1)\sigma_{n-3} \\ - (n-1)^2n(n+1)\sigma_{n-4} + \dots + (-1)^{n-2}3^2 \cdot 4 \cdot 5 \cdot 6 \dots (n+1)\sigma_0,$$

or

$$S_n = (-1)^{n-1}(n+1)! \left[(-1)^{n-1} \frac{n+2}{(n+1)!} \sigma_{n-1} + (-1)^{n-2} \frac{n+1}{n!} \sigma_{n-2} + \dots \right. \\ \left. + (-1)^\nu \frac{\nu+3}{(\nu+2)!} \sigma_\nu + \dots + \frac{3}{2} \sigma_0 \right].$$

Let

$$(-1)^\nu \frac{\nu+3}{(\nu+2)!} \sigma_\nu = t_\nu;$$

since $\sum_{\nu=0}^{\infty} t_\nu$ is absolutely convergent ($\sigma_\nu \rightarrow \sigma$), we may write

$$t_0 + t_1 + \dots + t_{n-1} = C - (t_n + t_{n+1} + \dots) \\ = C - \frac{1}{(n+1)!} \left[\frac{n+3}{n+2} \frac{(n+2)!}{n+3} t_n + \frac{n+4}{(n+2)(n+3)} \frac{(n+3)!}{n+4} t_{n+1} + \dots \right] \\ = C - \frac{(-1)^n}{(n+1)!} \left[\frac{n+3}{n+2} \sigma_n - \frac{n+4}{(n+2)(n+3)} \sigma_{n+1} + \dots \right].$$

Then

$$S_n = (-1)^{n-1}(n+1)! [t_0 + t_1 + \dots + t_{n-1}] \\ = (-1)^{n-1} \cdot C \cdot (n+1) + \left[\frac{n+3}{n+2} \sigma_n - \frac{n+4}{(n+2)(n+3)} \sigma_{n+1} + \dots \right] \\ = (-1)^{n-1} \cdot C \cdot (n+1)! + \frac{n+3}{n+2} \sigma_n \\ - \frac{1}{n+2} \left[\frac{n+4}{n+3} \sigma_{n+1} - \frac{n+5}{(n+3)(n+4)} \sigma_{n+2} + \dots \right] \\ = (-1)^{n-1} \cdot C \cdot (n+1)! + \frac{n+3}{n+2} \sigma_n - \frac{1}{n+2} O(1)$$

$$= (-1)^{n-1} \cdot C \cdot (n+1)! + \sigma_n + o(1).$$

This proves our assertion.

Obvious extensions can be made to the methods

$$\sigma_n = \left[\left(1 - \frac{1}{n+k} \right) S_n + \frac{1}{n+k} S_{n+1} \right],$$

or to iterations of these methods.

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