METHODS OF SUMMATION

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1. Methods of Rogosinski and Bernstein. In this note we shall discuss certain matrix methods of summation, though otherwise § 1 and § 2 are unrelated. In this section we wish to consider some of the properties of the method \((B^h)\), where we say that a series \(\sum_{\nu=0}^{\infty} u_{\nu}\) is summable \((B^h)\) when

\[
B^h_n = \sum_{\nu=0}^{n} u_{\nu} \cos \frac{\pi}{2} \left( \frac{\nu}{n + h} \right) \rightarrow S, \ n \rightarrow \infty.
\]

The method \((B^h)\) has been the subject of recent papers by Agnew [1], Karamata [5, 6], and Petersen [7]. It has been shown in the papers by Agnew and Petersen that for \(h > 1/2\) the method \((B^h)\) is equivalent to the arithmetic means of Cesaro \((C)\), and in the paper by Agnew that for \(0 < h < 1/2\) the method is equivalent to methods stronger than \((C)\).

We shall now construct examples after a method of Hurwitz [4], to show that for \(h < 0\) the method \((B^h)\) sums a series not summable \((C)\). Hence, since all series summable \((C)\) are summable \((B^h)\), we shall have proved that \((B^h)\) is stronger than \((C)\).

We shall first consider \(-1 < h < 0\), so that all the coefficients in any row are positive except the \(n\)th coefficient \(\cos \left\{ \pi n / [2(n + h)] \right\}\). We choose \(u_0 > 1\) and assume that the first \(m - 1\) terms of the series \(\sum_{\nu=0}^{\infty} u_{\nu}\) are known. Then we select \(u_m\) so that

\[
B^h_m = \sum_{\nu=0}^{m} u_{\nu} \cos \frac{\pi}{2} \left( \frac{\nu}{m + h} \right) = 0,
\]

or

\[
- u_m \cos \frac{\pi}{2} \left( \frac{m}{m + h} \right) = \sum_{\nu=0}^{m-1} u_{\nu} \cos \frac{\pi}{2} \left( \frac{\nu}{m + h} \right).
\]
All of the $u_{\nu}$ are positive; and since

$$\frac{u_m}{u_{m-2}} \geq \frac{\sin \frac{\pi}{2} \left( \frac{2 + h}{m + h} \right)}{-\sin \frac{\pi}{2} \left( \frac{h}{m + h} \right)} \simeq -\left( \frac{2}{h} + 1 \right)$$

for $-1 < h < 0$, the $u_{\nu}$ do not satisfy $u_n = o(n)$, and hence $\sum_{\nu=0}^\infty u_{\nu}$ is not summable (C); see [3].

If $h \leq -1$, we consider

$$B^h_m = \sum_{\nu=0}^{m-1} \left[ \cos \frac{\pi}{2} \left( \frac{\nu}{m + h} \right) - \cos \frac{\pi}{2} \left( \frac{\nu + 1}{m + h} \right) \right] S_\nu + \cos \frac{\pi}{2} \left( \frac{m}{m + h} \right) S_m.$$

Here again we select positive increasing $S_\nu$ so that $B^h_\nu = 0$ for $\nu \leq m - 1$. Under the assumption that $S_\nu \geq \nu$, $\nu \leq m - 1$, we shall show that $S_m \geq m$.

Observing that the first $m - 1$ coefficients of the $S_\nu$ are positive, we have (setting $\pi/[2(m + h)] = \theta$):

$$-\cos m \theta \geq \sum_{\nu=0}^{m-1} \left[ \cos \nu \theta - \cos (\nu + 1) \theta \right] \nu$$

$$= \sum_{\nu=0}^{m-1} \cos \nu \theta - (m - 1) \cos m \theta$$

$$= \Re \sum_{\nu=0}^{m-1} e^{i \nu \theta} - (m - 1) \cos m \theta$$

$$= \Re \frac{1 - e^{im \theta}}{1 - e^{i \theta}} - (m - 1) \cos m \theta$$

$$= \Re \frac{i (e^{-(i \theta)/2} - e^{i(m-1)/2})}{2 \sin \theta/2} - (m - 1) \cos m \theta$$

$$\geq \left( \frac{1}{2} - \frac{\pi}{2} h \right);$$

therefore,
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\[ S_m \geq \left( \frac{1}{2} - \frac{\pi}{2} h \right) \frac{m + h}{-h} \times \frac{2}{\pi} \geq q m, \quad q > 1. \]

Hence the series constructed does not satisfy the condition \( S_n = o(n) \), and is not summable (C).

2. A Nörlund method. The method defined by

\[ \sigma_n = \left( 1 - \frac{1}{n + 3} \right) S_n + \frac{1}{n + 3} S_{n+1} \]

has been used as an example in a recent paper by Agnew [2]. We shall treat this method in a manner similar to that in which the method

\[ t_n = (1 - a) S_{n-1} + a S_n \]

is treated in [7].

**Theorem.** If

\[ \sigma_n = \left[ \left( 1 - \frac{1}{n + 3} \right) S_n + \frac{1}{n + 3} S_{n+1} \right] \rightarrow \sigma, \]

then

\[ S_n = C \cdot (-1)^{n-1} (n + 1)! + \sigma'_n, \]

where \( \sigma'_n \) is convergent to \( \sigma \) and \( C \) is a constant.

**Proof.** Since (we may assume \( S_0 = 0 \))

\[
\begin{align*}
(n + 2) \sigma_{n-1} &= (n + 1) S_{n-1} + S_n \\
(n + 1) \sigma_{n-2} &= n S_{n-2} + S_{n-1} \\
& \quad \vdots \\
3 \sigma_0 &= 2 S_0 + S_1 
\end{align*}
\]

we have
\[ S_n = (n + 2) \sigma_{n-1} - (n + 1)^2 \sigma_{n-2} + n^2 (n + 1) \sigma_{n-3} - (n - 1)^2 n(n + 1) \sigma_{n-4} + \cdots + (-1)^{n-2} 3^2 \cdot 4 \cdot 5 \cdot 6 \cdots (n + 1) \sigma_0, \]

or

\[ S_n = (-1)^n (n + 1)! \left[ (-1)^n \frac{n + 2}{(n + 1)!} \sigma_{n-1} + (-1)^{n-2} \frac{n + 1}{n!} \sigma_{n-2} + \cdots \right. \]

\[ \left. + (-1)^{\nu} \frac{\nu + 3}{(\nu + 2)!} \sigma_{\nu} + \cdots + \frac{3}{2} \sigma_0 \right]. \]

Let

\[ (-1)^{\nu} \frac{\nu + 3}{(\nu + 2)!} \sigma_{\nu} = t_\nu; \]

since \( \sum_{\nu=0}^\infty t_\nu \) is absolutely convergent (\( \sigma_\nu \to \sigma \)), we may write

\[ t_0 + t_1 + \cdots + t_{n-1} = C - (t_n + t_{n+1} + \cdots) \]

\[ = C - \frac{1}{(n + 1)!} \left[ \frac{n + 3}{n + 2} \frac{(n + 2)!}{n + 3} t_n + \frac{n + 4}{(n + 2)(n + 3)} \frac{(n + 3)!}{n + 4} t_{n+1} + \cdots \right] \]

\[ = C - \frac{(-1)^n}{(n + 1)!} \left[ \frac{n + 3}{n + 2} \sigma_n - \frac{n + 4}{(n + 2)(n + 3)} \sigma_{n+1} + \cdots \right]. \]

Then

\[ S_n = (-1)^{n-1} (n + 1)! \left[ t_0 + t_1 + \cdots + t_{n-1} \right] \]

\[ = (-1)^{n-1} \cdot C \cdot (n + 1) + \left[ \frac{n + 3}{n + 2} \sigma_n - \frac{n + 4}{(n + 2)(n + 3)} \sigma_{n+1} + \cdots \right] \]

\[ = (-1)^{n-1} \cdot C \cdot (n + 1) + \frac{n + 3}{n + 2} \sigma_n \]

\[ - \frac{1}{n + 2} \left[ \frac{n + 4}{n + 3} \sigma_{n+1} - \frac{n + 5}{(n + 3)(n + 4)} \sigma_{n+2} + \cdots \right] \]

\[ = (-1)^{n-1} \cdot C \cdot (n + 1) + \frac{n + 3}{n + 2} \sigma_n - \frac{1}{n + 2} O(1) \]
This proves our assertion.

Obvious extensions can be made to the methods

\[
\sigma_n = \left[ \left( 1 - \frac{1}{n+k} \right) S_n + \frac{1}{n+k} S_{n+1} \right],
\]

or to iterations of these methods.

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