GENERALIZATIONS OF THE ROGERS-RAMANUJAN IDENTITIES

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1. Introduction. The first of the two Rogers-Ramanujan identities [1, Chap. 19] states that

\[
\prod_{\nu=0}^{\infty} \frac{1}{(1-x^{5\nu+1})(1-x^{5\nu+4})} = \sum_{\mu=0}^{\infty} \frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^\mu)},
\]

where the left side is the generating function for the number of partitions into parts not congruent to 0, ±2 (mod 5). This paper shows that as a generalization of (1) the generating function for the number of partitions into parts not congruent to 0, ±k (mod 2k + 1), where k is any positive integer, can be expressed as a sum similar to the one appearing in (1); in fact in general the \(x^{\mu^2}\) are replaced by polynomials \(G_k, \mu(x)\), so that we have the following theorem:

**Theorem 1.** The following identity holds:

\[
\prod_{\nu=0}^{\infty} \frac{1}{(1-x^{(2k+1)\nu+k+1})(1-x^{(2k+1)\nu+k})} = \sum_{\mu=0}^{\infty} \frac{G_{k, \mu}(x)}{(1-x)(1-x^2)\cdots(1-x^\mu)},
\]

where the left side is the generating function for the number of partitions into parts not congruent to 0, ±k (mod 2k + 1). The \(G_{k, \mu}(x)\) are polynomials in \(x\) and reduce to the monomial \(x^{\mu^2}\) for \(k = 2\), that is, for the Rogers-Ramanujan case.

While the right side of (1) is the generating function for the number of

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partitions into parts differing by at least 2, no similar interpretation of the right hand of (2) is possible. In particular, it follows from a theorem of the author [2] that the right side of (2) cannot be interpreted as the generating function for the number of partitions of $n$ into parts differing by at least $d$, each part being greater than or equal to $m$, unless $d = 2$, $m = 1$, that is, unless we have the Rogers-Ramanujan identity (1).

As a generalization of the second of the Rogers-Ramanujan identities:

\begin{equation}
\prod_{\nu=0}^{\infty} \frac{1}{(1-x^{5\nu+2})(1-x^{5\nu+3})} = \sum_{\mu=0}^{\infty} \frac{x^{\mu^{2}+\mu}}{(1-x)(1-x^{2})\cdots(1-x^{\mu})},
\end{equation}

we have again that not only the generating function for the number of partitions into parts not congruent to 0, ±1 (mod 5), but in general the one for the number of partitions into parts not congruent to 0, ±1 (mod $2k+1$) can be expressed as a sum; in fact again the $x^{\mu^{2}}$ are replaced by the same polynomials $G_{k,\mu}(x)$ appearing in (2), so that we have the following theorem:

**Theorem 2.** The following identity holds:

\begin{equation}
\prod_{\nu=0}^{\infty} \frac{1}{(1-x^{(2k+1)\nu+2})(1-x^{(2k+1)\nu+3})\cdots(1-x^{(2k+1)\nu+2k-1})} = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)x^{\mu}}{(1-x)(1-x^{2})\cdots(1-x^{\mu})}.
\end{equation}

More generally, it can be shown that identities involving the generating function for the number of partitions into parts not congruent to 0, ±$(k-r)$ (mod $2k+1$), where $0 \leq r \leq k-1$, can be obtained, of which (2) is the particular case where $r = k-1$, that is, for each modulus $2k+1$ there are $k$ identities.

2. **Proof of Theorem 1:** If we replace, in Jacobi's identity,

\begin{equation}
\prod_{\nu=0}^{\infty} (1-y^{2\nu+2}) \left[1+(z+z^{-1})y^{2\nu+1}+y^{4\nu+2}\right] = \sum_{\mu=-\infty}^{\infty} y^{\mu^{2}}z^{\mu},
\end{equation}

$y$ by $x^{(2k+1)/2}$ and $z$ by $-x^{1/2}$, we have
so that, dividing both sides of (6) by \((1 - x)(1 - x^2)(1 - x^3)\ldots\), we obtain

\[
(7) \quad \prod_{\nu=0}^{\infty} \frac{(1 - x^{(2k+1)\nu+k})(1 - x^{(2k+1)\nu+k+1})}{(1 - x^{(2k+1)\nu+1})(1 - x^{(2k+1)\nu+2}) \cdots (1 - x^{(2k+1)\nu+2k})}
\]

\[
\sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2 + \mu)/2} \cdot \prod_{s=1}^{\infty} (1 - x^s)
\]

To prove Theorem 1, we therefore have to show that the right side of (7) is the same as the right side of (2).

We use the auxiliary function

\[
(8) \quad C_{k,i}(y) = 1 - y^i x^i + \sum_{\mu=1}^{\infty} (-1)^{\mu} y^k \mu^2 x^{(2k+1)(\mu^2 + \mu)/2} - i\mu
\]

\[
(1 - y^i x^{(2\mu+1)i}) \cdot \frac{(1 - yx)(1 - yx^2) \cdots (1 - yx^\mu)}{(1 - x)(1 - x^2) \cdots (1 - x^\mu)}
\]

which was first used by Selberg [3] and is a generalization of the function used in some proofs of the Rogers-Ramanujan identities [1, Chap. 19]. The function (8) converges if \(|y| < 1\) and if \(k\) is real and \(\geq -1/2\). In our case \(k\) and \(i\) will be nonnegative integers. For \(i = k\) and \(y = 1\), (8) reduces to

\[
(9) \quad C_{k,k}(1) = \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2 + \mu)/2}.
\]

Since the \(C_{k,i}(y)\) satisfy the equation
it is easily seen that we can find a functional equation for the $C_{k,i}(y)$, which can be found to be of the form

\begin{equation}
C_{k,i}(y) = C_{k,i-1}(y) + y^{i-1}x^{i-1}(1-yx)C_{k,k-i+1}(yx),
\end{equation}

(10)

If we let

\begin{equation}
Q_k(y) = \frac{C_{k,k}(y)}{\prod_{s=1}^{\infty} (1-yx^s)},
\end{equation}

(11)

(10) reduces to

\begin{equation}
Q_k(y) = \sum_{\mu=1}^{k} A_{k,\mu}(y,x)Q_k(yx^\mu).
\end{equation}

(12)

If, for instance, $k = 3$, (12) becomes

\begin{equation}
Q_3(y) = (1+yx)Q_3(yx) + y^2x^2Q_3(yx^2) - y^3x^5Q_3(yx^3),
\end{equation}

(13)

while for $k = 4$ we would have

\begin{equation}
Q_4(y) = (1+yx)Q_4(yx) + y^2x^2(1+yx+yx^2)Q_4(yx^2)

- y^4x^7Q_4(yx^3) - y^6x^{13}Q_4(yx^4).
\end{equation}

(14)

In order to solve (12) for $Q_k(y)$ we try a solution of the form

\begin{equation}
Q_k(y) = \sum_{\mu=0}^{\infty} B_{k,\mu}(x)y^\mu,
\end{equation}

(15)

where $B_{k,0}(x) = Q_k(0) = 1$ by use of (11) and (8).

Putting (15) into (12) we obtain a difference equation for the $B_{k,\mu}(x)$. It can easily be verified that the $B_{k,\mu}(x)$ are of the form

\begin{equation}
B_{k,\mu}(x) = \frac{G_{k,\mu}(x)}{(1-x)(1-x^2)\cdots(1-x^\mu)},
\end{equation}

(16)
where the \( G_{k,\mu}(x) \) are polynomials in \( x \) and reduce to the monomial \( x^{\mu^2} \) for \( k = 2 \). In general these polynomials do not seem to possess any striking properties, even for small values of \( k \) and \( \mu \), as shall be illustrated below for \( k = 3 \) and \( k = 4 \).

Substituting now (16) into (15), and remembering (11), we obtain

\[
Q_k(y) = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)y^\mu}{(1-x)(1-x^2)\cdots(1-x^\mu)} = \frac{C_{k,k}(y)}{\prod_{s=1}^{\infty}(1-yx^s)},
\]

so that we have, in view of (9),

\[
\frac{G_{k,k}(1)}{\prod_{s=1}^{\infty}(1-x^s)} = \sum_{\mu=0}^{\infty} (-1)^\mu x^{(2k+1)\mu^2 + \mu}/2 \prod_{s=1}^{\infty}(1-x^s)
\]

\[
= \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)}{(1-x)(1-x^2)\cdots(1-x^\mu)},
\]

which completes the proof of the theorem.

In case \( k = 3 \), the difference equation for the \( B_{3,\mu}(x) \), which can easily be obtained from (13), is the following:

\[
B_{3,\mu}(x)(1-x^\mu) = B_{3,\mu-1}(x)x^\mu + B_{3,\mu-2}(x)x^{2\mu-2} - B_{3,\mu-3}(x)x^{3\mu-4},
\]

from which we calculate, remembering that \( B_{3,0}(x) = 1 \):

\[
G_{3,1}(x) = x,
\]

\[
G_{3,2}(x) = x^2,
\]

\[
G_{3,3}(x) = x^5 + x^6 - x^8,
\]

\[
G_{3,4}(x) = x^8 + x^{10} - x^{14},
\]

\[
G_{3,5}(x) = x^{13} + x^{14} + x^{15} - x^{18} - x^{19},
\]

\[
G_{3,6}(x) = x^{18} + x^{20} + x^{21} + x^{22} - x^{25} - x^{26} - x^{27} - x^{28} + x^{31},
\]

\[
G_{3,7}(x) = x^{25} + x^{26} + x^{27} + x^{28} + x^{29} - x^{32} - x^{33} - x^{34} - x^{35} - x^{36} + x^{42},
\]
It can easily be verified by induction that the degree of the \( G_3, \mu(x) \) is equal to
\[
\frac{5\mu^2 + \mu}{6} \quad \text{if } \mu \equiv 0 \text{ or } 1 \pmod{3},
\]
and is less than or equal to
\[
\frac{5\mu^2 - \mu - 6}{6} \quad \text{if } \mu \equiv 2 \pmod{3}.
\]

Similarly, it can be shown that the term with smallest exponent in each polynomial \( G_3, \mu(x) \) is \( x^\left\lfloor \frac{(\mu^2 + 1)}{2} \right\rfloor \), so that each polynomial has this power of \( x \) as a divisor and no higher power.

For \( k = 4 \), we obtain the difference equation for the \( B_4, \mu(x) \) from (14):
\[
(20) \quad B_4, \mu(x)(1 - x^\mu) = B_4, \mu - 1(x)x^\mu + B_4, \mu - 2(x)x^{2\mu - 2} + B_4, \mu - 3(x)x^{2\mu - 3}(x + 1) - B_4, \mu - 4(x)x^{3\mu - 5} - B_4, \mu - 6(x)x^{4\mu - 11},
\]
so that we obtain:
\[
G_4, 0(x) = 1, \\
G_4, 1(x) = x, \\
G_4, 2(x) = x^2, \\
G_4, 3(x) = x^3, \\
G_4, 4(x) = x^6 + x^7 + x^8 - x^9 - x^{10} - x^{11} + x^{13}, \\
G_4, 5(x) = x^9 + x^{10} + x^{11} - x^{14} - x^{15} - x^{16} + x^{20}, \\
G_4, 6(x) = x^{12} + x^{14} + x^{15} + x^{16} - x^{19} - 2x^{20} - x^{21} - x^{22} + x^{25} + x^{26}, \\
G_4, 7(x) = x^{17} + x^{18} + 2x^{19} + x^{20} + x^{21} - x^{23} - 2x^{24} - 2x^{25} - 2x^{26} - x^{27} + x^{30} + x^{31} + x^{32},
\]
and so on.

In this case the term with smallest exponent can be shown to equal \( x^{[(\mu^2+2)/3]} \), while for \( G_{\mu}(x) \) we would find the corresponding term to be \( x^{[(\mu^2+3)/4]} \) for \( \mu > 2 \), and so on.

3. Proof of Theorem 2. From the definition of \( C_{k,\mu}(x) \) we find

\[
(1-x)C_{k,\mu}(x) = \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+(2k-1)\mu)/2}.
\]

Substituting now, in Jacobi's identity (5), \( x^{(2k+1)/2} \) for \( y \) and \( -x^{(2k-1)/2} \) for \( z \), and dividing at the same time both sides by \((1-x)(1-x^2)(1-x^3)\ldots\), we obtain

\[
\prod_{\nu=0}^{\infty} \frac{1}{(1-x^{(2k+1)\nu+2})(1-x^{(2k+1)\nu+3})\ldots(1-x^{(2k+1)\nu+2k-1})}
= \frac{\sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+(2k-1)\mu)/2}}{\prod_{s=1}^{\infty} (1-x^s)}
= \frac{(1-x)C_{k,\mu}(x)}{\prod_{s=1}^{\infty} (1-x^s)} = Q_k(x) = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)x^\mu}{(1-x)(1-x^2)\ldots(1-x^\mu)},
\]

if we recall (11), (15), and (16).

Identities involving the generating function for the number of partitions into parts not congruent to 0, \( \pm(k-r) \pmod{2k+1} \), where \( 0 \leq r \leq k-1 \), can be obtained by noting that, using Jacobi's identity with \( y = x^{(2k+1)/2} \) and \( z = -x^{(2r+1)/2} \), we obtain

\[
\prod_{\nu=0}^{\infty} \left[ (1-x^{(2k+1)\nu+k-r})(1-x^{(2k+1)\nu+k+r+1})(1-x^{(2k+1)\nu+(2k+1)}) \right]
= \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+(2r+1)\mu)/2},
\]
where the right side, as can be verified, is expressible in terms of $C_{k,k}(y)$, which was shown already for $r = 0$ by Theorem 1 and for $r = k - 1$ by Theorem 2 and shall only be indicated here for $r = 1$, where we find

\[(23) \quad C_{k,k}(1) - x^{k-1}(1-x)(1-x^2)C_{k,k}(x^2) \]

\[= \sum_{\mu = -\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2 + 3\mu)/2}. \]

This method therefore allows us to find for each modulus $2k + 1$ exactly $k$ identities, that is, one for each value of $r$ in $0 \leq r \leq k - 1$.

References


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