

Pacific Journal of Mathematics

NOTE ON THE SCHWARZ TRIANGLE FUNCTIONS

JOSEPH LEHNER

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1. Introduction. In his classical investigation of the hypergeometric series, Schwarz discussed the function which maps the upper half w -plane onto a curvilinear triangle in the z -plane with angles $\delta\pi, \epsilon\pi, \eta\pi$ ($\delta + \epsilon + \eta < 1$). The inverse, $w = \phi(z)$, of this function is automorphic with respect to the group got by reflecting the triangle in its sides, reflecting the new figure in *its* free sides, and so on. In order that this process shall lead to a properly discontinuous group, it is necessary and sufficient that $1/\delta, 1/\epsilon, 1/\eta$ be positive integers or ∞ . We take in particular $\delta = 1/q, \epsilon = 1/2, \eta = 0$ ($q = 3, 4, 5, \dots$), and place the triangle in the upper half plane with vertices at $-\exp \pi i/q, i$, and $i\infty$. The group $\Gamma(\lambda)$ of transformations is then generated by

$$S : z \rightarrow z + \lambda \quad \text{and} \quad T : z \rightarrow -\frac{1}{z},$$

where $\lambda = 2 \cos \pi/q$ ($q = 3, 4, 5, \dots$). (We restrict λ to this countable set from now on.) The automorphic function $\phi_\lambda(z) = \phi(z)$ having a simple pole at $z = i\infty$ thus has the period λ , and we normalize its Fourier expansion as follows:

$$(1.1) \quad \phi_\lambda(z) = \phi(z) = x^{-1} + \sum_{n=0}^{\infty} c_n(\lambda) x^n, \quad x = \exp \frac{2\pi i z}{\lambda}.$$

This makes $\phi(z)$ unique except for an additive constant; ϕ is called a triangle function.

Concerning the Fourier coefficients $c_n(\lambda)$, we wish to make the following observations:

I. *All the Fourier coefficients of any triangle function ϕ_λ are rational numbers.*

II. *The Fourier coefficients $c_n(\lambda)$ have the asymptotic value*

Received March 2, 1953.

Pacific J. Math. 4 (1954), 243-249

$$(1.2) \quad c_n(\lambda) \sim \sqrt{\frac{1}{2\lambda}} \frac{e^{4\pi\sqrt{n}/\lambda}}{n^{3/4}}, \quad n \rightarrow \infty.$$

Both results can be extended to a wider class of Fuchsian groups; this will be done in future publications.¹

2. Proof of I. Let $z = \psi(w)$ be the function inverse to ϕ ; that is, ψ maps the upper half w -plane onto the triangle in the z -plane. It is well known [1, p. 304 f] that ψ is the quotient of two independent solutions of the hypergeometric equation

$$(2.1) \quad w(w-1) \frac{d^2z}{dw^2} + [(\alpha + \beta + 1)w - \gamma] \frac{dz}{dw} + \alpha\beta z = 0,$$

where

$$\alpha = \beta = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), \quad \gamma = 1 - \frac{1}{q}.$$

In this case ($\alpha = \beta$), Fricke [2, p. 115, (18)] has given an explicit representation of a system of independent solutions valid at $w = \infty$:

$$(2.2) \quad \begin{aligned} Z_1 &= w^{-\alpha} F(\alpha, \alpha - \gamma + 1, 1; 1/w), \\ Z_2 &= w^{-\alpha} [F_1(\alpha, \alpha - \gamma + 1; 1/w) - \log w \cdot F(\alpha, \alpha - \gamma + 1, 1; 1/w)], \end{aligned}$$

where F is the ordinary hypergeometric series, and F_1 is a series with coefficients rational in α, β [2, p. 114, (15)],

$$F_1(\alpha, \beta; u) = \frac{\alpha \cdot \beta}{1.1} \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{2}{1} \right) u + \dots.$$

Both series converge for $|w| > 1$.

For our purposes we take for $z = \psi(w)$ the combination

$$-\frac{2\pi iz}{\lambda} = -\frac{Z_2}{Z_1} = \log w - \frac{F_1}{F} = \log w + \frac{A_1}{w} + \frac{A_2}{w^2} + \dots,$$

¹When $\phi(z)$ is Klein's absolute modular invariant $J(z)$, (1.2) is an immediate consequence of the Petersson-Rademacher [3, p. 202; 4] series for $J(z)$.

where, as we see from (2.2), A_1, A_2, \dots , are rational numbers. Hence with $x = \exp 2\pi i z/\lambda$, we have

$$x^{-1} = w \cdot \exp\left(\frac{A_1}{w} + \frac{A_2}{w^2} + \dots\right) = w\left(1 + \frac{B_1}{w} + \frac{B_2}{w^2} + \dots\right),$$

where again the B_n are rational.

We now invert this equation, setting

$$(2.3) \quad w = \phi(z) = x^{-1}(1 + c_0x + c_1x^2 + \dots),$$

and have:

$$\begin{aligned} w^{-1} &= x(1 + d_0x + d_1x^2 + \dots), \\ x^{-1} &= x^{-1}(1 + c_0x + c_1x^2 + \dots)(1 + B_1x(1 + d_0x + d_1x^2 + \dots) \\ &\quad + B_2x^2(1 + d'_0x + d'_1x^2 + \dots) + \dots). \end{aligned}$$

The last equation determines the c_n uniquely in a step-by-step manner. They clearly are rational numbers. Furthermore, (2.3) agrees with (1.1). This proves I.

3. Proof of II. From (1.1) we have

$$c_n(\lambda) = \frac{1}{\lambda} \int_C \phi(z) e^{-2\pi inz/\lambda} dz \quad (n > 0),$$

where C is a path connecting any two points in the upper half plane at the same height and at a distance λ apart. We take C to be the horizontal line

$$z = x + \frac{i}{N}, \quad |x| \leq \frac{\lambda}{2};$$

$N > 0$ will eventually be taken of the order of \sqrt{n} .

The line C cuts a finite number of fundamental regions of $\Gamma(\lambda) = R_1, R_2, \dots, R_s$; the corresponding segments are l_1, l_2, \dots, l_s . Thus

$$\lambda c_n(\lambda) = \sum_{j=1}^s \int_{l_j} \phi(z) e^{-2\pi inz/\lambda} dz.$$

There is a unique substitution

$$(3.1) \quad z' = \frac{a_j z + b_j}{c_j z + d_j}$$

of $\Gamma(\lambda)$ which carries R_j into R_0 , the standard fundamental region with cusp at $i\infty$; the coefficients a_j, b_j, \dots are real, and $c_j \neq 0$. Thus because of the invariance on ϕ on Γ , we get

$$\lambda c_n(\lambda) = \sum_{j=1}^s \int_{l_j} \phi(z') e^{-2\pi i n z / \lambda} dz,$$

where z' lies in R_0 .

Now, by (1.1), write

$$\phi(z) = e^{-2\pi i z / \lambda} + \psi(z), \quad \psi(z) = \sum_0^{\infty} c_m e^{2\pi i m z / \lambda};$$

then

$$(3.2) \quad \lambda c_n(\lambda) = \sum_{j=1}^s \int_{l_j} e^{-2\pi i (z' + n z) / \lambda} dz + \sum_{j=1}^s \int_{l_j} \psi(z') e^{-2\pi i n z / \lambda} dz$$

$$= \sum_{j=1}^s H_j + S_2 = S_1 + S_2.$$

In the following estimates, A will denote a constant, not the same one at each appearance, independent of N and n but possibly depending on λ ; θ is an absolute constant of modulus less than unity.

We know that $\psi(z')$ is bounded in R_0 because ϕ is regular in the upper half plane except for a simple pole at $i\infty$; put $|\psi(z')| \leq A$. Hence

$$(3.3) \quad |S_2| \leq A e^{2\pi n / N\lambda} \int_C |dz| \leq A e^{2\pi n / N\lambda}.$$

The principal contribution to S_1 will come from the segment lying in the fundamental region, R_1 say, which is the map of R_0 by $T: z' = -1/z$. R_1 is bounded by an arc of the unit circle and by two arcs passing through the origin,

the right-hand one having the equation

$$\left(\frac{x-1}{\lambda}\right)^2 + y^2 = \frac{1}{\lambda^2} \quad (z = x + iy).$$

Hence the endpoints of l_1 are $\pm z_1$, where

$$z_1 = \frac{\theta A}{N^2} + \frac{i}{N}.$$

Let K be the circle, described counter clockwise, with center at the origin and passing through z_1, z_2 , and L the larger of the arcs connecting z_1, z_2 . We have

$$H_1 = \int_{l_1} = - \int_K \int_L = J_1 + J_2,$$

the integrands being the same as in the first term of the right member of (3.2).

The first integral on the right is calculated by the residue theorem. We have $z' = -1/z$, so

$$\begin{aligned} J_1 &= - \int_K e^{2\pi i(1/z - nz)/\lambda} dz = - 2\pi i \operatorname{Res}_{z=0} \sum_{\mu=0}^{\infty} \frac{1}{\mu!} \left(\frac{2\pi i}{\lambda z}\right)^{\mu} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left(\frac{-2\pi inz}{\lambda}\right)^{\nu} \\ &= \frac{2\pi}{\sqrt{n}} \sum_{\nu=0}^{\infty} \frac{(2\pi\sqrt{n}/\lambda)^{2\nu+1}}{\nu!(\nu+1)!} = \frac{2\pi}{\sqrt{n}} I_1\left(\frac{4\pi\sqrt{n}}{\lambda}\right), \end{aligned}$$

where I_1 is the Bessel function of the first kind with purely imaginary argument. To estimate J_2 , we note that on L we have

$$|z|^2 = x^2 + y^2 = |z_1|^2.$$

Thus

$$\begin{aligned} |J_2| &\leq \int_L |e^{2\pi i(1/z - nz)/\lambda} dz| < 2\pi |z_1| \max_L \exp \frac{2\pi}{\lambda} \left(\frac{y}{\alpha^2 + y^2} + ny\right) \\ &= 2\pi |z_1| \max_L \exp \frac{2\pi}{\lambda} \left(\frac{y}{|z_1|^2} + ny\right) = 2\pi |z_1| \exp \frac{2\pi}{\lambda N} \left(\frac{1}{|z_1|^2} + n\right), \end{aligned}$$

so that

$$J_2 = \theta A N^{-2} \exp 2\pi(n+1)/N\lambda.$$

Putting these results together, we find that

$$(3.4) \quad H_1 = \frac{2\pi}{\sqrt{n}} I_1\left(\frac{4\pi\sqrt{n}}{\lambda}\right) + \theta A \exp 2\pi n/N\lambda.$$

We now estimate the summands of S_1 for which $j \neq 1$. Here the decisive point is that, in (3.1), $|c_j| > 1$ if $j \neq 1$. This is because $1/|c_j|$ is the radius of an isometric circle. The largest isometric circle in the strip $|\Re z| \leq \lambda/2$ is the one corresponding to the transformation $T: z \rightarrow -1/z$, for which $c = 1$; all the others are smaller. From (3.1) we get, with $z' = x' + iy'$,

$$y' = \frac{y}{(c_j x + d_j)^2 + c_j^2 y^2} \leq \frac{1}{c_j^2 y} \leq \frac{1}{\gamma^2 y},$$

where $\gamma > 1$ is the minimum of $|c_2|, |c_3|, \dots, |c_s|$. Hence

$$\begin{aligned} |H_j| &\leq \int_{l_j} |e^{-2\pi i(z'+nz)/\lambda}| |dz| \leq \int_{l_j} e^{-2\pi(1/\gamma^2 y + ny)/\lambda} dx \\ &= |l_j| \cdot \exp \frac{2\pi}{\lambda} \left(\frac{N}{\gamma^2} + \frac{n}{N} \right) \end{aligned} \quad (j \neq 1),$$

where $|l_j|$ denotes the length of the segment l_j . Therefore,

$$(3.5) \quad \sum_{j=2}^s |H_j| < \lambda \exp \frac{2\pi}{\lambda} \left(\frac{N}{\gamma^2} + \frac{n}{N} \right).$$

From (3.2), (3.3), (3.4), and (3.5), we now obtain

$$\begin{aligned} c_n(\lambda) &= \frac{2\pi}{\sqrt{n}\lambda} I_1\left(\frac{4\pi\sqrt{n}}{\lambda}\right) + \theta A \exp 2\pi(n+1)/N\lambda \\ &\quad + \theta A \exp 2\pi \left(\frac{N}{\gamma^2} + \frac{n}{N} \right) / \lambda. \end{aligned}$$

The first term in the right member is asymptotic to

$$\frac{1}{\sqrt{2\lambda}} \cdot \frac{\exp 4\pi\sqrt{n}/\lambda}{n^{3/4}},$$

by a well-known formula for the Bessel function [5, p.373]. The last term is made as small as possible by the choice $N = \gamma\sqrt{n}$, in which case the exponent becomes $4\pi\sqrt{n}/\gamma\lambda$. Since $\gamma > 1$, this term, as well as the second one, is of lower order than the first term, and (1.2) follows.

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50; back numbers (Volumes 1, 2, 3) are available at \$2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

*To be succeeded in 1955, by H.L. Royden, Stanford University, Stanford, California.

**To be succeeded in 1955, by E.G. Straus, University of California, Los Angeles 24, Calif.

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