NOTE ON THE SCHWARZ TRIANGLE FUNCTIONS

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1. Introduction. In his classical investigation of the hypergeometric series, Schwarz discussed the function which maps the upper half $w$-plane onto a curvilinear triangle in the $z$-plane with angles $\delta \pi, \epsilon \pi, \eta \pi (\delta + \epsilon + \eta < 1)$. The inverse, $w = \phi(z)$, of this function is automorphic with respect to the group got by reflecting the triangle in its sides, reflecting the new figure in its free sides, and so on. In order that this process shall lead to a properly discontinuous group, it is necessary and sufficient that $1/\delta, 1/\epsilon, 1/\eta$ be positive integers or $\infty$. We take in particular $\delta = 1/q$, $\epsilon = 1/2$, $\eta = 0 (q = 3, 4, 5, \cdots)$, and place the triangle in the upper half plane with vertices at $-\exp \pi i/q$, $i$, and $i \infty$. The group $\Gamma(\lambda)$ of transformations is then generated by

$$S : z \rightarrow z + \lambda \quad \text{and} \quad T : z \rightarrow -\frac{1}{z},$$

where $\lambda = 2 \cos \pi/q (q = 3, 4, 5, \cdots)$. (We restrict $\lambda$ to this countable set from now on.) The automorphic function $\phi_\lambda(z) = \phi(z)$ having a simple pole at $z = i \infty$ thus has the period $\lambda$, and we normalize its Fourier expansion as follows:

$$(1.1) \quad \phi_\lambda(z) = \phi(z) = x^{-1} + \sum_{n=0}^{\infty} c_n(\lambda) x^n, \quad x = \exp \frac{2\pi i z}{\lambda}.$$ 

This makes $\phi(z)$ unique except for an additive constant; $\phi$ is called a triangle function.

Concerning the Fourier coefficients $c_n(\lambda)$, we wish to make the following observations:

I. All the Fourier coefficients of any triangle function $\phi_\lambda$ are rational numbers.

II. The Fourier coefficients $c_n(\lambda)$ have the asymptotic value
Both results can be extended to a wider class of Fuchsian groups; this will be done in future publications.

2. Proof of I. Let \( z = \psi(w) \) be the function inverse to \( \phi \); that is, \( \psi \) maps the upper half \( w \)-plane onto the triangle in the \( z \)-plane. It is well known [1, p. 304 f] that \( \psi \) is the quotient of two independent solutions of the hypergeometric equation

\[
2.1 \quad w(w-1) \frac{d^2 z}{dw^2} + \left[ (\alpha + \beta + 1)w - \gamma \right] \frac{dz}{dw} + \alpha \beta z = 0,
\]

where

\[
\alpha = \beta = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right), \quad \gamma = 1 - \frac{1}{q}.
\]

In this case (\( \alpha = \beta \)), Fricke [2, p. 115, (18)] has given an explicit representation of a system of independent solutions valid at \( w = \infty \):

\[
2.2 \quad Z_1 = w^{-\alpha} F(\alpha, \alpha - \gamma + 1, 1; 1/w),
\]

\[
Z_2 = w^{-\alpha} [F_1(\alpha, \alpha - \gamma + 1; 1/w) - \log w \cdot F(\alpha, \alpha - \gamma + 1, 1; 1/w)],
\]

where \( F \) is the ordinary hypergeometric series, and \( F_1 \) is a series with coefficients rational in \( \alpha, \beta \) [2, p. 114, (15)],

\[
F_1(\alpha, \beta; u) = \frac{\alpha \cdot \beta}{1,1} \left( \frac{1}{\alpha} + \frac{1}{\beta} - \frac{2}{1} \right) u + \cdots.
\]

Both series converge for \( |w| > 1 \).

For our purposes we take for \( z = \psi(w) \) the combination

\[
- \frac{2 \pi i z}{\lambda} = - \frac{Z_2}{Z_1} = \log w - \frac{F_1}{F} = \log w + \frac{A_1}{w} + \frac{A_2}{w^2} + \cdots,
\]

\[1\text{When } \phi(z) \text{ is Klein's absolute modular invariant } J(z), (1.2) \text{ is an immediate consequence of the Petersson-Rademacher [3, p. 202; 4] series for } J(z).\]
where, as we see from (2.2), $A_1, A_2, \cdots$, are rational numbers. Hence with $x = \exp 2\pi i z/\lambda$, we have

$$x^{-1} = w \cdot \exp \left( \frac{A_1}{w} + \frac{A_2}{w^2} + \cdots \right) = w \left( 1 + \frac{B_1}{w} + \frac{B_2}{w^2} + \cdots \right),$$

where again the $B_n$ are rational.

We now invert this equation, setting

$$(2.3) \quad w = \phi(z) = x^{-1} (1 + c_0 x + c_1 x^2 + \cdots),$$

and have:

$$w^{-1} = x (1 + d_0 x + d_1 x^2 + \cdots),$$

$$x^{-1} = x^{-1} (1 + c_0 x + c_1 x^2 + \cdots) (1 + B_1 x (1 + d_0 x + d_1 x^2 + \cdots)$$

$$+ B_2 x^2 (1 + d'_0 x + d'_1 x^2 + \cdots) + \cdots).$$

The last equation determines the $c_n$ uniquely in a step-by-step manner. They clearly are rational numbers. Furthermore, (2.3) agrees with (1.1). This proves I.

3. **Proof of II.** From (1.1) we have

$$c_n(\lambda) = \frac{1}{\lambda} \int_C \phi(z) e^{-2\pi i n z/\lambda} \, dz \quad (n > 0),$$

where $C$ is a path connecting any two points in the upper half plane at the same height and at a distance $\lambda$ apart. We take $C$ to be the horizontal line

$$z = x + \frac{i}{N}, \quad |x| \leq \frac{\lambda}{2};$$

$N > 0$ will eventually be taken of the order of $\sqrt{n}$.

The line $C$ cuts a finite number of fundamental regions of $\Gamma(\lambda) = R_1, R_2, \cdots, R_s$; the corresponding segments are $l_1, l_2, \cdots, l_s$. Thus

$$\lambda c_n(\lambda) = \sum_{j=1}^s \int_{l_j} \phi(z) e^{-2\pi i n z/\lambda} \, dz.$$
There is a unique substitution

\begin{equation}
(3.1) \quad z' = \frac{a_j z + b_j}{c_j z + d_j}
\end{equation}

of \( \Gamma(\lambda) \) which carries \( R_j \) into \( R_0 \), the standard fundamental region with cusp at \( i\infty \); the coefficients \( a_j, b_j, \ldots \) are real, and \( c_j \neq 0 \). Thus because of the invariance on \( \phi \) on \( \Gamma \), we get

\[
\lambda c_n(\lambda) = \sum_{j=1}^{s} \int_{I_j} \phi(z') e^{-2\pi i n z/\lambda} \, dz,
\]

where \( z' \) lies in \( R_0 \).

Now, by (1.1), write

\[
\phi(z) = e^{-2\pi i z/\lambda} + \psi(z), \quad \psi(z) = \sum_{0}^{\infty} c_m e^{2\pi i m z/\lambda};
\]

then

\begin{equation}
(3.2) \quad \lambda c_n(\lambda) = \sum_{j=1}^{s} \int_{I_j} e^{-2\pi i (z' + nz)/\lambda} \, dz + \sum_{j=1}^{s} \int_{I_j} \psi(z') e^{-2\pi i n z/\lambda} \, dz
\end{equation}

\[
= \sum_{j=1}^{s} H_j + S_2 = S_1 + S_2.
\]

In the following estimates, \( A \) will denote a constant, not the same one at each appearance, independent of \( N \) and \( n \) but possibly depending on \( \lambda \); \( \theta \) is an absolute constant of modulus less than unity.

We know that \( \psi(z') \) is bounded in \( R_0 \) because \( \phi \) is regular in the upper half plane except for a simple pole at \( i\infty \); put \( |\psi(z')| \leq A \). Hence

\begin{equation}
(3.3) \quad |S_2| \leq A e^{2\pi n / N\lambda} \int_{C} |dz| \leq A e^{2\pi n / N\lambda}.
\end{equation}

The principal contribution to \( S_1 \) will come from the segment lying in the fundamental region, \( R_1 \) say, which is the map of \( R_0 \) by \( T: z' = -1/z \). \( R_1 \) is bounded by an arc of the unit circle and by two arcs passing through the origin,
the right-hand one having the equation
\[
\left( \frac{x - 1}{\lambda} \right)^2 + y^2 = \frac{1}{\lambda^2} \quad (z = x + iy).
\]

Hence the endpoints of \( l_1 \) are \( \pm z_1 \), where
\[
z_1 = \frac{\theta A}{N^2} + \frac{i}{N}.
\]

Let \( K \) be the circle, described counter clockwise, with center at the origin and passing through \( z_1, z_2 \), and \( L \) the larger of the arcs connecting \( z_1, z_2 \). We have
\[
H_1 = \int_{l_1} - \int_K \int_L = J_1 + J_2,
\]
the integrands being the same as in the first term of the right member of (3.2).

The first integral on the right is calculated by the residue theorem. We have \( z' = -1/z \), so
\[
J_1 = - \int_K e^{2\pi i (1/z - nz)/\lambda} dz = -2\pi i \sum_{\mu=0}^{\infty} \frac{1}{\mu!} \left( \frac{2\pi i}{\lambda z} \right)^\mu \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left( \frac{-2\pi iz}{\lambda} \right)^\nu
\]
\[
= \frac{2\pi}{\sqrt{n}} \sum_{\nu=0}^{\infty} \frac{(2\pi \sqrt{n} / \lambda)^{2\nu+1}}{\nu!(\nu+1)!} = \frac{2\pi}{\sqrt{n}} I_1 \left( \frac{4\pi \sqrt{n}}{\lambda} \right),
\]
where \( I_1 \) is the Bessel function of the first kind with purely imaginary argument.

To estimate \( J_2 \), we note that on \( L \) we have
\[
|z|^2 = x^2 + y^2 = |z_1|^2.
\]
Thus
\[
|J_2| \leq \int_L |e^{2\pi i (1/z - nz)/\lambda} dz| < 2\pi |z_1| \max_L \frac{2\pi}{\lambda} \left( \frac{y}{\alpha^2 + y^2} + ny \right)
\]
\[
= 2\pi |z_1| \max_L \frac{2\pi}{\lambda} \left( \frac{y}{|z_1|^2} + ny \right) = 2\pi |z_1| \exp \frac{2\pi}{\lambda N} \left( \frac{1}{|z_1|^2} + n \right),
\]
so that

\[ J_2 = \theta A N^{-2} \exp \frac{2\pi(n+1)}{N\lambda}. \]

Putting these results together, we find that

\[
(3.4) \quad H_1 = \frac{2\pi}{\sqrt{n}} I_1 \left( \frac{4\pi \sqrt{n}}{\lambda} \right) + \theta A \exp \frac{2\pi n}{N\lambda}. \]

We now estimate the summands of \( S_1 \) for which \( j \neq 1 \). Here the decisive point is that, in (3.1), \( |c_j| > 1 \) if \( j \neq 1 \). This is because \( 1/|c_j| \) is the radius of an isometric circle. The largest isometric circle in the strip \( |R_z| \leq \lambda/2 \) is the one corresponding to the transformation \( T: z \rightarrow -1/z \), for which \( c = 1 \); all the others are smaller. From (3.1) we get, with \( z' = x' + iy' \),

\[ y' = \frac{y}{(c_j x + d_j)^2 + c_j^2 y^2} \leq \frac{1}{c_j^2 y} \leq \frac{1}{\gamma^2 y}, \]

where \( \gamma > 1 \) is the minimum of \( |c_2|, |c_3|, \ldots, |c_s| \). Hence

\[
|H_j| \leq \int_{l_j} |e^{-2\pi i(z'+nz)/\lambda}| dz' \leq \int_{l_j} e^{-2\pi n(y^2 + ny)/\lambda} dx
\]

\[ = |l_j| \cdot \exp \frac{2\pi}{\lambda} \left( \frac{N}{\gamma^2} + \frac{n}{N} \right) \quad (j \neq 1), \]

where \( |l_j| \) denotes the length of the segment \( l_j \). Therefore,

\[
(3.5) \quad \sum_{j=2}^{s} |H_j| < \lambda \exp \frac{2\pi}{\lambda} \left( \frac{N}{\gamma^2} + \frac{n}{N} \right). \]

From (3.2), (3.3), (3.4), and (3.5), we now obtain

\[
c_n(\lambda) = \frac{2\pi}{\sqrt{n}\lambda} I_1 \left( \frac{4\pi \sqrt{n}}{\lambda} \right) + \theta A \exp \frac{2\pi(n+1)}{N\lambda}
\]

\[
+ \theta A \exp \frac{2\pi}{\lambda} \left( \frac{N}{\gamma^2} + \frac{n}{N} \right)/\lambda. \]
The first term in the right member is asymptotic to
\[ \frac{1}{\sqrt{2\lambda}} \cdot \exp \frac{4\pi \sqrt{n}/\lambda}{n^{3/4}}, \]
by a well-known formula for the Bessel function [5, p. 373]. The last term is made as small as possible by the choice \( N = \gamma \sqrt{n} \), in which case the exponent becomes \( 4\pi \sqrt{n}/\gamma \lambda \). Since \( \gamma > 1 \), this term, as well as the second one, is of lower order than the first term, and (1.2) follows.

References


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