

Pacific Journal of Mathematics

A GENERALIZATION OF AN INEQUALITY DUE TO BEURLING

ARTHUR EUGENE LIVINGSTON

A GENERALIZATION OF AN INEQUALITY DUE TO BEURLING

ARTHUR E. LIVINGSTON

1. Introduction. In 1941, Arne Beurling gave a proof (unpublished) of the following result:

If $a_n \geq 0, b_n \geq 0$ for $n = 1, 2, \dots$, and

$$\sum_{m=1}^{\infty} ma_m^2 < \infty, \quad \sum_{n=1}^{\infty} nb_n^2 < \infty,$$

then

$$(1) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n / \log(m+n) \leq K \left(\sum_{m=1}^{\infty} ma_m^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} nb_n^2 \right)^{1/2}$$

with $0 < K < 4e$.

If we set

$$\alpha(x) = \int_1^x t^{-1} dt,$$

then the inequality (1) is of the form

$$(2) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n / \alpha(m+n) \leq K(\alpha) \left[\sum_{m=1}^{\infty} a_m^2 / \alpha'(m) \right]^{1/2} \left[\sum_{n=1}^{\infty} b_n^2 / \alpha'(n) \right]^{1/2},$$

and it is the purpose of this note to generalize this latter inequality. As an example of the type of result to be obtained, we quote the integral analogue of (2):

THEOREM 1. *Let $\alpha(x)$ be nonnegative, nondecreasing, and locally absolutely continuous on the interval $0 \leq x < \infty$. If $F(x) \geq 0, G(x) \geq 0$ for $0 \leq x < \infty$, and*

Received April 10, 1953. The author is a National Science Foundation Fellow.

Pacific J. Math. 4 (1954), 251-257

$$\int_0^\infty \frac{[F(x)]^p}{[\alpha'(x)]^{p-1}} dx < \infty, \int_0^\infty \frac{[G(y)]^q}{[\alpha'(y)]^{q-1}} dy < \infty,$$

where $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then

$$(3) \quad \int_0^\infty \int_0^\infty \frac{F(x)G(y)}{\alpha(x+y)} dx dy \\ \leq K(\alpha) \left\{ \int_0^\infty \frac{[F(x)]^p}{[\alpha'(x)]^{p-1}} dx \right\}^{1/p} \left\{ \int_0^\infty \frac{[G(y)]^q}{[\alpha'(y)]^{q-1}} dy \right\}^{1/p}$$

with $0 < K(\alpha) \leq p + q$.

If $\alpha(x+y) \geq \alpha(x) + \alpha(y)$, then $K(\alpha) \leq \pi/\sin(\pi/p)$. If $\alpha(0) = 0$, $\alpha(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $\alpha(x+y) \leq \alpha(x) + \alpha(y)$, then $K(\alpha) \geq \pi/\sin(\pi/p)$.

The author wishes to acknowledge that any novelty in the subject matter of this note is due entirely to Professor Beurling who suggested the very general Theorem 2 below.

2. The main result. This is:

THEOREM 2. Let $\alpha(x)$ be nonnegative, nondecreasing, and continuous from the right for $0 \leq x < \infty$. Let $f(x) \geq 0$, $g(x) \geq 0$ for $0 \leq x < \infty$. Let $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. If

$$\int_0^\infty [f(x)]^p d\alpha(x) < \infty, \int_0^\infty [g(y)]^q d\alpha(y) < \infty,$$

then

$$(4) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x+y)} d\alpha(x)d\alpha(y) \\ \leq K(\alpha) \left\{ \int_0^\infty [f(x)]^p d\alpha(x) \right\}^{1/p} \left\{ \int_0^\infty [g(y)]^q d\alpha(y) \right\}^{1/q}$$

with $0 < K(\alpha) \leq p + q$.

If $\alpha(x+y) \geq \alpha(x) + \alpha(y)$, then

$$(5) \quad K(\alpha) \leq \pi/\sin(\pi/p).$$

If $\alpha(0) = 0$, $\alpha(x) \rightarrow \infty$ as $x \rightarrow \infty$, $\alpha(x)$ is continuous for $0 \leq x < \infty$, and $\alpha(x+y) \leq \alpha(x) + \alpha(y)$, then

$$(6) \quad K(\alpha) \geq \pi/\sin(\pi/p).$$

Proof. We have

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x+y)} d\alpha(x)d\alpha(y) \\ &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x+y)} [\alpha(x)/\alpha(y)]^{1/pq} [\alpha(y)/\alpha(x)]^{1/pq} d\alpha(x)d\alpha(y) \\ &\leq P^{1/p}Q^{1/q}. \end{aligned}$$

by Hölder's Inequality [1, p. 11], where

$$\begin{aligned} P &= \int_0^\infty \int_0^\infty \frac{[f(x)]^p}{\alpha(x+y)} [\alpha(x)/\alpha(y)]^{1/q} d\alpha(x)d\alpha(y), \\ Q &= \int_0^\infty \int_0^\infty \frac{[g(y)]^q}{\alpha(x+y)} [\alpha(y)/\alpha(x)]^{1/p} d\alpha(x)d\alpha(y). \end{aligned}$$

Since $\alpha(x)$ is nondecreasing, we have

$$\alpha(x+y) \geq \max[\alpha(x), \alpha(y)].$$

Consequently,

$$\begin{aligned} J &= \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\alpha(x+y)} d\alpha(y) \leq \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\max[\alpha(x), \alpha(y)]} d\alpha(y) \\ &= [\alpha(x)]^{-1/p} \int_0^x [\alpha(y)]^{-1/q} d\alpha(y) + [\alpha(x)]^{1/q} \int_x^\infty [\alpha(y)]^{-1-1/q} d\alpha(y) \\ &\leq p\{1 - [\alpha(0)/\alpha(x)]^{1/p}\} + q\{1 - [\alpha(x)/\alpha(\infty)]^{1/q}\} \leq p + q. \end{aligned}$$

In a similar way, we find that

$$\int_0^\infty \frac{[\alpha(y)/\alpha(x)]^{1/p}}{\alpha(x+y)} d\alpha(x) \leq p+q.$$

Thus,

$$I \leq P^{1/p} Q^{1/q} \leq (p+q) \left\{ \int_0^\infty [f(x)]^p d\alpha(x) \right\}^{1/p} \left\{ \int_0^\infty [g(y)]^q d\alpha(y) \right\}^{1/q},$$

and this implies (4).

If $\alpha(x+y) \geq \alpha(x) + \alpha(y)$, then we have

$$\begin{aligned} J &= \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\alpha(x+y)} d\alpha(y) \leq \int_0^\infty \frac{[\alpha(x)/\alpha(y)]^{1/q}}{\alpha(x) + \alpha(y)} d\alpha(y) \\ &\leq \int_{\alpha(0)/\alpha(x)}^{\alpha(\infty)/\alpha(x)} t^{-1/q} (1+t)^{-1} dt \leq \int_0^\infty t^{-1/q} (1+t)^{-1} dt = \pi/\sin(\pi/p), \end{aligned}$$

and this implies (5).

If $\alpha(0) = 0$, $\alpha(\infty) = \infty$, $\alpha(x)$ is continuous for $0 \leq x < \infty$, and $\alpha(x+y) \leq \alpha(x) + \alpha(y)$, then

$$I \geq \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha(x) + \alpha(y)} d\alpha(x) d\alpha(y) = \int_0^\infty \int_0^\infty \frac{F(s)G(t)}{s+t} ds dt = I_1,$$

where we have made the changes of variable $\alpha(x) = s$, $\alpha(y) = t$ and set $F(s) = f(x)$, $G(t) = g(y)$. By Hilbert's Inequality [1, 226],

$$\begin{aligned} I_1 &\leq [\pi/\sin(\pi/p)] \left\{ \int_0^\infty [F(s)]^p ds \right\}^{1/p} \left\{ \int_0^\infty [G(t)]^q dt \right\}^{1/q} \\ &= [\pi/\sin(\pi/p)] \left\{ \int_0^\infty [f(x)]^p d\alpha(x) \right\}^{1/p} \left\{ \int_0^\infty [g(y)]^q d\alpha(y) \right\}^{1/q}, \end{aligned}$$

and the constant $\pi/\sin(\pi/p)$ is the best possible [1, p. 226]. This gives (6).

We note that the inequality (4), for $\alpha(x)$ continuous, could be obtained directly from Theorem 319 of [1, p. 229] as follows:

$$\begin{aligned}
 I &\leq \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max[\alpha(x), \alpha(y)]} d\alpha(x)d\alpha(y) = \int_0^\infty \int_0^\infty \frac{F(s)G(t)}{\max[s, t]} dsdt \\
 &\leq k \left\{ \int_0^\infty [F(s)]^p ds \right\}^{1/p} \left\{ \int_0^\infty [G(t)]^q dt \right\}^{1/q} \\
 &= k \left\{ \int_0^\infty [f(x)]^p d\alpha(x) \right\}^{1/p} \left\{ \int_0^\infty [g(y)]^q d\alpha(y) \right\}^{1/q},
 \end{aligned}$$

where

$$k = \int_0^\infty \frac{s^{-1/q} ds}{\max[s, 1]} = p + q.$$

We have made the changes of variable $\alpha(x) = s, \alpha(y) = t$ and set

$$\begin{aligned}
 F(s) &= f(x) \quad \text{for } \alpha(0) \leq s \leq \alpha(\infty), \\
 &= 0 \quad \text{otherwise,} \\
 G(t) &= g(y) \quad \text{for } \alpha(0) \leq t \leq \alpha(\infty), \\
 &= 0 \quad \text{otherwise.}
 \end{aligned}$$

3. Corollaries. If we set $f(x) = F(x)/\alpha'(x), g(y) = G(y)/\alpha'(y)$ in Theorem 2, we obtain Theorem 1.

As another application of Theorem 2, we deduce:

THEOREM 3. Let the sequence $\{\alpha_n\}_1^\infty$ be nonnegative and nondecreasing for $n = 1, 2, \dots$, and set $\alpha_0 = 0$. Let $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. If $a_n \geq 0, b_n \geq 0$ for $n = 1, 2, \dots$ and

$$\sum_{m=1}^\infty a_m^p / (\alpha_m - \alpha_{m-1})^{p-1} < \infty, \quad \sum_{n=1}^\infty b_n^q / (\alpha_n - \alpha_{n-1})^{q-1} < \infty,$$

then

$$\begin{aligned}
 (7) \quad &\sum_{m=1}^\infty \sum_{n=1}^\infty a_m b_n / \alpha_{m+n} \\
 &\leq K(\alpha) \left\{ \sum_{m=1}^\infty a_m^p / (\alpha_m - \alpha_{m-1})^{p-1} \right\}^{1/p} \left\{ \sum_{n=1}^\infty b_n^q / (\alpha_n - \alpha_{n-1})^{q-1} \right\}^{1/q}.
 \end{aligned}$$

with $0 < K(\alpha) \leq p + q$.

Proof. Let $\alpha(x)$, $0 \leq x < \infty$, be the polygonal function with vertices (n, α_n) , $n = 0, 1, \dots$. Set $f(x) = A_n \geq 0$, $g(x) = B_n \geq 0$ for $n - 1 \leq x < n$, $n = 1, 2, \dots$. By Theorem 2,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_m B_n \int_{m-1}^m \int_{n-1}^n \frac{d\alpha(x)d\alpha(y)}{\alpha(x+y)}$$

$$\leq K(\alpha) \left\{ \sum_{m=1}^{\infty} A_m^p (\alpha_m - \alpha_{m-1}) \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} B_n^q (\alpha_n - \alpha_{n-1}) \right\}^{1/q},$$

with $0 < K(\alpha) \leq p + q$. Since $\alpha(x)$ is nondecreasing, the double sum dominates

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_m B_n}{\alpha_{m+n}} (\alpha_m - \alpha_{m-1})(\alpha_n - \alpha_{n-1}).$$

Setting $A_m(\alpha_m - \alpha_{m-1}) = a_m$, $B_m(\alpha_m - \alpha_{m-1}) = b_m$ gives (7).

As a special case of Theorem 3, we take $\alpha_0 = 0$, $\alpha_n - \alpha_{n-1} = 1/n$ for $n = 1, 2, \dots$. Since, for $n = 2, 3, \dots$,

$$\alpha_n = \sum_{k=1}^n d^{-1} < 1 + \log n \leq \left(1 + \frac{1}{\log 2} \right) \log n,$$

we find that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n / \log(m+n) \leq M \left(\sum_{m=1}^{\infty} m^{p-1} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} n^{q-1} b_n^q \right)^{1/q},$$

with $0 < M < (p + q)(1 + 1/\log 2)$. For $p = 2$, this is the inequality (1) with a slightly smaller bound for the constant.

4. A generalization to several variables. The alternative proof offered for Theorem 2 suggests that the inequalities of this paper can be stated for N variables, $N \geq 2$.

For example, we have:

THEOREM 2'. Let $\alpha(x)$ be continuous, nonnegative, and nondecreasing for $0 \leq x < \infty$. Let $p_1 > 1, \dots, p_N > 1$ and $p_1^{-1} + \dots + p_N^{-1} = 1$. If $f_1(x) \geq 0, \dots, f_N(x) \geq 0$ for $0 \leq x < \infty$ and

$$\int_0^\infty [f_i(x)]^{p_i} d\alpha(x) < \infty$$

for $i = 1, \dots, N$, then

$$\int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^N f_i(x_i) d\alpha(x_i)}{[\alpha(\sum_{i=1}^N x_i)]^{N-1}} \leq K_N(\alpha) \prod_{i=1}^N \left\{ \int_0^\infty [f_i(x)]^{p_i} d\alpha(x) \right\}^{1/p_i}$$

with

$$0 < K_N(\alpha) \leq \int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{N-1} x_i^{-1/p_i} dx_i}{\{\max[x_1, \dots, x_{N-1}, 1]\}^{N-1}}.$$

If $\alpha(x+y) \geq \alpha(x) + \alpha(y)$, then

$$K_N(\alpha) \leq \int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{N-1} x_i^{-1/p_i} dx_i}{[1 + \sum_{i=1}^{N-1} x_i]^{N-1}} = M_N.$$

If $\alpha(0) = 0$, $\alpha(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $\alpha(x+y) \leq \alpha(x) + \alpha(y)$, then $K_N(\alpha) \geq M_N$.

The proof is patterned on that of Theorem 322 of [1, p. 231].

REFERENCE

1. G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge, England, 1952.

THE INSTITUTE FOR ADVANCED STUDY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

M.M. SCHIFFER*
Stanford University
Stanford, California

E. HEWITT
University of Washington
Seattle 5, Washington

R. P. DILWORTH
California Institute of Technology
Pasadena 4, California

E.F. BECKENBACH**
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

H. BUSEMANN
HERBERT FEDERER
MARSHALL HALL

P. R. HALMOS
HEINZ HOPF
R. D. JAMES

BØRGE JESSEN
PAUL LÉVY
GEORGE PÓLYA

J. J. STOKER
E. G. STRAUS
KÔSAKU YOSIDA

SPONSORS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA, BERKELEY
UNIVERSITY OF CALIFORNIA, DAVIS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD RESEARCH INSTITUTE
STANFORD UNIVERSITY
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
HUGHES AIRCRAFT COMPANY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, E.G. Straus, at the University of California Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50; back numbers (Volumes 1, 2, 3) are available at \$2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

*To be succeeded in 1955, by H.L. Royden, Stanford University, Stanford, California.

**To be succeeded in 1955, by E.G. Straus, University of California, Los Angeles 24, Calif.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

COPYRIGHT 1954 BY PACIFIC JOURNAL OF MATHEMATICS

Henry Ludwig Alder, <i>Generalizations of the Rogers-Ramanujan identities</i>	161
E. M. Beelsey, <i>Concerning total differentiability of functions of class P</i>	169
L. Carlitz, <i>The number of solutions of some special equations in a finite field</i>	207
Marshall Hall, <i>On a theorem of Jordan</i>	219
J. D. Hill, <i>Remarks on the Borel property</i>	227
Joseph Lehner, <i>Note on the Schwarz triangle functions</i>	243
Arthur Eugene Livingston, <i>A generalization of an inequality due to Beurling</i>	251
Edgar Reich, <i>An inequality for subordinate analytic functions</i>	259
Dan Robert Scholz, <i>Some minimum problems in the theory of functions</i>	275
J. C. Shepherdson, <i>On two problems of Kurepa</i>	301
Abraham Wald, <i>Congruent imbedding in F-metric spaces</i>	305
Gordon L. Walker, <i>Fermat's theorem for algebras</i>	317