COMMUTING SPECTRAL MEASURES ON HILBERT SPACE

JOHN WERMER
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1. Introduction. By a "spectral measure" on Hilbert space \( H \) we mean a family of bounded operators \( E(\sigma) \) on \( H \) defined for all Borel sets \( \sigma \) in the plane. We suppose:

(i) If \( \sigma_0 \) denotes the empty set and \( \sigma_1 \) the whole plane, then

\[
E(\sigma_0) = 0, \quad E(\sigma_1) = I,
\]

where \( I \) is the identity.

(ii) For all \( \sigma_1, \sigma_2 \),

\[
E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2);
\]

and for disjoint \( \sigma_1, \sigma_2 \),

\[
E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2).
\]

(iii) There exists a constant \( M \) with \( \|E(\sigma)\| \leq M \), all \( \sigma \). It follows that \( E(\sigma)^2 = E(\sigma) \) for each \( \sigma \), and \( E(\sigma_1)E(\sigma_2) = 0 \) if \( \sigma_1, \sigma_2 \) are disjoint.

Mackey has shown in [3], as part of the proof of Theorem 55 of [3], that if \( E(\sigma) \) is a spectral measure with the properties just stated, then there exists a bicontinuous operator \( A \) such that \( A^{-1}E(\sigma)A \) is self-adjoint for every \( \sigma \). In a special case this result was proved by Lorch in [2]. We shall prove:

**Theorem 1.** Let \( E(\sigma) \) and \( F(\eta) \) be two commuting spectral measures on \( H \); that is,

\[
E(\sigma)F(\eta) = F(\eta)E(\sigma)
\]

for every \( \sigma, \eta \). Then there exists a bicontinuous operator \( A \) such that \( A^{-1}E(\sigma)A \) and \( A^{-1}F(\eta)A \) are self-adjoint for every \( \sigma, \eta \).

As a corollary of Theorem 1, we shall obtain:

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Theorem 2. If $T_1, T_2$ are spectral operators on $H$, in the sense of Dunford [1], and $T_1 T_2 = T_2 T_1$, then $T_1 + T_2$ and $T_1 T_2$ are again spectral operators.

2. Lemmas. We shall use two lemmas in proving Theorem 1.

Lemma 1. Let $P_1, P_2, \ldots, P_n$ be operators on Hilbert space with

$$P_i P_j = 0 \quad (i \neq j), \quad P_i^2 = P_i, \quad \sum_{i=1}^n P_i = I.$$

Suppose that, for every set $\delta_1, \delta_2, \ldots, \delta_n$ of zeros and ones,

$$\left\| \sum_{i=1}^n \delta_i P_i \right\| \leq M.$$

Then for every $x$ we have

$$\frac{1}{4M^2} \| x \|^2 \leq \sum_{i=1}^n \| P_i x \|^2 \leq 4M^2 \| x \|^2.$$

This Lemma is proved in [3, p. 147]; we include the proof for completeness.

Proof. We note that

$$\sum_{i=1}^n \| P_i x \|^2 = \frac{1}{2^n} \sum \| \epsilon_1 P_1 x + \cdots + \epsilon_n P_n x \|^2,$$

where the sum is taken over all possible sets $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, where $\epsilon_i = \pm 1$. Hence

$$a_x = \| \epsilon'_1 P_1 x + \cdots + \epsilon'_n P_n x \|^2 \leq \sum_{i=1}^n \| P_i x \|^2 \leq \| \epsilon_1 P_1 x + \cdots + \epsilon_n P_n x \|^2 = b_x$$

for some choice of the $\epsilon'_i$ and $\epsilon_i$. Now

$$b_x = \left\| \sum_{i=1}^n \delta_i^+ P_i x - \sum_{i=1}^n \delta_i^- P_i x \right\|^2,$$
where the \( \delta_1^+ \) and the \( \delta_j^- \) are 1 or 0.

Hence

\[
\sum_{i=1}^{n} \| P_i x \|^2 \leq 4M^2 \cdot \| x \|^2.
\]

Let now \( P^+ = \sum P_i \), summed over those \( i \) with \( \varepsilon_i = 1 \); and let \( P^- = \sum P_i \), summed over those \( i \) with \( \varepsilon_i = -1 \). Then

\[
(P^+ - P^-)^2 = P^+ + P^- = I \quad \text{and} \quad \| P^+ x - P^- x \|^2 = a_x.
\]

hence

\[
\| x \|^2 = \| (P^+ - P^-)^2 x \|^2 \leq \| P^+ - P^- \|^2 \cdot \| P^+ x - P^- x \|^2.
\]

Now \( \| P^+ \| \leq M \) and \( \| P^- \| \leq M \); and so

\[
\| x \|^2 \leq (2M)^2 a_x \leq (2M)^2 \sum_{i=1}^{n} \| P_i x \|^2.
\]

**Lemma 2.** Let \( E(\sigma) \) and \( F(\eta) \) be commuting spectral measures on Hilbert space. Then there is a fixed \( K \) such that for any set \( \sigma_1, \sigma_2, \ldots, \sigma_n \) of disjoint Borel sets, and set \( \eta_1, \eta_2, \ldots, \eta_n \) of arbitrary Borel sets,

\[
\left\| \sum_{i=1}^{n} L(\sigma_i) F(\eta_i) \right\| \leq K.
\]

**Proof.** Fix \( x \). By (iii) there exist constants \( L \) and \( M \), with \( \| E(\sigma) \| \leq M \), \( \| F(\eta) \| \leq L \) for any \( \sigma, \eta \). Let \( \sigma_{n+1} \) be the complement of

\[
\bigcup_{i=1}^{n} \sigma_i.
\]

Then

\[
\left\| \sum_{i=1}^{n} E(\sigma_i) F(\eta_i) x \right\|^2 \leq 4M^2 \sum_{p=1}^{n+1} \left\| E(\sigma_p) \left( \sum_{i=1}^{n} E(\sigma_i) F(\eta_i) x \right) \right\|^2 = C
\]

by Lemma 1;
\[ C = 4M^2 \sum_{\nu=1}^{n} \| F(\sigma_\nu)E(\eta_\nu)x \|^2, \]

since \( E(\sigma_\nu)E(\sigma_i) = E(\sigma_\nu \cap \sigma_i) \);

\[ C = 4M^2 \sum_{\nu=1}^{n} \| F(\eta_\nu)E(\sigma_\nu)x \|^2, \]

by commutativity of the \( E(\sigma) \) and \( F(\eta) \);

\[ C \leq 4M^2 \cdot L^2 \sum_{\nu=1}^{n} \| E(\sigma_\nu)x \|^2, \]

since \( \| F(\eta_\nu) \| \leq L \);

\[ C \leq (4M^2)^2 \cdot L^2 \| x \|^2, \]

by Lemma 1. Hence

\[ \left\| \sum_{i=1}^{n} E(\sigma_i)F(\eta_i) \right\| \leq 4M^2 L. \]

In the proof of Theorem 1 we shall use the method of Mackey in [3], together with Lemmas 1 and 2.

3. Proof of Theorem 1. By a "partition" \( \pi \) of the plane we mean a finite family of Borel sets \( \sigma_1, \sigma_2, \ldots, \sigma_n \), mutually disjoint and with union equal to the whole plane. If \((x, y)\) denotes the given scalar product in \( H \), and

\[ \pi_1 = (\sigma_i)_{i=1}^{n} \quad \pi_2 = (\eta_j)_{j=1}^{m} \]

are two partitions, set

\[ (x, y)_{\pi_1, \pi_2} = \sum_{i=1}^{n} \sum_{j=1}^{m} (E(\sigma_i)F(\eta_j)x, E(\sigma_i)F(\eta_j)y). \]

It is easily verified that the quantity \((x, y)_{\pi_1, \pi_2}\) is a scalar product in \( H \). Further, it follows by Lemma 2 that the operators...
\[ P_{ij} = E(\sigma_i) F(\eta_j) \quad (i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, m, ) \]
satisfy the hypotheses of Lemma 1.

Hence Lemma 1 yields

\[
\frac{1}{4K^2} \| x \|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \| E(\sigma_i) F(\eta_j) x \|^2 \leq 4K^2 \| x \|^2,
\]

where \( K \) depends only on \( \max_{\sigma} \| E(\sigma) \| \) and \( \max_{\eta} \| F(\eta) \| \). But

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \| E(\sigma_i) F(\eta_j) x \|^2 = (x, x)_{\pi_1, \pi_2} = \| x \|^2_{\pi_1, \pi_2}.
\]

Finally, each \( E(\sigma_i) \) and \( F(\eta_j) \) \( (i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, m) \) is self-adjoint in the scalar product \( (x, y)_{\pi_1, \pi_2} \), as is readily verified.

For each pair of vectors \( x, y \in H \), now, let \( S_{\pi} \) be the disk in the complex plane consisting of all \( z \) with

\[ |z| \leq 4K^2 \| x \| \| y \|. \]

If \( S \) denotes the cartesian product of the disks \( S_{\pi} \) over all pairs \( x, y \), then \( S \) is a compact topological space, by Tychonoff's theorem. Further, as we saw above,

\[ \| x \|^2_{\pi_1, \pi_2} \leq 4K^2 \| x \|^2. \]

Hence by Schwarz's inequality, applied to the scalar product \( (x, y)_{\pi_1, \pi_2} \), we see that the number \( (x, y)_{\pi_1, \pi_2} \) lies in the disk \( S_{\pi} \) for every pair \( x, y \). Hence there is a point \( p_{\pi_1, \pi_2} \) in \( S \) whose \( x, y \)-coordinate is \( (x, y)_{\pi_1, \pi_2} \).

Let us now partially order the set of points \( p_{\pi_1, \pi_2} \) in \( S \) by saying that \( p_{\pi_1', \pi_2'} \) is "greater than" \( p_{\pi_1, \pi_2} \) (in symbols \( p_{\pi_1, \pi_2} < p_{\pi_1', \pi_2'} \)) if \( \pi_1' \) is a refinement of the partion \( \pi_1 \), and \( \pi_2' \) is a refinement of the partition \( \pi_2 \). This ordering makes the set of points \( p_{\pi_1, \pi_2} \) in \( S \) into a directed system. Since \( S \) is a compact space, this directed system has a point of accumulation \( p \). Let \( (x, y)_p \) denote the \( (x, y) \) coordinate of \( p \).

Then given a finite set of vector pairs \( (x_i, y_i), i = 1, 2, \ldots, n, \) and \( \epsilon > 0, \) and a pair \( \pi_1^0, \pi_2^0 \) of partitions, we have
\[ |(x_i, y_i)_p - (x_i, y_i)_{\pi_1, \pi_2}| < \varepsilon \quad (i = 1, 2, \ldots, n) \]

for some

\[ p_{\pi_1, \pi_2} > p_{\pi_1^0, \pi_2^0}. \]

Since \((x, y)_{\pi_1, \pi_2}\) is a scalar product for all \(\pi_1, \pi_2\) it thus follows that \((x, y)_p\) is a scalar product, and since the norm \(||x||_{\pi_1, \pi_2}\) is equivalent to the original norm with constants of equivalence independent of \(\pi_1, \pi_2\), it follows that

\[ ||x||_p = \sqrt{(x, x)_p} \]

is also equivalent to the original norm.

Finally, fix a Borel set \(\sigma\) and vectors \(x, y\). Let \(\pi_1^0\) be the partition defined by \(\sigma\) and its complement, and \(\pi_2^0\) be arbitrary. Then, if

\[ p_{\pi_1, \pi_2} > p_{\pi_1^0, \pi_2^0}, \]

we have

\[ (E(\sigma)x, y)_{\pi_1, \pi_2} = (x, E(\sigma)y)_{\pi_1, \pi_2}, \]

since \(\pi_1\) is a refinement of \(\pi_1^0\), and so \(\sigma\) is a finite union of sets involved in the partition \(\pi_1\). Thus

\[ (E(\sigma)x, y)_p = (x, E(\sigma)y)_p, \]

and so the \(E(\sigma)\) are self-adjoint with respect to the scalar product \((x, y)_p\), and similarly the \(F(\eta)\) are self-adjoint with respect to this scalar product.

Since \(||x||_p\) is equivalent to the given norm, it now follows that there exists a bi-continuous operator \(A\) with \((x, y)_p = (Ax, Ay)\), and hence \(AE(\sigma)A^{-1}\) and \(AF(\eta)A^{-1}\) are all self-adjoint.

4. **Proof of Theorem 2.** By Theorem 8 of [1], an operator \(T\) is spectral if and only if there exist two commuting operators \(S\) and \(N\) such that \(N\) is quasi-nilpotent and \(S\) admits a representation:

\[ S = \int \lambda E(d\lambda), \]

where \(E(d\lambda)\) denotes integration with respect to a certain spectral measure.
Such an $S$ is called in [1] a "scalar type operator."

Now, by hypothesis, $T_1$ and $T_2$ are commuting spectral operators. We write

$$T_1 = S_1 + N_1, \quad T_2 = S_2 + N_2,$$

in accordance with the preceding. Then by Theorem 5 of [1] the operators $S_1, S_2, N_1, N_2$ all commute with one another. We thus have

$$T_1 + T_2 = S_1 + S_2 + Q \quad \text{and} \quad T_1 T_2 = S_1 S_2 + Q',$$

where $Q$ and $Q'$ are quasi-nilpotent, $Q$ commutes with $S_1 + S_2$, and $Q'$ commutes with $S_1 S_2$. By Theorem 8, quoted above, it is thus sufficient to show that $S_1 + S_2$ and $S_1 S_2$ are spectral operators of type $0$; that is, of scalar type.

Let $E^1(\sigma)$ and $E^2(\sigma)$ be the spectral measures for $S_1$ and $S_2$, respectively. By Theorem 5 of [1] it follows, from the fact that $S_1 S_2 = S_2 S_1$, that $E^1(\sigma)$ and $E^2(\sigma)$ commute with one another for all $\sigma$. By our Theorem 1, then, there exists an operator $A$ such that the operators $AE^1(\sigma)A^{-1}$ and $AE^2(\sigma)A^{-1}$ are all self-adjoint. Hence

$$J_1 = AS_1 A^{-1} \quad \text{and} \quad J_2 = AS_2 A^{-1},$$

are normal operators. Also $J_1 J_2 = J_2 J_1$, since $S_1 S_2 = S_2 S_1$. It follows that $J_1 + J_2$ and $J_1 J_2$ are again normal operators, for they commute with their adjoints as we verify by direct computation, using the fact that $J_1$ and $J_2^*$ commute and $J_2$ and $J_1^*$ commute, since $J_1$ and $J_2$ commute.

Thus $A(S_1 + S_2)A^{-1}$ and $A(S_1 S_2)A^{-1}$ are normal operators and so of scalar type. But if $J$ is a scalar type operator and $A$ bi-continuous, then, as is easily seen, $A^{-1}JA$ is again a scalar type operator. Hence $S_1 + S_2$ and $S_1 S_2$ are scalar type operators, and all is proved.

**References**

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