ASYMPTOTIC LOWER BOUNDS FOR THE FREQUENCIES OF CERTAIN POLYGONAL MEMBRANES

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1. Background. Let the bounded, simply connected, open region $R$ of the $(x, y)$ plane have the boundary curve $C$. If a uniform elastic membrane of unit density is uniformly stretched upon $C$ with unit tension across each unit length, the square $\lambda = \lambda (R)$ of the fundamental frequency satisfies the conditions (subscripts denote differentiation)

\[
\begin{align*}
\Delta u &= u_{xx} + u_{yy} = -\lambda u \quad \text{in } R, \\
\lambda &= \text{minimum},
\end{align*}
\]

with the boundary condition

\[
(1b) \quad u(x, y) = 0 \quad \text{on } C.
\]

The solution $u$ of problem (1) is unique up to a constant factor. It is known [13, p. 24] that $\lambda$ is the minimum over all piecewise smooth functions $u$ satisfying (1b) of the Rayleigh quotient

\[
\rho (u) = \iint_{R} |\nabla u|^2 \, dx \, dy / \iint_{R} u^2 \, dx \, dy,
\]

where $|\nabla u|^2 = u_x^2 + u_y^2$. In many practical methods for approximating $\lambda$ one essentially determines $\rho (u)$ for functions $u$ satisfying (1b) which are close to a solution of the boundary value problem (1). See [9, p. 112; 6, p. 276; 11, and 12]. By (2) these approximations are known to be upper bounds for $\lambda$; they can be made arbitrarily good with sufficient labor. It is obviously of equal importance to obtain close lower bounds for $\lambda$; cf. [14].

The lower bounds for $\lambda$ given by Pólya and Szegő [13] are ordinarily far

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from close. Those obtainable from \( \rho(u) \), \( \iint_R u^2 \, dx \, dy \), and \( \iint_R |\Delta u|^2 \, dx \, dy \) by methods due to Temple [15], D. H. Weinstein [17], Wielandt [18], and Kato [8] (for expositions see [3] and [16]) are arbitrarily good, but presuppose knowledge of a lower bound for the second eigenvalue \( \lambda_2 \) of the problem (1). The same is true of Davis’s proposals in [4]. It is possible, following Aronszajn and Zeichner [1], to get close lower bounds for \( \lambda \) by minimizing \( \rho(u) \) over a class of functions \( u \) permitted some discontinuity in \( R \) (method of A. Weinstein); the author has no knowledge of the practicability of the method.

A common method of approximating \( \lambda \) is to replace the boundary value problem (1) by a similar problem in finite differences. Divide the plane into squares of side \( h \) by the network of lines \( x = \mu h, y = \nu h \) \( (\mu, \nu = 0, \pm 1, \pm 2, \cdots). \) The points \( (\mu h, \nu h) \) are the nodes of the net. A half-square is an isosceles right triangle whose vertices are three nodes of one square of the net. Assume that

\[ R \text{ is the union of a finite number of squares and half-squares.} \]

Then every interior node of \( R \) has four neighboring nodes in \( R \cup C \).

Define \( \Delta_h \), a finite-difference approximation to \( \Delta \), by the relation

\[ h^2 \Delta_h v(x, y) = v(x + h, y) + v(x - h, y) + v(x, y + h) + v(x, y - h) - 4v(x, y). \]

Let \( \lambda_h \) be the least number satisfying the following difference equation for a net function \( v \) defined on the nodes \( (x, y) \) of the net:

\[ \Delta_h v = -\lambda_h v \text{ at the nodes in } R, \]

with the boundary condition

\[ v = 0 \text{ at the nodes on } C. \]

One can interpret \( \lambda_h \) as the square of the fundamental frequency of a network of massless strings with uniform tension \( h \), fastened to \( C \), and supporting a particle of mass \( h^2 \) at each node. That is, a certain lumping of the distributed masses and tensions of problem (1) yields problem (4).

It is easily verified for a rectangular region of commensurable sides \( \pi/p, \pi/q \), and for \( h \) such that (3) holds, that one has \( u = v = \sin px \sin qy \), and that

\[ \frac{\lambda_h}{\lambda} = \frac{\sin^2 \left( \frac{ph}{2} \right) + \sin^2 \left( \frac{qh}{2} \right)}{\left( \frac{ph}{2} \right)^2 + \left( \frac{qh}{2} \right)^2} = 1 - \frac{p^4 + q^4}{p^2 + q^2} \frac{h^2}{12} + o(h^2) \quad (h \to 0). \]

Hence \( \lambda_h < \lambda \) for all \( h \), and one can use \( \lambda_h \) as a lower bound for \( \lambda \). However,
since $\lambda$ is known exactly for rectangular regions, relation (5) contributes nothing to its computation. For general regions $R$, it was stated \[3, p.405\] in 1949 that nothing could be said about the relation of $\lambda_h$ to $\lambda$.

2. A new result. An asymptotic relation resembling (5) will now be established for any convex polygonal region $R$ satisfying (3). Such regions are polygons of at most eight sides, having interior vertex angles of $45^\circ$, $90^\circ$, or $135^\circ$. The following theorem\(^1\) will be proved in § 3 by use of the lemmas of § 4:

**Theorem.** Let $R$ be a convex region which is a finite union of squares and half-squares for all $h$ under consideration. Let $u$ solve problem (1) for $R$, and let

$$a = a(R) = \frac{\iint_R (u^2_{xx} + u^2_{yy}) \, dx \, dy}{\iint_R (u^2_x + u^2_y) \, dx \, dy}.$$  

Then, as $h \to 0$, one has

$$\frac{\lambda_h}{\lambda} \leq 1 - \frac{a}{12} h^2 + o(h^2) \quad (h \to 0).$$

It is a consequence of the theorem that, for all sufficiently small $h$, say for $h \leq h_0$, $\lambda_h$ is a lower bound for $\lambda$. The ordinary finite-difference method thus complements any method based on Rayleigh quotients; and, since $\lambda_h \to \lambda$ as $h \to 0$, together two such methods can confine $\lambda$ to an arbitrarily short interval. In particular, Pólya \[11 \text{ and } 12\] devises modified finite-difference approximations to problem (1) which furnish upper bounds to $\lambda$ for all $h$. Hence arbitrarily good two-sided bounds to $\lambda$ can be found by finite-difference methods alone.

The constant $a$ of the theorem is the best possible for a rectangle $R$ of sides $\pi/p$, $\pi/q$. For this region, we have $a = (p^4 + q^4) \cdot (p^2 + q^2)^{-1}$, and (6) is seen by (5) to be actually an equality up to terms $o(h^2)$.

Using heuristic reasoning, Milne \[9, p.238, (97.5)\] finds an approximate formula which, specialized to the fundamental eigenvalue and set in our notation, says

$$\frac{\lambda_h}{\lambda} \approx 1 - \frac{\lambda h^2}{24} + o(h^2) \quad (h \to 0).$$

\(^1\)The author gratefully acknowledges many helpful conversations with his colleague Dr. Wolfgang Wasow on the subject of this paper.
For a rectangle of sides $\pi/p, \pi/q$, the coefficient of $-h^2/12$ in (7) is $(p^2 + q^2)/2$. Since

$$\frac{p^2 + q^2}{2} + \frac{(p^2 - q^2)^2}{p^2 + q^2} = \frac{p^4 + q^4}{p^2 + q^2},$$

the coefficient of $h^2$ in (7) is low for all rectangles with $p \neq q$, and exact for squares. Hence (7) cannot ordinarily be expected to be exact in its $h^2$ term.

The use of the theorem to bound $\lambda$ is limited by our lack of knowledge of $h_0$. However, it is the author's conjecture that, for the regions $R$ of the theorem, $\lambda_h < \lambda$ for all $h$.

The convexity of $R$ is vital to the statement and proof of the theorem; in fact, by the remark after Lemma 4, $a = \infty$ for nonconvex polygons. A heuristic argument, supported by the numerical example of §5, has in fact convinced the author that, for nonconvex polygons, $\lambda_h > \lambda$ for all sufficiently small $h$.

The restriction of $R$ and $h$ to satisfy (3) is less essential, but is used in two ways: (i) to be sure that no interior node has a neighboring node outside $R$; (ii) to prove that $\Gamma = 0$ in Lemma 7. With an appropriate alteration of $\Delta_h$ near $C$, and with a modification of Lemma 7, one can extend the present method to obtain formulas of type (6) without assuming (3) —and even for convex regions $R$ bounded by piecewise analytic curves $C$. See [5]. Analogous results can be expected in $n$ dimensions.

3. Proof of the theorem. Let $K$ be the class of functions $u$ which vanish on $C$, such that $(uu_x)_x$ and $(uu_y)_y$ are continuous in $R \cup C$. Applying Gauss's divergence formula (27) with $p = uu_x, q = uu_y$, one finds that, for all $u$ in $K$, Green's formula is valid in the form

$$\iint_R |\nabla u|^2\,dx\,dy = -\iint_R u\Delta u\,dx\,dy.$$

Hence, for all $u \in K, \rho(u)$ in (2) can be rewritten with $-\iint_R u\Delta u\,dx\,dy$ in the numerator.

Since, by Lemma 1, the function $u$ which minimizes (2) and solves (1) belongs to $K$, and since any function in $K$ is piecewise smooth, one may alternatively define $\lambda$ as the minimum, over all functions in $K$, of the quotient

$$\rho(u) = -\iint_R u\Delta u\,dx\,dy / \iint_R u^2\,dx\,dy.$$
Analogously, without having to worry about function classes, one can show that \( \lambda_h \) is the minimum, over all net functions \( v \) satisfying (4b), of the quotient

\[
(8) \quad \rho_h(v) = -h^2 \sum_{N_h} v \Delta_h v / h^2 \sum_{N_h} v^2,
\]

where the sums are extended over all nodes \( N_h \) of the net inside \( R \).

The key to proving the theorem is to set the solution \( u \) of problem (1) into the Rayleigh quotient (8) of problem (4). It will be shown that

\[
(9) \quad \frac{\rho_h(u)}{\lambda} = 1 - \frac{1}{12} ah^2 + o(h^2) \quad (h \to 0).
\]

Since \( \lambda_h \leq \rho_h(u) \), the theorem follows from (9). Henceforth \( u \) will always denote a solution of problem (1).

The denominator of \( \rho_h(u) \) is a Riemann sum for \( \iiint_R u^2 \, dx \, dy \). Since \( u^2 \) is continuous and hence Riemann integrable over \( R \),

\[
(10) \quad h^2 \sum_{N_h} u^2 = \iint_R u^2 \, dx \, dy + o(1) \quad (h \to 0).
\]

(It can be shown that one can replace \( o(1) \) by \( o(h^2) \) in (10), but we shall not need to do this.)

The nodes \( N_h \) inside \( R \) are divided into two classes:

- \( N_h' \): those at a distance \( h \) from some 135° vertex of \( C \);
- \( N_h'' \): the other nodes of \( N_h \).

Split the numerator of \( \rho_h(u) \) accordingly:

\[
(11) \quad -h^2 \sum_{N_h} u \Delta_h u = -h^2 \sum_{N_h'} u \Delta_h u - h^2 \sum_{N_h''} u \Delta_h u = S'_h(u) + S''_h(u).
\]

To estimate \( S'_h(u) \) note that, since there are at most eight 135° vertices, the number of nodes in \( N_h' \) is at most 8, for any \( h \). At any node in \( N_h' \),

\[
h^2 |u \Delta_h u| \leq h^2 \left( \frac{u - u_i}{h} \right) \sum_{i=1}^{4} \left| \frac{u - u_i}{h} \right| \leq 4h^2 \max |\nabla u|^2,
\]
where the maximum of $|\nabla u|^2$ is taken for all points $(x, y)$ within a distance $2h$ of some $135^\circ$ vertex. Hence, by Lemma 2, as $h \to 0$ through values such that (3) holds,

$$(12) \quad |S'_h(u)| \leq 32h^2 \max |\nabla u|^2 = o(h^2) \quad (h \to 0).$$

Now, using the notation and assertion of Lemma 5, one obtains

$$(13) \quad S''_h(u) = -h^2 \sum \sum_{N''_h} u \Delta u - \frac{h^4}{12} \sum \sum_{N''_h} u (u''_{xxx} + u''_{yyy}).$$

Since $u$ satisfies (1a),

$$(14) \quad -h^2 \sum \sum_{N''_h} u \Delta u = \lambda h^2 \sum \sum_{N''_h} u^2 = \lambda h^2 \sum \sum_{N''_h} u^2 + o(h^2) \quad (h \to 0);$$

the last step is correct because $u(x, y) \to 0$ as $(x, y) \to C$.

Combining (13) and (14), one finds that, as $h \to 0$,

$$(15) \quad S''_h(u) = \lambda h^2 \sum \sum_{N''_h} u^2 - \frac{h^4}{12} \sum \sum_{N''_h} u (u''_{xxx} + u''_{yyy}) + o(h^2)$$

by Lemma 6. The integrals used in this proof exist, by Lemma 3. Using (11), (12), (15), and Lemma 7, one finds that

$$(16) \quad -h^2 \sum \sum_{N_h} u \Delta_h u$$

$$= \lambda h^2 \sum \sum_{N_h} u^2 - \frac{h^2}{12} \iint_R (u''_{xxx} + u''_{yyy}) dxdy + o(h^2) \quad (h \to 0).$$

Dividing (16) by the denominator of $\rho_h(u)$, one gets
\[ \rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iiint_R \left( u_{xx}^2 + u_{yy}^2 \right) dx dy}{\int_R u^2 dx dy} + o(h^2). \]

Hence, by (10),

\[ \rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iiint_R \left( u_{xx}^2 + u_{yy}^2 \right) dx dy}{\int_R u^2 dx dy} + o(h^2) \quad (h \to 0). \]

If one divides (17) by \( \lambda \), and notes from (2) that \( \lambda \int_R u^2 dx dy = \int_R \| \nabla u \|^2 dx dy \), it is seen that

\[ \frac{\rho_h(u)}{\lambda} = 1 - \frac{h^2}{12} \frac{\iiint_R \left( u_{xx}^2 + u_{yy}^2 \right) dx dy}{\int_R \| \nabla u \|^2 dx dy} + o(h^2) \quad (h \to 0). \]

By the definition of \( a \) we have proved (9) and hence the theorem.

4. Some lemmas. Lemma 1, suggested to the author by Professor Max Shiffman, is used to establish Lemmas 2 to 7, which were applied to prove the theorem. In all the lemmas \( R \) is the convex union of squares and half-squares of the network, while \( u = u(x, y) \) is a function solving problem (1) in \( R \).

**Lemma 1.** The function \( u \) is an analytic function of \( x \) and \( y \) in \( R \cup C \), except at the \( 135^\circ \) vertices of \( C \). Let \( r, \theta \) be local polar coordinates centered at a \( 135^\circ \) vertex \( P_k \), with \( 0 < \theta < 3\pi/4 \) in \( R \). Then

\[ u = \gamma_k r^{4/3} \sin \left( \frac{4\theta}{3} \right) + r^{7/3} E_k(r, \theta), \]

where \( \gamma_k \) is a constant, and where \( E_k(r, \theta) \), together with all its derivatives, is bounded in a neighborhood of \( P_k \).

**Proof.** By reflection one can continue \( u \) antisymmetrically across each straight segment of \( C \), and (1a) is satisfied by the extended \( u \) at all points of \( R \cup C \) except the \( 135^\circ \) vertices. The first sentence of the lemma then follows from [2, p. 179].

For \( (\xi, \eta) \in R \), write \( t = \xi + i\eta \). For each \( t \), let \( w = w(z, t) \) be an analytic function of the complex variable \( z = x + iy \) which maps \( R \) into the unit circle \( |w| < 1 \), with \( f(t, t) = 0 \). To study \( f \) near a vertex \( z_k \) of \( C \), one may assume
that $f(z_k, t) = 1$. Let the interior vertex angle of $C$ at $z_k$ be $\pi/\alpha_k$ ($\alpha_k = 4, 2, \text{ or } 4/3$). It is a property of the Schwarz-Christoffel transformation [10, p. 189] that

(19) \[ f(z, t) = 1 + (z - z_k)^{\alpha_k} g_k(z, t), \]

where $g_k$ is an analytic function of $z$ regular at $z_k$.

Let $G(z, t) = G(x, y, \xi, \eta)$ be Green’s function for $\Delta u$ in $R$. Now $G(z, t) = -(2\pi)^{-1} \log |f(z, t)|$; see [10, p. 181]. It then follows from (19) that, in the notation of the lemma, when $\alpha_k = 4/3$,

(20) \[ G(z, t) = \gamma_k(t) r^{4/3} \sin \left(4\theta/3\right) + r^{7/3} E_k(r, \theta, t). \]

Moreover, $\gamma_k(t)$ and $E_k(r, \theta, t)$ are integrable over $R$, since the only discontinuity of $G(z, t)$ is a logarithmic one at $t = z$.

The function $u$ is representable by the integral [2, pp. 182-3]

(21) \[ u(x, y) = \lambda \int_{R} G(x, y, \xi, \eta) u(\xi, \eta) \, d\xi \, d\eta. \]

Substituting (20) into (21) proves (18) and the lemma.

**Lemma 2.** $|\nabla u(x, y)| \to 0$ as $(x, y) \to$ any 135° vertex of $C$.

**Proof.** By (18), $|\nabla u| = O(r^{1/3})$, as $(x, y) \to$ any 135° vertex of $C$.

**Lemma 3.** The functions $u_x^2, u_x^2u_{xx}, uu_{xxx}, u_y^2, u_y^2u_{yy}, \text{ and } uu_{yyy}$ are Lebesgue-integrable in $R$.

**Proof.** By Lemma 1 these functions are continuous in $R \cup C$, except at the 135° vertices $P_k$. At these vertices (18) implies that they are $O(r^{-4/3})$ and are hence integrable.

**Lemma 4.** The Lebesgue integrals $\int_C u_yu_{yy} \, dx$ and $\int_C u_xu_{xx} \, dy$ exist.

**Proof.** Analogous to that of Lemma 3.

**Remark.** Lemmas 2, 3, and 4 are false for polygonal regions $R$ which are not convex, since in general the exponent in (18) is $\alpha_k$, where $\pi/\alpha_k$ is the interior angle at the vertex $P_k$.

**Lemma 5.** At each node $(x, y)$ in $R$ of the network of section 1, one has
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(22) \[ \Delta_h u = \Delta u + \frac{1}{12} h^2 (u_{xxxx} + u_{yyyy}), \]

where

\[
\begin{align*}
\phi'_{xxxx} &= u_{xxxx}(x + \theta'h, y), \quad -1 < \theta' < 1; \\
\phi''_{yyyy} &= u_{yyyy}(x, y + \theta''h), \quad -1 < \theta'' < 1.
\end{align*}
\]

Proof. By Lemma 1, \( u_{xxxx} \) is continuous in the open line segment from \((x - h, y)\) to \((x + h, y)\) (though infinite at any \(135^\circ\) vertex). Since \( u \) is continuous in \( R \cup C \), it follows from Taylor's formula [7, p. 357] that, if we fix \( y \) and set \( \phi(x) = u(x, y) \),

\[
\phi(x + h) + \phi(x - h) - 2\phi(x) = h^2 \phi''(x) + \frac{1}{24} h^4 [\phi''''(x + \theta_1 h) + \phi''''(x - \theta_2 h)],
\]

where \( 0 < \theta_i < 1 \) \((i = 1, 2)\). By the continuity of \( \phi'''' \), the last bracket equals \( 2\phi''''(x + \theta'h) \), where \(-1 < \theta' < 1\).

A similar formula for \( \psi(y) = u(x, y) \), when added to the above and divided by \( h^2 \), yields (22) and (23).

Lemma 6. Define \( N_h'' \) as in §3. For each node \((x, y)\) in \( N_h'' \), use the notation of (23). Then, as \( h \to 0 \) over values such that (3) holds, one has

(24) \[ h^2 \sum_{N_h''} u(u_{xxxx} + u_{yyyy}) = \iint_R u(u_{xxxx} + u_{yyyy}) \, dx \, dy + o(1) \,(h \to 0). \]

Proof. For all \((x, y)\) in the entire plane \( E_2 \) define

\[
f(x, y) = \begin{cases} 
  u(u_{xxxx} + u_{yyyy}), & \text{if } (x, y) \in R; \\
  0, & \text{elsewhere}.
\end{cases}
\]

By the proof of Lemma 3 one sees that \( f(x, y) \) is \( O(r^{-4/3}) \) in the neighborhood of each \(135^\circ\) vertex \( P_k \) of \( C \), and continuous elsewhere. Divide the nodes \((x, y) = (\mu h, \nu h)\) of \( N_h'' \subset R \) into four classes \( K^{(i)} \) \((i = 1, 2, 3, 4)\) according to the parity of \((\mu, \nu)\). Fix any class \( K^{(i)} \). For each vertex \((x, y)\) in \( K^{(i)} \) let \( S(x, y) \) be the union of the four closed network squares of \( E_2 \) which contain \((x, y)\). The area
of each \( S(x, y) \) is \( 4h^2 \); ordinarily certain of the \( S(x, y) \) contain points not in \( R \). Define

\[
f_h^{(i)}(ξ, η) = \begin{cases} 
  u(x, y) (u''_{xxx} + u''_{yyy}), & \text{for } (ξ, η) \in S(x, y); \\
  0, & \text{for } (ξ, η) \notin \bigcup S(x, y).
\end{cases}
\]

Then \( f_h^{(i)}(ξ, η) \to f(ξ, η) \), as \( h \to 0 \), for almost all \((ξ, η)\) in the plane. Using the fact that no node of \( N_h'' \) is adjacent to a 135° vertex of \( C \), one can show that for all \( i \), uniformly in \( h \), \( |f_h^{(i)}(ξ, η)| \leq F(ξ, η) \), where \( F \) is an integrable function in \( E_2 \).

Each term of the sum (24) for which \((x, y) \in K^{(i)}\) is equal to

\[
\frac{1}{4} \iint_{S(x, y)} f_h^{(i)}(ξ, η) dξ dη.
\]

Hence, applying Lebesgue’s convergence theorem, one sees that, as \( h \to 0 \), for each \( i \),

\[
\sum_{N_h'' \cap K^{(i)}} u(u''_{xxx} + u''_{yyy}) = \frac{1}{4} \iint_{E_2} f_h^{(i)}(ξ, η) dξ dη
\]

(25)

\[
\to \frac{1}{4} \iint_{E_2} f(ξ, η) dξ dη \quad (h \to 0).
\]

Summing (25) over \( i = 1, 2, 3, 4 \) proves (24) and the lemma.

**Lemma 7.** One has

(26)

\[
\iint_{R} u(u''_{xxx} + u''_{yyy}) \, dx dy = \iint_{R} (u^2_{xx} + u^2_{yy}) \, dx dy.
\]

*Proof.* The following applications of Gauss’s divergence theorem in the form

(27)

\[
\iint_{R} (p_x + q_y) \, dx dy = \int_{C} (pdy - qdx)
\]

can be justified by integrating over the region \( R^* \) interior to a smooth convex curve \( C^* \) inside \( R \), and then letting \( C^* \to C \) appropriately. The continuity of
the integrals in the limit follows from Lemmas 1, 3, and 4.

In the divergence theorem for \( p = uu_{xxx}, q = uu_{yyy} \), the line integral vanishes, and one finds

\[
\iint_R u (u_{xxxx} + u_{yyyy}) \, dx \, dy = - \iint_R (u_x u_{xxx} + u_y u_{yyy}) \, dx \, dy.
\]

A second application of the divergence theorem with \( p = u_x u_{xx}, q = u_y u_{yy} \), combined with (28), shows that

\[
\iint_R u (u_{xxxx} + u_{yyyy}) \, dx \, dy = \iint_R (u_x^2 + u_y^2) \, dx \, dy + \Gamma,
\]

where \( \Gamma = \int_C (u_y u_{yy} \, dx - u_x u_{xx} \, dy) \).

By (1a), \( u_{xx} = -u_{yy} \) on \( C \), whence \( \Gamma = \int_C u_{yy} (u_y \, dx + u_x \, dy) \). On the segments of \( C \) parallel to the axes, \( u_{xx} = u_{yy} = 0 \), so that there the contribution to \( \Gamma \) is zero.

Now the vector \( \nabla u = (u_x, u_y) \) is perpendicular to \( C \). On the segments of \( C \) making a 45° or 135° angle with the x-axis, \( (u_y, u_x) \) is parallel to \( (u_x, u_y) \), whence \( (u_y, u_x) \) is perpendicular to \( C \). Thus \( u_{yy} \, dx + u_x \, dy = 0 \) when \( (dx, dy) \) is tangent to \( C \), so that the contribution to \( \Gamma \) from these 45° and 135° segments of \( C \) is also zero.

Hence \( \Gamma = 0 \), and the lemma follows from (29).

5. **Numerical example.** Let \( R_1 \) be the six-sided, nonconvex, \( L \)-shaped region whose closure is the union of the three unit squares

\[
\begin{align*}
-1 & \leq x \leq 0, \quad 0 \leq y \leq 1; \\
0 & \leq x \leq 1, \quad 0 \leq y \leq 1; \\
0 & \leq x \leq 1, \quad -1 \leq y \leq 0.
\end{align*}
\]

The fundamental frequencies \( \lambda_h = \lambda_h (R_1) \) and corresponding net functions \( v \) were computed by B.F. Handy on the SWAC (National Bureau of Standards Western Automatic Computer) for \( 1/h = 3, 4, \ldots, 8 \). The computation used a power method; for some initial net function \( v_0 \), \( (h^2 \Delta_h + 5I)^m v_0 \) was determined for large positive integers \( m \), where \( I \) is the identity operator. On the basis of Collatz's inclusion theorem [3, p. 289], the values in the accompanying table are believed to have errors less than \( 5 \times 10^{-6} \). Observe that \( \lambda_h (R_1) \) is less for \( h = 1/8 \) than for \( h = 1/7 \).
Since $R_1$ is not convex, the theorem of § 2 does not apply, but a heuristic argument suggests that $\lambda_h(R_1) - \lambda(R_1) = O(h^{4/3})$. A least-squares fit to the values of $\lambda_h(R_1)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_1) = \alpha_1 + \beta_1 h^{4/3} + \gamma_1 h^2 = \phi_1(h)$$

yielded the values

(30) $\alpha_1 = 9.63632, \quad \beta_1 = 2.40286, \quad \gamma_1 = -5.97212$.

The maximum of $|\lambda_h(R_1) - \phi_1(h)|$ for the five values of $h$ is .00013. Hence $\alpha_1$ is a working estimate of $\lambda(R_1)$.

The fact that $\beta_1 > 0$ in (30) supports the author's conjecture that, for nonconvex polygonal domains satisfying (3), $\lambda_h > \lambda$ for all sufficiently small $h$.

The table also gives Handy's values for the second eigenvalues of $R_1$, which are the fundamental eigenvalues $\lambda_h(R_2)$ of the trapezoidal halfdomain $R_2$ of $R_1$ for which $x > y$. Since the theorem does apply to $R_2$, a least-squares fit to the values of $\lambda_h(R_2)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_2) = \alpha_2 + \beta_2 h^2 = \phi_2(h)$$

seemed appropriate, and yielded the values

$\alpha_2 = 15.19980, \quad \beta_2 = -13.22219$.

The maximum of $|\lambda_h(R_2) - \phi_2(h)|$ for the five values of $h$ was .00010. Hence $\alpha_2$ is a working estimate of $\lambda(R_2)$.

The value of $\beta_2$ is negative, in agreement with (6), but the quantity
$-12\beta_2/\alpha_2 = 10.4387$ is something like one-fifth larger than an estimate of the corresponding quantity $a(R_2)$ of the theorem. One therefore suspects that $a$ is not the best possible constant in (6) for the region $R_2$.

In the table, note the relative closeness of the values of $\lambda_h(R_2)$ to the working estimate, $\alpha_2$, of $\lambda(R_2)$, even for a coarse net. Thus the value 12 for $\lambda_{1/2}(R_2)$, which is obtained by pencil and paper from a simple quadratic equation, is comparable to the lower bounds 12.1 and $5\pi^2/4$ obtained respectively by comparison with $\lambda$ for the circular membrane of equal area [13, p. 8] and with $\lambda$ for the rectangular region $0 < x < 1$; $-1 < y < 1$. The value $\lambda_{1/3}(R_2) = 13.737$ requires getting the least eigenvalue of a 7th-order matrix, a relatively easy procedure with a desk machine.

The monotonicity of $\lambda_h(R_2)$ supports the author's conjecture\(^2\) that, for the $R$ of the theorem, $\lambda_h < \lambda$ for all $h$.

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\(^2\)See page 470.

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