Pacific Journal of Mathematics

ASYMPTOTIC LOWER BOUNDS FOR THE FREQUENCIES OF CERTAIN POLYGONAL MEMBRANES

GEORGE E. FORSYTHE

Vol. 4, No. 3 July 1954

ASYMPTOTIC LOWER BOUNDS FOR THE FREQUENCIES OF CERTAIN POLYGONAL MEMBRANES

George E. Forsythe

1. Background. Let the bounded, simply connected, open region R of the (x, y) plane have the boundary curve C. If a uniform elastic membrane of unit density is uniformly stretched upon C with unit tension across each unit length, the square $\lambda = \lambda(R)$ of the fundamental frequency satisfies the conditions (subscripts denote differentiation)

$$(1a) \qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u & \text{in } R, \\ \lambda = \text{minimum}, \end{cases}$$

with the boundary condition

$$(1b) \qquad u(x, y) = 0 \quad \text{on } C.$$

The solution u of problem (1) is unique up to a constant factor. It is known [**13**, p. 24] that λ is the minimum over all piecewise smooth functions u satisfying (1b) of the Rayleigh quotient

$$(2) \qquad \rho(u) = \iint_R |\nabla u|^2 dxdy \Big/ \iint_R u^2 dxdy,$$

where $|\nabla u|^2 = u_x^2 + u_y^2$. In many practical methods for approximating λ one essentially determines $\rho(u)$ for functions u satisfying (1b) which are close to a solution of the boundary value problem (1). See [**9**, p. 112; **6**, p. 276; **11**, and **12**]. By (2) these approximations are known to be *upper bounds* for λ; they can be made arbitrarily good with sufficient labor. It is obviously of equal importance to obtain close lower bounds for λ; cf. [**14**].

The lower bounds for λ given by Pólya and Szegö [**13**] are ordinarily far

Presented to the American Mathematical Society May 2, 1953, under a slightly different title; received by the editors May 15, 1953. The work on this report was sponsored in part by the Office of Naval Research, USN.

Pacific J. Math. 4 (1954), 467-480

from close. Those obtainable from $\rho(u)$, $\iint_R u^2 dxdy$, and $\iint_R |\Delta u|^2 dxdy$ by methods due to Temple [**15**], D. H. Weinstein [**17**], Wielandt [**18**], and Kato [**8**] (for expositions see [**3**] and [**16**]) are arbitrarily good, but presuppose knowledge of a lower bound for the second eigenvalue λ_2 of the problem (1). The same is true of Davis's proposals in [**4**]. It is possible, following Aronszajn and Zeichner [**1**], to get close lower bounds for λ by minimizing $\rho(u)$ over a class of functions u permitted some discontinuity in R (method of A. Weinstein); the author has no knowledge of the practicability of the method.

A common method of approximating λ is to replace the boundary value problem (1) by a similar problem in finite differences. Divide the plane into *squares* of side h by the network of lines $x = \mu h$, $y = \nu h$ (μ, $\nu = 0, \pm 1, \pm 2, \cdots$). The points $(\mu h, \nu h)$ are the *nodes* of the net. A *half-square* is an isosceles right triangle whose vertices are three nodes of one square of the net. Assume that

(3) R is the union of a finite number of squares and half-squares.

Then every interior node of R has four neighboring nodes in $R \cup C$.

Define Δ_h, a finite-difference approximation to Δ, by the relation

$$h^2 \Delta_h v(x,y) = v(x+h,y) + v(x-h,y) + v(x,y+h) + v(x,y-h) - 4v(x,y).$$

Let λ_h be the least number satisfying the following difference equation for a net function v defined on the nodes (x,y) of the net:

$$\Delta_h v = -\lambda_h v \quad \text{at the nodes in } R, \tag{4a}$$

with the boundary condition

$$v = 0 \quad \text{at the nodes on } C. \tag{4b}$$

One can interpret λ_h as the square of the fundamental frequency of a network of massless strings with uniform tension h, fastened to C, and supporting a particle of mass h^2 at each node. That is, a certain lumping of the distributed masses and tensions of problem (1) yields problem (4).

It is easily verified for a rectangular region of commensurable sides π/p, π/q, and for h such that (3) holds, that one has $u = v = \sin px \sin qy$, and that

$$\frac{\lambda_h}{\lambda} = \frac{\sin^2(ph/2) + \sin^2(qh/2)}{(ph/2)^2 + (qh/2)^2} = 1 - \frac{p^4 + q^4}{p^2 + q^2}\,\frac{h^2}{12} + o(h^2) \quad (h \longrightarrow 0). \tag{5}$$

Hence $\lambda_h < \lambda$ for all h, and one can use λ_h as a lower bound for λ. However, since λ is known exactly for rectangular regions, relation (5) contributes nothing to its computation. For general regions R, it was stated [**3**, p.405] in 1949 that nothing could be said about the relation of λ_h to λ.

2. A new result. An asymptotic relation resembling (5) will now be established for any *convex* polygonal region R satisfying (3). Such regions are polygons of at most eight sides, having interior vertex angles of 45°, 90°, or 135°. The following theorem[1] will be proved in § 3 by use of the lemmas of § 4:

THEOREM. *Let R be a convex region which is a finite union of squares and half-squares for all h under consideration. Let u solve problem (1) for R, and let*

$$a = a(R) = \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R (u_x^2 + u_y^2)\,dxdy}.$$

Then, as $h \longrightarrow 0$, one has

$$\frac{\lambda_h}{\lambda} \leq 1 - \frac{a}{12} h^2 + o(h^2) \qquad (h \longrightarrow 0). \tag{6}$$

It is a consequence of the theorem that, for all sufficiently small h, say for $h \leq h_0$, λ_h is a *lower bound* for λ. The ordinary finite-difference method thus complements any method based on Rayleigh quotients; and, since $\lambda_h \longrightarrow \lambda$ as $h \longrightarrow 0$, together two such methods can confine λ to an arbitrarily short interval. In particular, Pólya [**11** and **12**] devises modified finite-difference approximations to problem (1) which furnish upper bounds to λ for all h. Hence arbitrarily good two-sided bounds to λ can be found by finite-difference methods alone.

The constant a of the theorem is the best possible for a rectangle R of sides π/p, π/q. For this region, we have $a = (p^4 + q^4) \cdot (p^2 + q^2)^{-1}$, and (6) is seen by (5) to be actually an equality up to terms $o(h^2)$.

Using heuristic reasoning, Milne [**9**, p. 238, (97.5)] finds an approximate formula which, specialized to the fundamental eigenvalue and set in our notation, says

$$\frac{\lambda_h}{\lambda} \doteq 1 - \frac{\lambda h^2}{24} + o(h^2) \qquad (h \longrightarrow 0). \tag{7}$$

---

[1] The author gratefully acknowledges many helpful conversations with his colleague Dr. Wolfgang Wasow on the subject of this paper.

For a rectangle of sides π/p, π/q, the coefficient of $-h^2/12$ in (7) is $(p^2+q^2)/2$. Since

$$\frac{p^2+q^2}{2} + \frac{(p^2-q^2)^2}{p^2+q^2} = \frac{p^4+q^4}{p^2+q^2},$$

the coefficient of h^2 in (7) is low for all rectangles with $p \neq q$, and exact for squares. Hence (7) cannot ordinarily be expected to be exact in its h^2 term.

The use of the theorem to bound λ is limited by our lack of knowledge of h_0. However, it is the author's conjecture that, for the regions R of the theorem, $\lambda_h < \lambda$ for all h.

The convexity of R is vital to the statement and proof of the theorem; in fact, by the remark after Lemma 4, $a = \infty$ for nonconvex polygons. A heuristic argument, supported by the numerical example of § 5, has in fact convinced the author that, for nonconvex polygons, $\lambda_h > \lambda$ for all sufficiently small h.

The restriction of R and h to satisfy (3) is less essential, but is used in two ways: (i) to be sure that no interior node has a neighboring node outside R; (ii) to prove that $\Gamma = 0$ in Lemma 7. With an appropriate alteration of Δ_h near C, and with a modification of Lemma 7, one can extend the present method to obtain formulas of type (6) without assuming (3)—and even for convex regions R bounded by piecewise analytic curves C. See [5]. Analogous results can be expected in n dimensions.

3. Proof of the theorem. Let K be the class of functions u which vanish on C, such that $(uu_x)_x$ and $(uu_y)_y$ are continuous in $R \cup C$. Applying Gauss's divergence formula (27) with $p = uu_x$, $q = uu_y$, one finds that, for all u in K, Green's formula is valid in the form

$$\iint_R |\nabla u|^2 dxdy = -\iint_R u\Delta u dxdy.$$

Hence, for all $u \in K$, $\rho(u)$ in (2) can be rewritten with $-\iint_R u\Delta u\, dxdy$ in the numerator.

Since, by Lemma 1, the function u which minimizes (2) and solves (1) belongs to K, and since any function in K is piecewise smooth, one may alternatively define λ as the minimum, over all functions in K, of the quotient

$$\rho(u) = -\iint_R u\Delta u dxdy \Big/ \iint_R u^2 dxdy.$$

Analogously, without having to worry about function classes, one can show that λ_h is the minimum, over all net functions v satisfying (4b), of the quotient

$$(8) \qquad \rho_h(v) = -h^2 \sum_{N_h}\sum v\Delta_h v \Big/ h^2 \sum_{N_h}\sum v^2,$$

where the sums are extended over all nodes N_h of the net inside R.

The key to proving the theorem is to set the solution u of problem (1) into the Rayleigh quotient (8) of problem (4). It will be shown that

$$(9) \qquad \frac{\rho_h(u)}{\lambda} = 1 - \frac{1}{12} ah^2 + o(h^2) \qquad (h \longrightarrow 0).$$

Since $\lambda_h \leq \rho_h(u)$, the theorem follows from (9). Henceforth u will always denote a solution of problem (1).

The denominator of $\rho_h(u)$ is a Riemann sum for $\iint_R u^2\,dxdy$. Since u^2 is continuous and hence Riemann integrable over R,

$$(10) \qquad h^2 \sum_{N_h}\sum u^2 = \iint_R u^2\,dxdy + o(1) \qquad (h \longrightarrow 0).$$

(It can be shown that one can replace $o(1)$ by $o(h^2)$ in (10), but we shall not need to do this.)

The nodes N_h inside R are divided into two classes:

N_h' : those at a distance h from some 135° vertex of C;

N_h'': the other nodes of N_h.

Split the numerator of $\rho_h(u)$ accordingly:

$$(11) \qquad -h^2 \sum_{N_h}\sum u\Delta_h u = -h^2 \sum_{N_h'}\sum u\Delta_h u - h^2 \sum_{N_h''}\sum u\Delta_h u = S_h'(u) + S_h''(u).$$

To estimate $S_h'(u)$ note that, since there are at most eight 135° vertices, the number of nodes in N_h' is at most 8, for any h. At any node in N_h',

$$h^2|u\Delta_h u| \leq h^2\left(\frac{u-0}{h}\right) \sum_{i=1}^{4} \left|\frac{u-u_i}{h}\right| \leq 4h^2 \max\ |\nabla u|^2,$$

where the maximum of $|\nabla u|^2$ is taken for all points (x, y) within a distance $2h$ of some 135° vertex. Hence, by Lemma 2, as $h \longrightarrow 0$ through values such that (3) holds,

$$|S_h'(u)| \leq 32h^2 \max |\nabla u|^2 = o(h^2) \qquad (h \longrightarrow 0). \tag{12}$$

Now, using the notation and assertion of Lemma 5, one obtains

$$S_h''(u) = -h^2 \sum_{N_h''}\sum u\Delta u - \frac{h^4}{12} \sum_{N_h''}\sum u(u'_{xxxx} + u''_{yyyy}). \tag{13}$$

Since u satisfies (la),

$$-h^2 \sum_{N_h''}\sum u\Delta u = \lambda h^2 \sum_{N_h''}\sum u^2 = \lambda h^2 \sum_{N_h}\sum u^2 + o(h^2) \quad (h \longrightarrow 0); \tag{14}$$

the last step is correct because $u(x, y) \longrightarrow 0$ as $(x, y) \longrightarrow C$.

Combining (13) and (14), one finds that, as $h \longrightarrow 0$,

$$\begin{aligned} S_h''(u) &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^4}{12} \sum_{N_h''}\sum u(u'_{xxxx} + u''_{yyyy}) + o(h^2) \\ &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^2}{12} \iint_R u(u_{xxxx} + u_{yyyy})\,dxdy + o(h^2), \end{aligned} \tag{15}$$

by Lemma 6. The integrals used in this proof exist, by Lemma 3. Using (11), (12), (15), and Lemma 7, one finds that

$$\begin{aligned} -h^2 \sum_{N_h}\sum u\Delta_h u & \\ = \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^2}{12} \iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy + o(h^2) & \qquad (h \longrightarrow 0). \end{aligned} \tag{16}$$

Dividing (16) by the denominator of $\rho_h(u)$, one gets

$$\rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{h^2 \sum\sum_{N_h} u^2} + o(h^2).$$

Hence, by (10),

$$\rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R u^2\,dxdy} + o(h^2) \qquad (h \longrightarrow 0). \tag{17}$$

If one divides (17) by λ, and notes from (2) that $\lambda \iint_R u^2 dxdy = \iint_R |\nabla u|^2 dxdy$, it is seen that

$$\frac{\rho_h(u)}{\lambda} = 1 - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R |\nabla u|^2\,dxdy} + o(h^2) \qquad (h \longrightarrow 0).$$

By the definition of a we have proved (9) and hence the theorem.

4. Some lemmas. Lemma 1, suggested to the author by Professor Max Shiffman, is used to establish Lemmas 2 to 7, which were applied to prove the theorem. In all the lemmas R is the convex union of squares and half-squares of the network, while $u = u(x, y)$ is a function solving problem (1) in R.

LEMMA 1. *The function u is an analytic function of x and y in $R \cup C$, except at the* 135° *vertices of C. Let r, θ be local polar coordinates centered at a* 135° *vertex P_k, with $0 < \theta < 3\pi/4$ in R. Then*

$$u = \gamma_k r^{4/3} \sin(4\theta/3) + r^{7/3} E_k(r, \theta), \tag{18}$$

where γ_k is a constant, and where $E_k(r, \theta)$, together with all its derivatives, is bounded in a neighborhood of P_k.

Proof. By reflection one can continue u antisymmetrically across each straight segment of C, and (1a) is satisfied by the extended u at all points of $R \cup C$ except the 135° vertices. The first sentence of the lemma then follows from [2, p. 179].

For $(\xi, \eta) \in R$, write $t = \xi + i\eta$. For each t, let $w = f(z, t)$ be an analytic function of the complex variable $z = x + iy$ which maps R into the unit circle $|w| < 1$, with $f(t, t) = 0$. To study f near a vertex z_k of C, one may assume that $f(z_k, t) = 1$. Let the interior vertex angle of C at z_k be π/α_k ($\alpha_k = 4, 2$, or $4/3$). It is a property of the Schwarz-Christoffel transformation [**10**, p. 189] that

$$f(z,t) = 1 + (z - z_k)^{\alpha_k} g_k(z,t), \tag{19}$$

where g_k is an analytic function of z regular at z_k.

Let $G(z,t) = G(x,y;\xi,\eta)$ be Green's function for Δu in R. Now $G(z,t) = -(2\pi)^{-1} \log |f(z,t)|$; see [**10**, p. 181]. It then follows from (19) that, in the notation of the lemma, when $\alpha_k = 4/3$,

$$G(z,t) = \gamma_k(t) r^{4/3} \sin (4\theta/3) + r^{7/3} E_k(r,\theta,t). \tag{20}$$

Moreover, $\gamma_k(t)$ and $E_k(r,\theta,t)$ are integrable over R, since the only discontinuity of $G(z,t)$ is a logarithmic one at $t = z$.

The function u is representable by the integral [**2**, pp. 182-3]

$$u(x,y) = \lambda \iint_R G(x,y;\xi,\eta) u(\xi,\eta)\, d\xi d\eta. \tag{21}$$

Substituting (20) into (21) proves (18) and the lemma.

LEMMA 2. *$|\nabla u(x,y)| \longrightarrow 0$ as $(x,y) \longrightarrow$ any $135°$ vertex of C.*

Proof. By (18), $|\nabla u| = O(r^{1/3})$, as $(x,y) \longrightarrow$ any $135°$ vertex of C.

LEMMA 3. *The functions u_{xx}^2, $u_x u_{xxx}$, $u u_{xxxx}$, u_{yy}^2, $u_y u_{yyy}$, and $u u_{yyyy}$ are Lebesgue-integrable in R.*

Proof. By Lemma 1 these functions are continuous in $R \cup C$, except at the $135°$ vertices P_k. At these vertices (18) implies that they are $O(r^{-4/3})$ and are hence integrable.

LEMMA 4. *The Lebesgue integrals $\int_C u_y u_{yy} dx$ and $\int_C u_x u_{xx} dy$ exist.*

Proof. Analogous to that of Lemma 3.

REMARK. Lemmas 2, 3, and 4 are false for polygonal regions R which are not convex, since in general the exponent in (18) is α_k, where π/α_k is the interior angle at the vertex P_k.

LEMMA 5. *At each node (x,y) in R of the network of section* 1, *one has*

$$\Delta_h u = \Delta u + \frac{1}{12} h^2 (u'_{xxxx} + u''_{yyyy}), \tag{22}$$

where

$$\begin{cases} u'_{xxxx} = u_{xxxx}(x + \theta' h, y), \quad -1 < \theta' < 1; \\ u''_{yyyy} = u_{yyyy}(x, y + \theta'' h), \quad -1 < \theta'' < 1. \end{cases} \tag{23}$$

Proof. By Lemma 1, u_{xxxx} is continuous in the open line segment from $(x - h, y)$ to $(x + h, y)$ (though infinite at any 135° vertex). Since u is continuous in $R \cup C$, it follows from Taylor's formula [7, p. 357] that, if we fix y and set $\phi(x) = u(x, y)$,

$$\phi(x + h) + \phi(x - h) - 2\phi(x)$$

$$= h^2 \phi''(x) + \frac{1}{24} h^4 [\phi''''(x + \theta_1 h) + \theta''''(x - \theta_2 h)],$$

where $0 < \theta_i < 1$ $(i = 1, 2)$. By the continuity of ϕ'''', the last bracket equals $2\phi''''(x + \theta' h)$, where $-1 < \theta' < 1$.

A similar formula for $\psi(y) = u(x, y)$, when added to the above and divided by h^2, yields (22) and (23).

LEMMA 6. *Define N''_h as in § 3. For each node (x, y) in N''_h, use the notation of (23). Then, as $h \longrightarrow 0$ over values such that (3) holds, one has*

$$h^2 \sum_{N''_h}\sum u(u'_{xxxx} + u''_{yyyy}) = \iint_R u(u_{xxxx} + u_{yyyy}) dxdy + o(1) \quad (h \longrightarrow 0). \tag{24}$$

Proof. For all (x, y) in the entire plane E_2 define

$$f(x, y) = \begin{cases} u(u_{xxxx} + u_{yyyy}), & \text{if } (x, y) \in R; \\ 0, & \text{elsewhere}. \end{cases}$$

By the proof of Lemma 3 one sees that $f(x, y)$ is $O(r^{-4/3})$ in the neighborhood of each 135° vertex P_k of C, and continuous elsewhere. Divide the nodes $(x, y) = (\mu h, \nu h)$ of $N''_h \subset R$ into four classes $K^{(i)}$ $(i = 1, 2, 3, 4)$ according to the parity of (μ, ν). Fix any class $K^{(i)}$. For each vertex (x, y) in $K^{(i)}$ let $S(x, y)$ be the union of the four closed network squares of E_2 which contain (x, y). The area of each $S(x, y)$ is $4h^2$; ordinarily certain of the $S(x, y)$ contain points not in R. Define

$$f_h^{(i)}(\xi, \eta) = \begin{cases} u(x, y)(u'_{xxxx} + u''_{yyyy}), & \text{for } (\xi, \eta) \in S(x, y); \\ \\ 0, & \text{for } (\xi, \eta) \notin \bigcup S(x, y). \end{cases}$$

Then $f_h^{(i)}(\xi, \eta) \longrightarrow f(\xi, \eta)$, as $h \longrightarrow 0$, for almost all (ξ, η) in the plane. Using the fact that no node of N''_h is adjacent to a $135°$ vertex of C, one can show that for all i, uniformly in h, $|f_h^{(i)}(\xi, \eta)| \leq F(\xi, \eta)$, where F is an integrable function in E_2.

Each term of the sum (24) for which $(x, y) \in K^{(i)}$ is equal to

$$\frac{1}{4} \iint_{S(x, y)} f_h^{(i)}(\xi, \eta) d\xi d\eta.$$

Hence, applying Lebesgue's convergence theorem, one sees that, as $h \longrightarrow 0$, for each i,

$$(25) \qquad \begin{aligned} \sum_{N''_h \cap K^{(i)}} \sum u(u'_{xxxx} + u''_{yyyy}) &= \frac{1}{4} \iint_{E_2} f_h^{(i)}(\xi, \eta) d\xi d\eta \\ &\longrightarrow \frac{1}{4} \iint_{E_2} f(\xi, \eta) d\xi d\eta \qquad (h \longrightarrow 0). \end{aligned}$$

Summing (25) over $i = 1, 2, 3, 4$ proves (24) and the lemma.

LEMMA 7. *One has*

$$(26) \qquad \iint_R u(u_{xxxx} + u_{yyyy}) dxdy = \iint_R (u_{xx}^2 + u_{yy}^2) dxdy.$$

Proof. The following applications of Gauss's divergence theorem in the form

$$(27) \qquad \iint_R (p_x + q_y) dxdy = \int_C (pdy - qdx)$$

can be justified by integrating over the region R^* interior to a smooth convex curve C^* inside R, and then letting $C^* \longrightarrow C$ appropriately. The continuity of the integrals in the limit follows from Lemmas 1, 3, and 4.

In the divergence theorem for $p = uu_{xxx}$, $q = uu_{yyy}$, the line integral vanishes, and one finds

$$(28) \qquad \iint_R u\,(u_{xxxx} + u_{yyyy})\,dxdy = -\iint_R (u_x u_{xxx} + u_y u_{yyy})\,dxdy.$$

A second application of the divergence theorem with $p = u_x u_{xx}$, $q = u_y u_{yy}$, combined with (28), shows that

$$(29) \qquad \iint_R u\,(u_{xxxx} + u_{yyyy})\,dxdy = \iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy + \Gamma,$$

where $\Gamma = \int_C (u_y u_{yy} dx - u_x u_{xx} dy)$.

By (1a), $u_{xx} = -u_{yy}$ on C, whence $\Gamma = \int_C u_{yy}(u_y dx + u_x dy)$. On the segments of C parallel to the axes, $u_{xx} = u_{yy} = 0$, so that there the contribution to Γ is zero.

Now the vector $\nabla u = (u_x, u_y)$ is perpendicular to C. On the segments of C making a 45° or 135° angle with the x-axis, (u_y, u_x) is parallel to (u_x, u_y), whence (u_y, u_x) is perpendicular to C. Thus $u_y dx + u_x dy \equiv 0$ when (dx, dy) is tangent to C, so that the contribution to Γ from these 45° and 135° segments of C is also zero.

Hence $\Gamma = 0$, and the lemma follows from (29).

5. Numerical example. Let R_1 be the six-sided, nonconvex, L-shaped region whose closure is the union of the three unit squares

$$\begin{cases} -1 \le x \le 0, & 0 \le y \le 1; \\ 0 \le x \le 1, & 0 \le y \le 1; \\ 0 \le x \le 1, & -1 \le y \le 0. \end{cases}$$

The fundamental frequencies $\lambda_h = \lambda_h(R_1)$ and corresponding net functions v were computed by B. F. Handy on the SWAC (National Bureau of Standards Western Automatic Computer) for $1/h = 3, 4, \cdots, 8$. The computation used a *power method*; for some initial net function v_0, $(h^2\Delta_h + 5I)^m v_0$ was determined for large positive integers m, where I is the identity operator. On the basis of Collatz's inclusion theorem [**3**, p. 289], the values in the accompanying table are believed to have errors less than 5×10^{-6}. Observe that $\lambda_h(R_1)$ is less for $h = 1/8$ than for $h = 1/7$.

TABLE

h	$\lambda_h(R_1)$	$\lambda_h(R_2)$
1/2	9.07180	12.00000
1/3	9.52514	13.73700
1/4	9.64143	14.37340
1/5	9.67860	14.67081
1/6	9.69083	14.83259
1/7	9.69384	14.93003
1/8	9.69316	14.99315

Since R_1 is not convex, the theorem of § 2 does not apply, but a heuristic argument suggests that $\lambda_h(R_1) - \lambda(R_1) = O(h^{4/3})$. A least-squares fit to the values of $\lambda_h(R_1)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_1) \doteq \alpha_1 + \beta_1 h^{4/3} + \gamma_1 h^2 = \phi_1(h)$$

yielded the values

$$(30) \qquad \alpha_1 = 9.63632, \qquad \beta_1 = 2.40286, \qquad \gamma_1 = -5.97212.$$

The maximum of $|\lambda_h(R_1) - \phi_1(h)|$ for the five values of h is .00013. Hence α_1 is a working estimate of $\lambda(R_1)$.

The fact that $\beta_1 > 0$ in (30) supports the author's conjecture that, for nonconvex polygonal domains satisfying (3), $\lambda_h > \lambda$ for all sufficiently small h.

The table also gives Handy's values for the second eigenvalues of R_1, which are the fundamental eigenvalues $\lambda_h(R_2)$ of the trapezoidal halfdomain R_2 of R_1 for which $x > y$. Since the theorem does apply to R_2, a least-squares fit to the values of $\lambda_h(R_2)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_2) \doteq \alpha_2 + \beta_2 h^2 = \phi_2(h)$$

seemed appropriate, and yielded the values

$$\alpha_2 = 15.19980, \qquad \beta_2 = -13.22219.$$

The maximum of $|\lambda_h(R_2) - \phi_2(h)|$ for the five values of h was .00010. Hence α_2 is a working estimate of $\lambda(R_2)$.

The value of β_2 is negative, in agreement with (6), but the quantity

$-12\beta_2/\alpha_2 = 10.4387$ is something like one-fifth larger than an estimate of the corresponding quantity $a(R_2)$ of the theorem. One therefore suspects that a is not the best possible constant in (6) for the region R_2.

In the table, note the relative closeness of the values of $\lambda_h(R_2)$ to the working estimate, α_2, of $\lambda(R_2)$, even for a coarse net. Thus the value 12 for $\lambda_{1/2}(R_2)$, which is obtained by pencil and paper from a simple quadratic equation, is comparable to the lower bounds 12.1 and $5\pi^2/4$ obtained respectively by comparison with λ for the circular membrane of equal area [**13**, p. 8] and with λ for the rectangular region $0 < x < 1$; $-1 < y < 1$. The value $\lambda_{1/3}(R_2) = 13.737$ requires getting the least eigenvalue of a 7th-order matrix, a relatively easy procedure with a desk machine.

The monotonicity of $\lambda_h(R_2)$ supports the author's conjecture[2] that, for the R of the theorem, $\lambda_h < \lambda$ for all h.

[2]See page 470.

REFERENCES

1. N. Aronszajn and A. Zeichner, *A new type of auxiliary problem for approximation of eigenvalues by Weinstein's method*, Technical Report 5, Oklahoma A. and M. College, Department of Mathematics, 1951.

2. Dorothy L. Bernstein, *Existence theorems in partial differential equations*, Annals of Mathematics Studies, No. 23, Princeton University Press, 1950.

3. Lothar Collatz, *Eigenwertaufgaben mit technischen Anwendungen*, Akademische Verlagsges., Leipzig, 1949.

4. Chandler Davis, *Estimating eigenvalues*, Proc. Amer. Math. Soc. 3 (1952), 942-947.

5. George E. Forsythe, *Asymptotic lower bounds for the fundamental frequency of convex membranes*, multilithed typescript, National Bureau of Standards, Los Angeles, November 1953, 21 pp. (To appear in the Pacific Journal of Mathematics.)

6. L. V. Kantorovič and V. I. Krylov, *Približennye metody vysšego analiza* (Approximate methods of higher analysis), 3rd edit., Moscow-Leningrad, 1950.

7. Wilfred Kaplan, *Advanced calculus*, Addison-Wesley, Cambridge, 1952.

8. Tosio Kato, *On the upper and lower bounds of eigenvalues*, J. Phys. Soc. Japan 4 (1949), 334-339.

9. William Edmund Milne, *Numerical solution of differential equations*, Wiley, New York and Chapman-Hall, London, 1953.

10. Zeev Nehari, *Conformal mapping*, McGraw-Hill, New York, Toronto, and London, 1952.

11. Georges Pólya, *Sur une interprétation de la méthode des différences finies qui peut fournir des bornes supérieures ou inférieures*, C. R. Acad. Sci. Paris 235 (1952), 995-997.

12. ———, *Estimates for eigenvalues*, typescript, 1952. To appear in a von Mises "Festschrift."

13. G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Annals of Mathematics Studies, No. 27, Princeton University Press, 1951.

14. Mario G. Salvadori, *Numerical computation of buckling loads by finite differences*, Trans. Amer. Soc. Civil Engrs. **116** (1951), 590-636.

15. G. Temple, *The computation of characteristic numbers and characteristic functions*, Proc. London Math. Soc. (2) 29 (1929), 257-280.

16. ———, *The accuracy of Rayleigh's method of calculating the natural frequencies of vibrating systems*, Proc. Roy. Soc. London. Ser. A. **211** (1952), 204-224.

17. D. H. Weinstein, *Modified Ritz method*, Proc. Nat. Acad. Sci. U.S.A. **20** (1934), 529-532.

18. Helmut Wielandt, *Die Einschliessung von Eigenwerten normaler Matrizen*, Math. Ann. **121** (1949), 234-241.

NATIONAL BUREAU OF STANDARDS, LOS ANGELES

(Now with the University of California, Los Angeles)

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, E.G. Straus, at the University of California Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50; back numbers (Volumes 1, 2, 3) are available at $2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

*To be succeeded in 1955, by H.L. Royden, Stanford University, Stanford, California.

**To be succeeded in 1955, by E.G. Straus, University of California, Los Angeles 24, Calif.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics
Vol. 4, No. 3 July, 1954