Pacific Journal of Mathematics

IN THIS ISSUE—

Vol. 4, No. 3 July, 1954

PACIFIC JOURNAL OF MATHEMATICS

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50; back numbers (Volumes 1, 2, 3) are available at $2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

UNIVERSITY OF CALIFORNIA PRESS · BERKELEY AND LOS ANGELES

SPECTRAL OPERATORS

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1. Introduction. The present paper and the five following it by S. Kakutani, J. Wermer, W. G. Bade, and J. Schwartz are all related; in them we discuss different aspects of the problem of the complete reduction of an operator. A spectral operator is a linear operator on a complex Banach space which has a resolution of the identity.[1] It is shown that a bounded operator T is spectral if and only if it has a canonical decomposition of the form

$$T = S + N,$$

where S is a scalar type operator and N is a generalized nilpotent commuting with S. By a scalar type operator is meant a spectral operator S with resolution of the identity E which satisfies the equation

$$S = \int_{\sigma(S)} \lambda E(d\lambda).$$

The scalar part S of T and the radical part N of T are uniquely determined by T. For analytic functions f one has an operational calculus given by the formula

$$f(T) = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{\sigma(T)} f^{(n)}(\lambda) E(d\lambda).$$

Some spectral operators are of type m; that is, the above formula reduces to

$$f(T) = \sum_{n=0}^{m} \frac{N^n}{n!} \int_{\sigma(T)} f^{(n)}(\lambda) E(d\lambda),$$

and in Hilbert space conditions on the resolvent are given which are equivalent to the statement that the spectral operator T is of type m. Spectral operators T

[1] Formal definitions will be given later.

Received March 4, 1953. The research contained in this paper was done under Contract onr 609(04) with the Office of Naval Research.

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have the property that for every x the analytic function $(\lambda I - T)^{-1}x$ has only single-valued analytic extensions and thus has a maximal extension defined on an open set $\rho(x)$. The spectrum $\sigma(x)$ is defined as the complement of $\rho(x)$. In terms of these concepts it is shown that if E is a resolution of the identity for T, then, for closed sets σ,

$$E(\sigma)\mathfrak{X} = [x \mid \sigma(x) \subset \sigma],$$

This (Theorem 4) is a basic theorem; from it one deduces that the resolution of the identity is unique, as well as the fact that every bounded operator commuting with T commutes with $E(\sigma)$, a fact proved for normal operators on Hilbert space by B. Fuglede [7].[2] Let $\mathfrak{A}(T, U, \cdots, V)$ be the full B-algebra generated by the operators $T, U, \cdots, V$; then we have the following decomposition theorems. If T is spectral and S its scalar part, then, as a vector direct sum,

$$\mathfrak{A}(T,S) = \mathfrak{A}(S) \oplus R,$$

where R is the radical in $\mathfrak{A}(T,S)$. Furthermore, $\mathfrak{A}(S)$ is equivalent to that subalgebra of $C(\sigma(T))$ consisting of uniform limits of rational functions. The algebra $\mathfrak{A}$, which is generated by a spectral operator T and the projections $E(\sigma)$ in its resolution of the identity, is equivalent to

$$C(\mathfrak{M}) \oplus R,$$

where $\mathfrak{M}$ is the compact structure space of $\mathfrak{A}$ and R is the radical in $\mathfrak{A}$. Along these lines we mention the decomposition of the full B-algebra $\mathfrak{A}(\tau)$ determined by a family τ of commuting spectral operators together with their resolutions of the identity. If there is a bounded Boolean algebra of projections in $\mathfrak{X}$ containing all of the projections found among the resolutions of the identity of operators in τ, then

$$\mathfrak{A}(\tau) = \mathfrak{A} \oplus R,$$

where $\mathfrak{A}$ is equivalent to the space $C(\mathfrak{M})$ of continuous functions on the space $\mathfrak{M}$ of maximal ideals in $\mathfrak{A}(\tau)$ (or in $\mathfrak{A}$) and R is the radical in $\mathfrak{A}(\tau)$. Furthermore, the adjoint of every operator in $\mathfrak{A}(\tau)$ is a spectral operator. If $\mathfrak{X}$ is reflexive, then every operator in $\mathfrak{A}(\tau)$ is a spectral operator. Thus in a reflexive space the sum and product of two commuting spectral operators is a spectral

[2] That this conjecture of von Neumann, which was first proved by Fuglede, is a corollary of Theorem 4 was pointed out to the author by J. Schwartz.

operator provided that there is a bounded Boolean algebra containing both resolutions of the identity. W. G. Bade [**2**] has generalized this by showing that the weakly closed algebra generated by a bounded Boolean algebra of projections in a reflexive space consists entirely of scalar type spectral operators. In this paper Bade has also given sufficient conditions for the strong limit of scalar type spectral operators to be of scalar type. If X is Hilbert space J. Werner [**16**] has shown that the sum and product of two commuting spectral operators is again a spectral operator. However, S. Kakutani [**10**] has constructed an example of two commuting operators, each of scalar type, such that their sum is not a spectral operator. W. G. Bade [**1**] has shown which portions of the theory are valid for unbounded operators and has developed the operational calculus for this case. J. Schwartz [**12**] has shown that, on a finite interval, the members of a large class of boundary-value problems determine spectral operators. These operators need not be purely differential operators but may also involve difference or integral operators.

2. Notation. By an *admissible domain* is meant an open set bounded by a finite number of rectifiable Jordan curves. By an *admissible contour* is meant the boundary of an admissible domain. The class of complex-valued functions analytic and single-valued on some admissible domain containing the spectrum $\sigma(T)$ of the linear operator T is denoted by $F(T)$ or $F(\sigma(T))$. For $f \in F(T)$ the operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi i} \int_C f(\lambda) T(\lambda) d\lambda,$$

where C is the boundary of some admissible domain containing the spectrum of T upon whose closure f is single-valued and analytic and where $T(\lambda) = (\lambda I - T)^{-1}$ is the resolvent of T. The mapping, given by the above formula, of the algebra of analytic functions into an algebra of operators is a homomorphism (See, for example, [**3**] or [**14**].) which assigns the operators I, T to the functions $1, \lambda$, respectively. It has the property that $\sigma(f(T)) = f(\sigma(T))$. If $f(\lambda) = 1$ for λ in a component of its domain, and $f(\lambda) = 0$ for λ in the remaining components, then $f(T)$ is the projection

$$E(\sigma) = \frac{1}{2\pi i} \int_G T(\lambda) d\lambda,$$

where G is the boundary of that component upon which $f(\lambda) = 1$ and where σ is that part of the spectrum $\sigma(T)$ of T bounded by G. It is clear that such a projection is associated with every subset σ of $\sigma(T)$ which is both open and closed in $\sigma(T)$. From the fact that the map $f \longrightarrow f(T)$ is a homomorphism it follows that the map $\sigma \longrightarrow E(\sigma)$ is a homomorphism of the Boolean algebra $\mathfrak{B}_0$ of open and closed sets in $\sigma(T)$ into a Boolean algebra of projection operators. It has the property (see [2])

$$\sigma(T, E(\sigma)\mathfrak{X}) \subset \sigma, \quad \sigma \in \mathfrak{B}_0,$$

where here we have used the notation $\sigma(T, E(\sigma)\mathfrak{X})$ for the spectrum of T when considered as an operator in $E(\sigma)\mathfrak{X}$. Similarly $\rho(T, E(\sigma)\mathfrak{X})$ is the resolvent set of T when considered as an operator in $E(\sigma)\mathfrak{X}$ and $\rho(T)$ is $\rho(T, \mathfrak{X})$. The symbol $B(\mathfrak{X})$ will be used for the algebra of all bounded linear transformations in the B-space $\mathfrak{X}$.

3. Spectral operators. Let $\mathfrak{B}$ be a Boolean algebra of subsets of a set p. We suppose that p and the void set $\emptyset$ are both in $\mathfrak{B}$. A homomorphic map E of $\mathfrak{B}$ into a Boolean algebra of projection operators in the complex B-space $\mathfrak{X}$ is called a *spectral measure* in $\mathfrak{X}$ provided that it is bounded and $E(p) = I$. A spectral measure has then, by definition, the properties

$$(\alpha) \begin{cases} E(\sigma)E(\delta) = E(\sigma\delta), \quad E(\sigma) \cup E(\delta) = E(\sigma \cup \delta), & \sigma, \delta \in \mathfrak{B}, \\ E(\sigma') = I - E(\sigma), \quad E(\emptyset) = 0, \quad E(p) = I, & \sigma \in \mathfrak{B}, \\ |E(\sigma)| \leq K, & \sigma \in \mathfrak{B}. \end{cases}$$

In the conditions (α) the union of two commuting projection operators is understood to be defined by the equation

$$A \cup B = A + B - AB.$$

This union is a projection whose range is the closed linear manifold determined by the ranges of A and B.

An operator $T \in B(\mathfrak{X})$ is said to be a *spectral operator of class* $(\mathfrak{B}, \Gamma)$ in case

(β) $\mathfrak{B}$ is a Boolean algebra of sets in the complex plane p;

(γ) Γ is a linear manifold in $\mathfrak{X}^*$ which is total; that is, $\Gamma x = 0$ only when $x = 0$;

(δ) there is a spectral measure E in $\mathfrak{X}$ with domain $\mathfrak{B}$ such that

$$TE(\sigma) = E(\sigma)T, \quad \sigma(T, E_\sigma \mathfrak{X}) \subset \bar{\sigma}, \quad \sigma \in \mathfrak{B};$$

and

(ϵ) for every $x \in \mathfrak{X}$, $x^* \in \Gamma$, the function $x^*E(\sigma)x$ is countably additive on $\mathfrak{B}$.

The condition (ϵ) means that if $\{\sigma_n\}$ is a sequence of disjoint sets in $\mathfrak{B}$ whose union σ is also in $\mathfrak{B}$ then

$$\sum_n x^*E(\sigma_n)x = x^*E(\sigma)x, \quad x \in \mathfrak{X}, \ x^* \in \Gamma.$$

In case $\mathfrak{B}$ is a σ-field and $\Gamma = \mathfrak{X}^*$, the Orlicz-Banach-Pettis theorem (see [**11**, Theorem 2.32] or [**5**, p. 322]) shows that the operator-valued set function $E(\sigma)$, $\sigma \in \mathfrak{B}$, is countably additive on $\mathfrak{B}$ in the strong operator topology.

An operator $T \in B(\mathfrak{X})$ is said to be a *spectral operator of class* (Γ), or simply an operator of class (Γ), in case it is a spectral operator of class ($\mathfrak{B}, \Gamma$), where $\mathfrak{B}$ is the set of all Borel sets in the plane. An operator is said to be a *spectral operator* in case it is a spectral operator of class (Γ) for some Γ satisfying (γ). If T is a spectral operator of type ($\mathfrak{B}, \Gamma$), then any spectral measure in $\mathfrak{X}$ with domain $\mathfrak{B}$ which satisfies (δ) and (ϵ) is called a *resolution of the identity* for T.

THEOREM 1. *Let E be a resolution of the identity for the spectral operator T. Then*

$$E(\sigma(T)) = I.$$

Proof. Let σ be a closed subset of the resolvent set $\rho = \rho(T)$. Then, in view of (δ), we see that the spectrum of T as an operator in $E_\sigma \mathfrak{X}$ is void and hence (see [**15**]) $E_\sigma = 0$. Since ρ is a denumerable union of closed sets we have from (ϵ) that

$$x^*E_\rho x = 0, \quad x \in \mathfrak{X}, \quad x^* \in \Gamma,$$

and from (γ) that $E_\rho = 0$, and hence $E(\sigma(T)) = I$.

For $\lambda \in \rho(T)$ we write, as usual, $T(\lambda)$ for $(\lambda I - T)^{-1}$. In the next theorem we shall show that, for spectral operators, every analytic extension of $T(\lambda)x$ is necessarily single-valued. That this is not the case for an arbitrary operator T is elegantly shown by the following example due to S. Kakutani.

Consider the space $\mathfrak{X}$ of functions f analytic in the unit circle $|z| \leq 1$ and for which

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \sum_{n=0}^{\infty} |c_n|^2 = |f|^2.$$

In this space define T by

$$T(f, z) = \frac{f(z) - f(0)}{z}.$$

The spectrum of T is the set of z with $|z| \leq 1$, and for $\lambda \in \rho(T)$ the function $T(\lambda)(g, z)$ may be calculated by solving the equation

$$(\lambda I - T)f = g$$

for $f(z)$. An elementary calculation gives

$$f(z) = \frac{zg(z) - f(0)}{\lambda z - 1}.$$

Since $f(z)$ is analytic when $z = \lambda^{-1}$ we must have

$$f(0) = \lambda^{-1} g(\lambda^{-1}),$$

so that

$$T(\lambda)(g, z) = \frac{zg(z) - \lambda^{-1} g(\lambda^{-1})}{\lambda(z - \lambda^{-1})}.$$

Thus the vector-valued analytic function $T(\lambda)g$, $\lambda \in \rho(T)$, will have multiple-valued extensions if the function g has a multiple-valued analytic continuation outside the unit circle.

In order to describe the situation discussed in the next theorem certain concepts are introduced. By an *analytic extension* of $T(\xi)x$ will be meant a function f defined and analytic on an open set $D(f) \supset \rho(T)$ and such that

$$(\xi I - T)f(\xi) = x$$

for every ξ in $D(f)$. It is clear that, for such an extension,

$$f(\xi) = T(\xi)x$$

for ξ in $\rho(T)$. The function $T(\xi)x$ is said to have the *single-valued extension property* provided that for every pair f, g of analytic extensions of $T(\xi)x$ we have $f(\xi) = g(\xi)$ for every ξ in $D(f)D(g)$. The union of the sets $D(f)$ as f varies over all analytic extensions of $T(\xi)x$ is called the *resolvent set* of x and is denoted by $\rho(x)$. The *spectrum* $\sigma(x)$ *of* x is defined to be the complement of $\rho(x)$. It is clear that if $T(\xi)x$ has the single-valued extension property then there is a maximal extension $x(\cdot)$ whose domain is $\rho(x)$. In this case $x(\xi)$ is a single-valued analytic function with domain $\rho(x)$ and with $x(\xi) = T(\xi)x$, $\xi \in \rho(T)$.

THEOREM 2. *If T is a spectral operator in $\mathfrak{X}$, then for every $x \in \mathfrak{X}$ the function $T(\xi)x$ has the single-valued extension property.*

Proof. Let f, g be two extensions of $T(\xi)x$ and define

$$h(\xi) = f(\xi) - g(\xi), \qquad \xi \in D(f)D(g).$$

We suppose, in order to make an indirect proof, that for some $\xi_0 \in D(f)D(g)$ we have $h(\xi_0) \neq 0$. Thus there is a neighborhood $N(\xi_0)$ of ξ_0 with $N(\xi_0) \subset D(f)D(g)$ and

$$\text{(i)} \qquad h(\xi) \neq 0, \quad (\xi I - T)h(\xi) = 0, \quad \xi \in N(\xi_0).$$

The desired contradiction may be obtained from these equations and the following lemma.

LEMMA 1. *Let E be a resolution of the identity for the spectral operator T. Let σ be a closed set of complex numbers with $\xi_0 \notin \sigma$. If $(\xi_0 I - T)x_0 = 0$ then*

$$E(\sigma)x_0 = 0, \quad E(\{\xi_0\})x_0 = x_0,$$

where $\{\xi_0\}$ is the set consisting of the single point ξ_0.

Proof. Let $T_\sigma(\xi)$ be the resolvent of T as an operator in $E(\sigma)\mathfrak{X}$, so that

$$T_\sigma(\xi_0)(\xi_0 I - T)E(\sigma) = E(\sigma).$$

But since

$$(\xi_0 I - T)E(\sigma)x_0 = E(\sigma)(\xi_0 I - T)x_0 = 0,$$

we have $E(\sigma)x_0 = 0$. Now let

$$\sigma_n = [\xi \mid |\xi - \xi_0| \geq 1/n],$$

so that $E(\sigma_n)x_0 = 0$; by (δ), (ϵ), therefore,

$$x^*(I - E(\{\xi_0\}))x_0 = \lim_n x^*E(\sigma_n)x_0 = 0, \qquad (x^* \in \Gamma).$$

Condition (γ) thus shows that $E(\{\xi_0\})x_0 = x_0$, and the lemma is proved.

Returning now to the proof of Theorem 2, let

$$\xi_n \neq \xi_0, \quad \xi_n \in N(\xi_0), \quad \xi_n \longrightarrow \xi_0.$$

Then $h(\xi_n) \longrightarrow h(\xi_0)$, and the lemma together with (i) gives

$$0 = E(\{\xi_0\})h(\xi_n) \longrightarrow E(\{\xi_0\})h(\xi_0) = h(\xi_0),$$

which is a contradiction to (i) and proves the theorem.

THEOREM 3. *If T is a spectral operator, the spectrum $\sigma(x)$ is void if and only if $x = 0$.*

Proof. Using Theorem 2, we see that if $\sigma(x)$ is void then $x(\xi)$ is everywhere defined, single-valued, and hence entire. Since, as $\xi \longrightarrow \infty$, we have

$$x^*x(\xi) = x^*T(\xi)x \longrightarrow 0,$$

we see that $x^*x(\xi) = 0$ for all ξ. Hence

$$x^*x = x^*(\xi I - T)x(\xi) = 0,$$

and $x = 0$.

THEOREM 4. *Let T be a spectral operator with resolution of the identity E, and let σ be a closed set of complex numbers. Then*

$$E(\sigma)\mathfrak{X} = [x \mid \sigma(x) \subset \sigma].$$

Proof. Let $E(\sigma)x = x$, and let $T_\sigma(\xi)$ be the resolvent of T as an operator in $E(\sigma)\mathfrak{X}$. Then (δ) shows that

$$T_\sigma(\xi)E_\sigma x = T_\sigma(\xi)x$$

is an analytic extension of $T(\xi)x$ to σ', the complement of σ. Thus $\rho(x) \supset \sigma'$, $\sigma(x) \subset \sigma$. Conversely, assume that $\sigma(x) \subset \sigma$ and let σ_1 be a closed subset of the complement σ' of σ. Then $T_{\sigma_1}(\xi)E(\sigma_1)x$ is an extension of $T(\xi)E(\sigma_1)x$ to σ_1'. Also $E(\sigma_1)x(\xi)$ is an extension of $T(\xi)E(\sigma_1)x$ to $\rho(x)$. Thus, from Theorem 2, it is seen that

$$E(\sigma_1)x(\xi) = T_{\sigma_1}(\xi)E(\sigma_1)x, \quad \xi \in \rho(x)\sigma_1'.$$

Since σ, σ_1 are disjoint compact sets, there is an admissible contour C_1 with σ_1 inside C_1 and σ outside. Now let C be a large circle surrounding $\sigma(T)$ so that, since $x(\xi)$ is analytic and single-valued on and within C_1, we have

$$E(\sigma_1)x = \frac{1}{2\pi i}\int_C T(\xi)E(\sigma_1)x\,d\xi = \frac{1}{2\pi i}\int_C T_{\sigma_1}(\xi)E(\sigma_1)x\,d\xi$$

$$= \frac{1}{2\pi i}\int_{C_1} T_{\sigma_1}(\xi)E(\sigma_1)x\,d\xi = \frac{1}{2\pi i}\int_{C_1} E(\sigma_1)x(\xi)\,d\xi = 0.$$

Let σ_n be an increasing sequence of closed sets whose union is σ'. Then

$$x^*E(\sigma')x = \lim_n x^*E(\sigma_n)x = 0, \qquad (x^* \in \Gamma),$$

and so (α), (γ) show that $E(\sigma')x = 0$, $E(\sigma)x = x$.

THEOREM 5. *Let T be a spectral operator and A a bounded linear transformation which commutes with T. Then A commutes with every resolution of the identity for T.*

Proof. Let σ, σ_1 be disjoint closed sets of complex numbers and let E be a resolution of the identity for T. Since

$$AT(\xi)x = T(\xi)Ax,$$

we see that

$$\rho(Ax) \supset \rho(x), \quad \sigma(Ax) \subset \sigma(x).$$

Thus Theorem 4 shows that

$$E(\sigma)AE(\sigma) = AE(\sigma), \quad E(\sigma)AE(\sigma_1) = E(\sigma)E(\sigma_1)AE(\sigma_1) = 0.$$

Statements (γ), (ϵ) show then that $E(\sigma)AE(\sigma') = 0$, and hence

$$E(\sigma)A = E(\sigma)A[E(\sigma) + E(\sigma')] = E(\sigma)AE(\sigma) + E(\sigma)AE(\sigma') = AE(\sigma).$$

THEOREM 6. *If T is a spectral operator, its resolution of the identity is unique.*

Proof. If E, A are both resolutions of the identity for T, and σ is a closed set of complex numbers, then Theorem 4 gives

$$A(\sigma)E(\sigma) = E(\sigma), \quad E(\sigma)A(\sigma) = A(\sigma),$$

and (δ) together with Theorem 5 gives

$$A(\sigma)E(\sigma) = E(\sigma)A(\sigma).$$

Thus for closed sets σ, $A(\sigma) = E(\sigma)$, and (γ), (ϵ) show that this same equality holds for every Borel set σ.

THEOREM 7. *Let E be a spectral measure whose domain consists of the Borel sets in the plane and which vanishes on the complement of the compact set σ. Then, for every scalar function f continuous on σ, the Riemann integral $\int_\sigma f(\lambda)E(d\lambda)$ exists in the uniform operator topology, and*

$$\left|\int_\sigma f(\lambda)E(d\lambda)\right| \leq \sup_\lambda |f(\lambda)| v(E),$$

where $v(E)$ is a constant depending only upon E. Furthermore, for any two continuous functions f and g we have

$$\left[\int f(\lambda)E(d\lambda)\right]\left[\int g(\lambda)E(d\lambda)\right] = \int f(\lambda)g(\lambda)E(d\lambda).$$

Proof. Let $\delta > 0$ be such that $|f(\lambda) - f(\lambda')| < \epsilon$ if $|\lambda - \lambda'| < 2\delta$, and and let $\pi = (\sigma_i, \lambda_i)$, $\pi' = (\sigma_j', \lambda_j')$ be two partitionings of σ with norms at most δ. Then for $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$, and the operator

$$U(\pi) \equiv \Sigma f(\lambda_i)E(\sigma_i),$$

we have the inequality

$$|x^*(U(\pi) - U(\pi'))x| \leq \sum_i \sum_j |f(\lambda_i) - f(\lambda_j')|\,|x^*E(\sigma_i\sigma_j')x| \leq \epsilon \operatorname*{var}_\sigma x^*E(\sigma)x.$$

But

$$\operatorname*{var}_{\sigma} x^*E(\sigma)x \leq 4 \operatorname*{l.u.b.}_{\sigma} |x^*E(\sigma)x| \leq 4K|x| \, |x^*|,$$

where K is an upper bound for $|E(\sigma)|$. Thus

$$|U(\pi) - U(\pi')| = \operatorname*{l.u.b.}_{|x|=|x^*|=1} |x^*(U(\pi) - U(\pi'))x| \leq 4\epsilon K.$$

The final assertion is seen by using (α) to obtain the equation

$$\left[\sum f(\lambda_i)E(\sigma_i)\right]\left[\sum_j f(\lambda_j)E(\sigma_j)\right] = \sum f(\lambda_i)g(\lambda_i)E(\sigma_i).$$

LEMMA 2. *Let $\mathfrak{A}$ be a commutative subalgebra of $B(\mathfrak{X})$ which contains I and the inverse of any of its elements provided that the inverse exists as an element of $B(\mathfrak{X})$. Let $T, E \in \mathfrak{A}$, $E^2 = E$, and let $\mathfrak{M} = (m)$ be the set of maximal ideals in $\mathfrak{A}$. Then*[3]

$$\sigma(T, E\mathfrak{X}) = [\lambda \mid \lambda = T(m),\ m \in \mathfrak{M},\ E(m) = 1].$$

Proof. The symbol $B(\mathfrak{X})$, as always, is used for the algebra of all bounded linear operators in the space $\mathfrak{X}$. It is normed by the bound of the operator. For an element T_0 of an algebra $\mathfrak{A}_0$ with unit E_0, we write $\sigma(T_0, \mathfrak{A}_0)$ for the spectrum of T_0 as an element of $\mathfrak{A}_0$. This is the complement of the set of those λ for which $\lambda E_0 - T_0$ has an inverse in $\mathfrak{A}_0$. According to our hypothesis, then, we have

(i) $$\sigma(T, \mathfrak{A}) = \sigma(T, \mathfrak{X}) = \sigma(T).$$

Let $\mathfrak{A}_E = \mathfrak{A}E$, and note that this is a subalgebra of $\mathfrak{A}$ with unit E. Each $V \in \mathfrak{A}_E$ maps $E\mathfrak{X}$ into itself and as an operator in $E\mathfrak{X}$ has the spectrum $\sigma(V, E\mathfrak{X})$. Just as in (i) above we have

(ii) $$\sigma(V, \mathfrak{A}_E) = \sigma(V, E\mathfrak{X}).$$

To see this, let

$$V_0 = (\lambda I - V)E \in \mathfrak{A}_E,$$

[3] The difference algebra $\mathfrak{A} - m$ is the complex number system [8]. We write, using Gelfand's notation, $U(m)$ for the complex number corresponding to an element $U \in \mathfrak{A}$ under the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A} - m$.

and suppose that V_0 has an inverse as an operator in $E\mathfrak{X}$. Define

$$W = V_0E + E',$$

so that $W^{-1} \in B(\mathfrak{X})$. Thus $W^{-1} \in \mathfrak{A}$, $W^{-1}E \in \mathfrak{A}_E$; and since $V_0W^{-1}E = E$ it is seen that V_0 has an inverse as an element of $\mathfrak{A}_E$. This proves that $\rho(V, E\mathfrak{X}) \subset \rho(V, \mathfrak{A}_E)$. The converse inequality being obvious, we have proved (ii). Now let $\mathfrak{M} = (m)$ be the set of maximal ideals in $\mathfrak{A}$, and $\mathfrak{M}_E = (m_E)$ the set of maximal ideals in $\mathfrak{A}_E$. We shall next show that

$$\text{(iii)} \qquad \mathfrak{M}_E = [mE \mid m \in \mathfrak{M},\ E(m) = 1];$$

that is, the maximal ideals in $\mathfrak{A}_E$ are precisely those of the form $m_E = mE$, where m is a maximal ideal in $\mathfrak{M}$ for which $E(m) = 1$. Since $E^2 = E$, we have $E(m)$ always 0 or 1, and so the statement $E(m) = 1$ is equivalent to the statement $E \notin m$. To prove (iii), let m be a maximal ideal in $\mathfrak{A}$ with $E \notin m$. The set $m_E = mE$ is clearly a proper ideal in $\mathfrak{A}_E$. To see that m_E is maximal, let n_E be a proper ideal in $\mathfrak{A}_E$ which contains m_E, and let n be the set of all $V \in \mathfrak{A}$ for which $VE \in n_E$. Then n is a proper ideal in $\mathfrak{A}$ which contains m. Since m is maximal, we have $m = n$ and hence $m_E = n_E$. Conversely, let m_E be a maximal ideal in $\mathfrak{A}_E$; then $m = m_E + \mathfrak{A}E'$ is a proper ideal in $\mathfrak{A}$ with $mE = m_E$. To see that m is maximal, suppose that n is a proper ideal in $\mathfrak{A}$ containing m properly. Then we shall show that $n_E = nE$ is a proper ideal in $\mathfrak{A}_E$ which contains m_E properly. Let $U \in n$, $U \notin m$. Then $UE \in n_E$. Since $E' \in m$, we have $UE' \in m$ and hence $UE \notin m$. Therefore, since $m_E \subset m$, we have $UE \notin m_E$, and this proves (iii). Thus we may say that for any $m \in \mathfrak{M}$ for which $E(m) = 1$ the difference algebras $\mathfrak{A} - m$, $\mathfrak{A}E - mE$ are both isometrically isomorphic to the complex number system. There are, therefore, uniquely determined complex numbers $T(m)$, $TE(mE)$ for which

$$T - T(m)I \in m, \quad TE - (TE)(mE)E \in mE.$$

From the first of these relations it follows that $TE - T(m)E \in mE$, and from the second, therefore, that $T(m) = (TE)(mE)$. But as m varies over all points in $\mathfrak{M}$ for which $E(m) = 1$, we see from (iii) that mE varies over all maximal ideals in AE and hence $(TE)(mE) = T(m)$ varies over the spectrum of TE as an element of $\mathfrak{A}E$. Hence the desired conclusion follows from (ii).

DEFINITION 1. An operator S is said to be of *scalar type* in case it is a spectral operator and satisfies the equation

$$S = \int \lambda E(d\lambda),$$

where E is the resolution of the identity for S. According to Theorem 1, $S(e) = 0$ if $e \subset \rho(S)$ so that the integral over the compact set $\sigma(S)$ exists in the uniform topology of operators.

THEOREM 8. *An operator T is a spectral operator of class (Γ) if and only if it is the sum $T = S + N$ of a scalar type operator S of class (Γ) and a generalized nilpotent operator N commuting with S. Furthermore, this decomposition is unique and T and S have the same spectrum and the same resolution of the identity.*

Proof. We shall first show that the sum $T = S + N$ of an arbitrary spectral operator S of class (Γ) and a generalized nilpotent N commuting with S is itself a spectral operator of class (Γ). Let E be the resolution of the identity for S, and let σ be a Borel set of complex numbers. Then, by Theorem 5, $NE(\sigma) = E(\sigma)N$. Let $\mathfrak{A}$ be the smallest commutative subalgebra of $B(\mathfrak{X})$ containing N, S, $E(\sigma)$, I, and also containing the inverse of any of its elements provided that the inverse exists as an element in $B(\mathfrak{X})$. Then, as established in equation (ii) during the proof of Lemma 2, we have

$$\sigma(S + N, E(\sigma)\mathfrak{X}) = \sigma(S + N, \mathfrak{A}E(\sigma)).$$

Thus if $\mathfrak{M}(\sigma)$ is the set of maximal ideals in $AE(\sigma)$, we have

$$(*) \qquad \sigma(T, E(\sigma)\mathfrak{X}) = [\lambda \mid \lambda = S(m) + N(m),\ m \in \mathfrak{M}(\sigma)]$$
$$= [\lambda \mid \lambda = S(m),\ m \in \mathfrak{M}_\sigma]$$
$$= \sigma(S, \mathfrak{A}E(\sigma)) = \sigma(S, E(\sigma)\mathfrak{X}) \subset \bar{\sigma}.$$

Thus T is a spectral operator of class (Γ), and its resolution of the identity is also E. Conversely, let T be a spectral operator of class (Γ) with resolution of the identity E. Using Theorem 7, define

$$S = \int \lambda E(d\lambda), \quad N = T - S.$$

Clearly S and N commute. It will first be shown that N is a generalized nilpotent. Let $\mathfrak{A}$ be the algebra generated by T, $E(\sigma)$ (σ a Borel set), N, I, and with the property that $U^{-1} \in \mathfrak{A}$ if $U \in \mathfrak{A}$ and $U^{-1} \in B(\mathfrak{X})$. Let $\mathfrak{M} = (m)$ be the set of maximal ideals in $\mathfrak{A}$. Then $E(\delta)(m)$ is a zero-one valued additive set function, and hence determines uniquely a complex number $\lambda(m)$ with the property that $E(\delta_m)(m) = 1$ provided that δ_m is a neighborhood of $\lambda(m)$. Thus for every neighborhood δ_m of $\lambda(m)$ we have

$$S(m) = \int \lambda E(d\lambda)(m) = \int_{\delta_m} \lambda E(d\lambda)(m) = \lambda(m).$$

Since $E(\delta_m)(m) = 1$ if δ_m is a neighborhood of $\lambda(m)$, it follows from Lemma 2 that

$$T(m) \in \sigma(T, E(\delta_m)\mathfrak{X}) \subset \delta_m,$$

and hence

$$T(m) = \lambda(m) = S(m), \quad N(m) = 0.$$

Thus by a theorem of Gelfand[4], N is a generalized nilpotent. It will next be shown that S is a scalar type operator. For this it is sufficient to show that E is the resolution of the identity for S. According to Lemma 2,

$$\begin{aligned}\sigma(S, E(\delta)\mathfrak{X}) &= [\lambda \mid \lambda = S(m), m \in \mathfrak{M}, \ E(\delta)(m) = 1] \\ &= [\lambda \mid \lambda = T(m), m \in \mathfrak{M}, E(\delta)(m) = 1] \\ &= \sigma(T, E(\delta)\mathfrak{X}) \subset \bar{\delta},\end{aligned}$$

and this shows that E is the resolution of the identity for S. Finally it remains to be shown that S and N are uniquely determined by T. Let $T = S_1 + N_1$, where S_1 is of scalar type and N_1 is a generalized nilpotent commuting with S_1. Let E_1 be the resolution of the identity for S_1. Then, by Theorem 5,

$$N_1 E_1(\sigma) = E_1(\sigma) N_1,$$

so that $E_1(\sigma)$ commutes with T. It was established in (*) above that $\sigma(T, E_1(\sigma)\mathfrak{X}) \subset \bar{\sigma}$, and hence E_1 is a resolution of the identity. By Theorem 6, we have $E(\sigma) = E_1(\sigma)$, and hence $S = S_1$, $N = N_1$.

DEFINITION 2. The decomposition, given in Theorem 8, of a spectral operator $T = S + N$ into a sum of a scalar type operator S and a generalized nilpotent N commuting with S is called the *canonical decomposition* of T. The

[4] I. Gelfand [8] has shown that N is a generalized nilpotent if and only if N belongs to every maximal ideal.

operator S is called the *scalar part* of T, and N is called the *generalized nilpotent part*, or the *radical part*, of T.

LEMMA 3. *Let E be the resolution of the identity for the spectral operator T, and let N be its radical part. Then in the uniform topology of operators, and uniformly with respect to ξ in any closed set $\rho \subset \rho(T)$, we have*

$$T(\xi) = \sum_{n=0}^{\infty} N^n \int \frac{E(d\lambda)}{(\xi - \lambda)^{n+1}}.$$

Proof. By Theorem 7 the integral exists in the uniform operator topology, and

$$\left| \int \frac{E(d\lambda)}{(\xi - \lambda)^{n+1}} \right| \leq r^{n+1} v(E),$$

where $r = \max |\xi - \lambda|^{-1}$, the maximum being taken over $\lambda \in \sigma(T)$, $\xi \in \rho$. Since N is a generalized nilpotent,

$$\sqrt[n]{|N^n|} \longrightarrow 0,$$

and hence the series

$$\sum |N^n| r^{n+1}$$

converges. Thus the series

$$U = \sum_{n=0}^{\infty} N^n \int \frac{E(d\lambda)}{(\xi - \lambda)^{n+1}}$$

converges in the uniform operator topology, and uniformly with respect to $\xi \in \rho$. From Theorem 7 we have

$$(\xi I - S) \int \frac{E(d\lambda)}{(\xi - \lambda)^{n+1}} = \left[\int (\xi - \lambda) E(d\lambda) \right] \left[\int \frac{E(d\lambda)}{(\xi - \lambda)^{n+1}} \right] = \int \frac{E(d\lambda)}{(\xi - \lambda)^{n}},$$

and so, if S is the scalar part of T,

$$(\xi I - T) U = (\xi I - S - N) \sum_{0}^{\infty} N^n \int \frac{E(d\lambda)}{(\xi - \lambda)^{n+1}}$$

$$= \sum_{0}^{\infty} \left\{ N^n \int \frac{E(d\lambda)}{(\xi-\lambda)^n} - N^{n+1} \int \frac{E(d\lambda)}{(\xi-\lambda)^{n+1}} \right\} = I.$$

This proves the lemma.

THEOREM 9. *Let T be a spectral operator and N its radical part. Then for every scalar function f analytic and single-valued on the spectrum $\sigma(T)$ we have, in the uniform topology of operators,*

$$f(T) = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int f^{(n)}(\lambda) E(d\lambda).$$

Proof. Let C be an admissible rectifiable Jordan curve in $\rho(T)$ containing $\sigma(T)$ in its interior and such that f is analytic on and within C. Then, using Lemma 3, we have

$$f(T) = \frac{1}{2\pi i} \int_C f(\xi) T(\xi) d\xi = \sum_{n=0}^{\infty} N^n \int_C f(\xi) \int_{\sigma(T)} \frac{E(d\lambda)}{(\xi-\lambda)^{n+1}}$$

$$= \sum_{n=0}^{\infty} N^n \int_{\sigma(T)} \left[\int_C \frac{f(\xi)d\xi}{(\xi-\lambda)^{n+1}} \right] E(d\lambda)$$

$$= \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{\sigma(T)} f^{(n)}(\lambda) E(d\lambda).$$

DEFINITION 3. An operator T is said to be of *type m* in case it is a spectral operator with resolution of the identity E and

$$f(T) = \sum_{n=0}^{m} \frac{N^n}{n!} \int f^{(n)}(\lambda) E(d\lambda), \quad f \in F(T).$$

THEOREM 10. *Let N be the radical part of the spectral operator T; then T is of type m if and only if $N^{m+1} = 0$.*

Proof. If $N^{m+1} = 0$ then clearly the formula of Theorem 9 reduces to that of Definition 3. Conversely, if T is of type m we see, by placing

$$f(\lambda) = \lambda^{m+1}/(m+1)!$$

in these two formulas, that

$$0 = N^{m+1} \int E(d\lambda) = N^{m+1}.$$

COROLLARY. *A spectral operator is of scalar type if and only if it is of type* 0.

We shall next endeavor to characterize operators of finite type in terms of the rate of growth of the resolvent. To this end we introduce the following definition.

DEFINITION 4. Let E be the resolution of the identity for the spectral operator T. If $\xi \notin \sigma(T, E(\sigma)\mathfrak{X})$, and in particular if $\xi \notin \bar{\sigma}$, *the operator* $T_\sigma(\xi)$ is defined on $\mathfrak{X}$ as follows. For each x in $\mathfrak{X}$, $T_\sigma(\xi)x$ is that uniquely determined point $y \in E(\sigma)\mathfrak{X}$ for which $(\xi I - T)y = E(\sigma)x$. Thus $T_\sigma(\xi)$ is a bounded linear operator in $\mathfrak{X}$ formed by first projecting with $E(\sigma)$ and then operating with the inverse of $(\xi I - T)$ in $E(\sigma)\mathfrak{X}$.

THEOREM 11. *In Hilbert space a spectral operator T is of type $m-1$ if and only if there is a constant K such that, for every Borel set σ,*

$$(*) \qquad |\operatorname{dis}(\xi, \sigma)^m T_\sigma(\xi)| \le K, \quad \xi \notin \sigma, \quad |\xi| \le |T| + 1.$$

Proof. In view of Theorem 10 it is sufficient to prove that the condition $(*)$ is equivalent to the condition $N^m = 0$. If $N^m = 0$, and $\xi \notin \bar{\sigma}$, then

$$T_\sigma(\xi) = \sum_{n=0}^{m-1} N^n \int_\sigma \frac{E(d\lambda)}{(\lambda - \xi)^{n+1}},$$

from which the condition $(*)$ follows.

The converse will require the following lemma.

LEMMA 4. *Let T be a spectral operator in Hilbert space $\mathfrak{X}$ and let E be its resolution of the identity. Then there is a constant M such that for any finite collection A_j $(j = 1, 2, \cdots, n)$ of bounded operators in $\mathfrak{X}$ which commute with T, and any collection σ_j $(j = 1, 2, \cdots, n)$ of disjoint Borel sets, we have*

$$\left| \sum_{j=1}^{n} A_j E(\sigma_j) \right| \le M \sup_{1 \le j \le n} |A_j|.$$

Proof. It is known (see [**16**]) that there is a linear one-to-one map B with $B\mathfrak{X} = \mathfrak{X}$, with B and B^{-1} both continuous and such that for each Borel set σ the projection

$$P(\sigma) = BE(\sigma)B^{-1}$$

is self-adjoint. If $B_j = BA_jB^{-1}$ then

$$B\left\{\sum_{j=1}^{n} A_jE(\sigma_j)\right\}B^{-1} = \sum_{j=1}^{n} B_jP(\sigma_j).$$

By Theorem 5, A_j commutes with $E(\sigma)$ and hence B_j commutes with $P(\sigma)$. Thus

$$\left|\sum_{j=1}^{n} B_jP(\sigma_j)x\right|^2 = \left|\sum_{j=1}^{n} P(\sigma_j)B_jx\right|^2 = \sum_{j=1}^{n} |P(\sigma_j)B_jx|^2$$

$$\leq \sup_j |B_j|^2 \sum_{j=1}^{n} |P(\sigma_j)x|^2 \leq \sup_j |B_j|^2 |x|^2,$$

which proves the lemma.

Now let $T = S + N$ be the canonical form of the spectral operator T which we assume enjoys the property $(*)$ of the theorem. Since

$$(T - \xi I)^m = (S + N - \xi I)^m = \sum_{r=0}^{m} \binom{m}{r} (S - \xi I)^{m-r} N^r,$$

and

$$\int_{\sigma(T)} (S - \xi I)^p E(d\xi) = 0 \qquad (p \geq 1),$$

we have

$$N^m = \int_{\sigma(T)} (T - \xi I)^m E(d\xi).$$

Now let $\sigma(T)$ be partitioned into the Borel sets σ_j $(j = 1, 2, \cdots, n(\delta))$, each of diameter at most $\delta > 0$, and let $\xi_j \in \sigma_j$ $(j = 1, 2, \cdots, n(\delta))$. Let C_j be the circle with center ξ_j and radius 2δ. Then since the distance from a point λ on C_j to

σ_j is at least δ we have

$$|(\lambda - \xi_j)^m T_{\sigma_j}(\lambda)| \leq 2^m K, \quad \lambda \in C_j.$$

Let

$$\lambda_k^{(j)} = \xi_j + 2\delta \exp(2k\pi i/p),$$

so that

$$\sum_{j=1}^{n(\delta)} (T - \xi_j I)^m E(\sigma_j) = \sum_{j=1}^{n(\delta)} \frac{1}{2\pi i} \int_{C_j} (\lambda - \xi_j)^m T_{\sigma_j}(\lambda) d\lambda$$

$$= \lim_p \sum_{j=1}^{n(\delta)} \frac{1}{2\pi i} \left\{ \sum_{k=1}^{p} (\lambda_k^{(j)} - \xi_j)^m T_{\sigma_j}(\lambda_k^{(j)})(\lambda_k^{(j)} - \lambda_{k-1}^{(j)}) \right\} E(\sigma_j).$$

But

$$\sup_j \left| \frac{1}{2\pi i} \sum_{k=1}^{p} (\lambda_k^j - \xi_j)^m T_{\sigma_j}(\lambda_k^j)(\lambda_k^{(j)} - \lambda_{k-1}^{(j)}) \right|$$

$$\leq 2^{m-1} K \pi^{-1} \sup_j \sum_{k=1}^{p} |\lambda_k^{(j)} - \lambda_{k-1}^{(j)}| < 2^{m+1} K\delta,$$

and by Lemma 4 therefore

$$\left| \sum_{j=1}^{n(\delta)} (T - \xi_j I)^m E(\sigma_j) \right| \leq 2^{m+1} MK\delta,$$

which shows that

$$N^m = \int (T - \xi I)^m E(d\xi) = 0.$$

THEOREM 12. *In Hilbert space a spectral operator T whose spectrum is nowhere dense is of type $m - 1$ if and only if its resolvent has at most* mth *order rate of growth for ξ near the spectrum.*

Proof. This theorem is an immediate corollary of Theorem 11.

4. Algebras of spectral operators. In this section we shall characterize commutative algebras of spectral operators. To this end we shall need the following preliminary lemmas.

LEMMA 5. *If T is of class (Γ) with resolution of the identity $E(T)$, and $f \in F(T)$, then $f(T)$ is of class (Γ) and its resolution of the identity is given by the formula*

$$E(f(T), \sigma) = E(T, f^{-1}(\sigma)).$$

Proof. The foregoing formula clearly yields a spectral measure commuting with $f(T)$. Also $x^*E(f(T), \sigma)x$ is countably additive if $x^* \in \Gamma$. Now if $\lambda_0 \notin \bar{\sigma}$ then the function

$$h(\lambda) \equiv \frac{1}{\lambda_0 - f(\lambda)}$$

is analytic on the closure of $f^{-1}(\sigma)$ and hence if C is an admissible contour surrounding the closure of $f^{-1}(\sigma)$ we have

$$\left(\frac{1}{2\pi i}\int_C h(\lambda)T_{f^{-1}(\sigma)}(\lambda)d\lambda\right)(\lambda_0 I - f(T))E(T, f^{-1}(\sigma)) = E(T, f^{-1}(\sigma)),$$

which shows that

$$\sigma(f(T), E(f(T), \sigma)\mathfrak{X}) \subset \bar{\sigma},$$

and this completes the proof of the lemma.

At this point we introduce the notion of an integral which will be needed later. For the purposes of the following theorem the Riemann integral will suffice, but for subsequent work the next lemma will be needed for a more general integral. Accordingly let $\mathfrak{M}$ be a set, $\mathfrak{B}$ a field of its subsets with $\mathfrak{M} \in \mathfrak{B}$, and let $\mathfrak{B}(\mathfrak{M})$ be the normed linear space of all complex bounded functions on $\mathfrak{M}$ which are measurable $\mathfrak{B}$. The norm in $\mathfrak{B}(\mathfrak{M})$ is given by $|f| = \sup_m |f(m)|$. Let E be an additive operator-valued function on $\mathfrak{B}$ with

$$|E(e)| \leq M, \qquad e \in \mathfrak{B}.$$

For a finitely valued function

$$f = \sum_{i=1}^{n} \alpha_i \psi_{e_i} \in \mathfrak{B}(\mathfrak{M})$$

we define the integral

$$\int_M f(m)E(dm) = \sum_{i=1}^{n} \alpha_i E(e_i)$$

and note that this definition is independent of the representation of f. Also

$$\left| \int_M f(m)E(dm) \right| = \sup_{|x|=|x^*|=1} |\sum \alpha_i x^* E(\sigma_i)x|$$

$$\leq \sup_i |\alpha_i| \operatorname*{var}_{e \in \mathfrak{B}} x^*E(e)x \leq \sup_i |\alpha_i| \, 4 \sup_{e \in \mathfrak{B}} |x^*E(e)x| \leq 4\, M \sup_m |f(m)|.$$

Thus if $f \in \mathfrak{B}(\mathfrak{M})$ is the limit in $\mathfrak{B}(\mathfrak{M})$ of two sequences $\{f_n\}$ and $\{g_n\}$ of finitely valued functions in $\mathfrak{B}(\mathfrak{M})$ then

$$\lim_n \int_{\mathfrak{M}} f_n(m)E(dm) = \lim_n \int_{\mathfrak{M}} g_n(m)E(dm),$$

and this limit is taken as the *definition of the integral*

$$\int_{\mathfrak{M}} f(m)E(dm).$$

It is clear that in case $\mathfrak{M}$ is a compact set in the plane and f is continuous the integral as defined coincides with the Riemann integral. In case E is a spectral measure on $\mathfrak{B}$ for which $x^*E(e)x$ is countably additive on $\mathfrak{B}$ for each $x \in \mathfrak{X}$ and each x^* in a total linear manifold $\Gamma \subset \mathfrak{X}^*$, we say that E is a *spectral measure of class* $(\mathfrak{B}, \Gamma)$.

LEMMA 6. *Let $\mathfrak{B}$ be a σ-field of subsets of a set $\mathfrak{M}$ with $\mathfrak{M} \in \mathfrak{B}$. Let E be a spectral measure of class $(\mathfrak{B}, \Gamma)$, and for $f \in \mathfrak{B}(\mathfrak{M})$ let*

$$S(f) \equiv \int_{\mathfrak{M}} f(m)E(dm).$$

Then there is a constant $v(E)$ such that

$$|S(f)| \leq v(E)|f|, \qquad f \in \mathfrak{B}(\mathfrak{M}).$$

Also for every $f \in \mathfrak{B}(\mathfrak{M})$ the operator $S(f)$ is a scalar type operator of class (Γ) whose resolution of the identity $E(S)$ is given by the equation

$$E(S, e) = E(f^{-1}(e)).$$

Proof. The first conclusion follows from the foregoing definition of the integral. Now if E is a spectral measure the map $f \longrightarrow S(f)$ of $\mathfrak{B}(\mathfrak{M})$ into $B(\mathfrak{X})$ is a homomorphism; that is, it preserves multiplication as well as addition. Thus, if $\lambda_0 \notin \bar{\sigma}$, the operator

$$U = \int_{\mathfrak{M}} (\lambda_0 - f(m))^{-1} \psi_{f^{-1}(\sigma)}(m) E(dm)$$

satisfies the equation

$$(\lambda_0 I - S(f)) U = E(f^{-1}(\sigma)),$$

which shows that

$$\sigma(S(f),\ E(f^{-1}(\sigma))\mathfrak{X}) \subset \bar{\sigma}.$$

Thus S is a spectral operator whose resolution of the identity is given by

$$E(S, \sigma) = E(f^{-1}(\sigma)).$$

To see that S is a scalar type operator we decompose the closure of $f(\mathfrak{M})$ into a finite number of disjoint parts σ_i, each of diameter at most ϵ. Let $\lambda_i \in \sigma_i$. Then

$$|\sum \lambda_i \psi_{\sigma_i}(f(m)) - f(m)| < \epsilon, \qquad m \in \mathfrak{M},$$

and so

$$S(f) = \lim_{\epsilon \to 0} \sum \lambda_i E(f^{-1}(\sigma_i)) = \lim_{\epsilon \to 0} \int \sum_i \lambda_i \psi_{\sigma_i}(\lambda) E(S, d\lambda) = \int \lambda E(S, d\lambda),$$

which proves that S is of scalar type.

DEFINITION 5. If $T, U, \cdots, V$ are in $B(\mathfrak{X})$, the symbol $\mathfrak{A}(T, U, \cdots, V)$ will stand for the smallest subalgebra of $B(\mathfrak{X})$ which is closed in the norm topology of $B(\mathfrak{X})$, which contains $T, U, \cdots, V$, and I, and which contains the inverse W^{-1} of any of its elements provided that the inverse exists as an element of $B(\mathfrak{X})$. The algebra $\mathfrak{A}(U, T, \cdots, V)$ will sometimes be called the *full algebra generated by* $U, T, \cdots, V$. If σ is a compact set in the complex plane, the symbol $CR(\sigma)$ will stand for the algebra of all complex functions $f(\lambda)$, $\lambda \in \sigma$ which may be approximated uniformly on σ by rational functions. The norm in $CR(\sigma)$ is

$$|f| = \max_{\lambda \in \sigma} |f(\lambda)|,$$

so that $CR(\sigma)$ is a subalgebra of $C(\sigma)$. Two B-algebras are said to be *equivalent* in case they are topologically and algebraically isomorphic.

THEOREM 13. *Let T be a spectral operator and S its scalar part. Then, as a vector direct sum,*

$$\mathfrak{A}(T,S) = \mathfrak{A}(S) \oplus \mathfrak{R},$$

where $\mathfrak{R}$ is the radical in $\mathfrak{A}(T,S)$. Furthermore, $\mathfrak{A}(S)$ is equivalent to $CR(\sigma(T))$, and every operator in $\mathfrak{A}(T,S)$ is a spectral operator.

Proof. If f is rational and analytic on $\sigma(T) = \sigma(S)$, then $f(\sigma(S)) = \sigma(f(S))$ and thus

$$\max_{\lambda \in \sigma(S)} |f(\lambda)| \leq |f(S)| \leq \max_{\lambda \in \sigma(S)} |f(\lambda)| v(E).$$

Thus $\mathfrak{A}(S)$ is equivalent to $CR(\sigma(S))$. Since $\mathfrak{A}(S)$ has no radical it is seen that $\mathfrak{A}(S) \oplus \mathfrak{R}$ is a direct vector sum contained in $\mathfrak{A}(T,S)$. Now let N be the radical part of T. It follows from Theorems 8 and 9 and Lemma 6 that the canonical decomposition of $f(T)$ for $f \in F(T)$ is

$$f(T) = f(S) + N_1.$$

Hence in particular if T^{-1} exists its canonical decomposition is

(i) $$T^{-1} = S^{-1} + N_2.$$

Also

$$T^n = S^n + N_3, \quad T^n S^m = S^{n+m} + N_4,$$

and thus for a polynomial P in T and S we have

$$P(T,S) = Q(S) + N_5,$$

where Q is a polynomial and N_5 a generalized nilpotent. Since for m in the space $\mathfrak{M}$ of maximal ideals of $\mathfrak{A}(T,S)$ we have

$$P(S(m), S(m)) = P(T(m), S(m)) = Q(S(m)),$$

it is seen that $Q(S) = P(S,S)$ and thus

$$\text{(ii)} \qquad P(T,S) = P(S,S) + N_5 .$$

If P_1 is also a polynomial in two variables, the operator

$$R(T,S) = P(T,S)P_1(T,S)^{-1}$$

will be defined as an element of $\mathfrak{A}(T,S)$ if and only if $P_1(\lambda,\lambda) \neq 0$ for $\lambda \in \sigma(T)$. In this case we see from (i) and (ii) that

$$\text{(iii)} \qquad R(T,S) = R(S,S) + N_6 .$$

Since $R(S,S)$ is of type 0, this is the canonical form for $R(T,S)$. An arbitrary $U \in \mathfrak{A}(T,S)$ is a limit, $U = \lim R_n$, of rational functions R_n in T and S. Since

$$\sigma(T) = \sigma(S) = S(\mathfrak{M}) = T(\mathfrak{M}),$$

and $T(m) = S(m)$, we have

$$\sup_{\lambda \in \sigma(S)} |R_n(\lambda,\lambda) - R_p(\lambda,\lambda)| = \sup_m |\{R_n(T,S) - R_p(T,S)\}(m)|$$

$$\leq |R_n(T,S) - R_p(T,S)| \longrightarrow 0.$$

Hence $R_n(\lambda,\lambda)$ converges uniformly on $\sigma(S)$ to a function $f \in CR(\sigma(S))$. Thus $R_n(S,S) \longrightarrow f(S)$ in $\mathfrak{A}(S)$, and $U \in \mathfrak{A}(S) \oplus \mathfrak{N}$. It follows from Lemma 6 that every operator in $\mathfrak{A}(S)$ is a scalar type operator and thus it is seen, by Theorem 8, that every operator in $\mathfrak{A}(T,S)$ is a spectral operator.

THEOREM 14. *Let E be the resolution of the identity of the spectral operator T. Let $\mathfrak{M}$ be the space of maximal ideals is the algebra*

$$\mathfrak{A} = \mathfrak{A}(E(\sigma), \ \sigma \text{ a Borel set}).$$

Let R_1 be the radical in the algebra

$$\mathfrak{A}_1 = \mathfrak{A}(T, E(\sigma), \ \sigma \text{ a Borel set}).$$

Then $\mathfrak{A}$ *is equivalent to* $C(\mathfrak{M})$, *and*

$$\mathfrak{A}_1 = \mathfrak{A} \oplus \mathfrak{R}_1.$$

Furthermore, every operator in $\mathfrak{A}_1$ *is a spectral operator.*

Proof. Elements of the form

$$\text{(i)} \qquad U = \sum_{i=1}^{n} \alpha_i E(\sigma_i), \; E(\sigma_i) \neq 0, \quad \sigma_i \sigma_j = \emptyset, \; i \neq j, \; \bigcup_i \sigma_i = \sigma(T)$$

are dense in $\mathfrak{A}$ since if such an element has an inverse the inverse is again of the same form. Furthermore, if $E(\sigma_i) \notin m \in \mathfrak{M}$ then $U(m) = \alpha_i$. Thus, using Lemma 6, we have

$$\sup_m |U(m)| = \sup_i |\alpha_i| \leq |U| \leq \sup_i |\alpha_i| v(E) = \sup_m |U(m)| v(E),$$

and therefore

$$\sup_m |U(m)| \leq |U| \leq \sup_m |U(m)| v(E), \quad U \in \mathfrak{A},$$

which shows that $\mathfrak{A}$ is equivalent to a subalgebra of $C(\mathfrak{M})$. Since the projections $E(\sigma)$ generate $\mathfrak{A}$, they distinguish between points in $\mathfrak{M}$. Also it is clear that the element

$$V = \sum \bar{\alpha}_i E(\sigma_i)$$

is related to the operator U given in (i) by

$$\overline{U(m)} = V(m), \quad m \in \mathfrak{M}.$$

Thus, by the Stone-Weierstrass theorem, $\mathfrak{A}$ is equivalent to $C(\mathfrak{M})$. Hence $\mathfrak{A} \oplus \mathfrak{R}$ is a vector direct sum and a subalgebra of $\mathfrak{A}_1$. It is also closed in $\mathfrak{A}_1$ since if m_1 is a maximal ideal in $\mathfrak{A}_1$ we have, for an arbitrary operator $U = S + N$ with $S \in \mathfrak{A}$, $N \in \mathfrak{R}$,

$$|S| v(E)^{-1} \leq \sup_{m_1} |S(m_1)| = \sup_m |U(m_1)| \leq |U| \leq |S| + |N|.$$

Also since $\mathfrak{A} \simeq C(\mathfrak{M})$ it is seen that $\mathfrak{A} \oplus \mathfrak{R}_1$ is a full algebra of operators;

that is, it contains the inverse W^{-1} of any of its elements provided that W^{-1} exists as an element of $B(\mathfrak{X})$. Thus $\mathfrak{A}_1 \subset \mathfrak{A} \oplus \mathfrak{R}_1 \subset \mathfrak{A}_1$. Finally, to see that every operator in $\mathfrak{A}_1$ is a spectral operator it will, in view of Theorem 8, suffice to show that every $U \in \mathfrak{A}$ is a scalar type operator. Consider a finitely valued measurable function

$$f(\lambda) = \sum \alpha_i \psi_{\sigma_i}(\lambda), \quad \lambda \in \sigma(S).$$

We may suppose that $\sigma_i \sigma_j = \emptyset$ $(i \neq j)$, and $\cup \sigma_i = \sigma(S)$, so that the values of f are the numbers α_i. The operator

$$f(S) = \int_{\sigma(S)} f(\lambda) E(d\lambda) = \sum_i \alpha_i E(\sigma_i),$$

as was shown above, has the property that except for λ in a set σ with $E(\sigma) = 0$ we have $|f(\lambda)| \leq f(S)$. Thus if we define the norm

$$|f|_E \equiv E\text{-ess. sup } |f(\lambda)| \equiv \inf_{E(\sigma) = I} \ \sup_{\lambda \in \sigma} |f(\lambda)|,$$

the operator $f(S)$ satisfies the inequality

$$|f|_E \leq |f(S)| \leq |f|_E \, v(E).$$

The general operator U in $\mathfrak{A}$ is the limit of a sequence $f_n(S)$, where $f_n(\lambda)$ is a finitely valued measurable function. Thus

$$f(\lambda) = \lim f_n(\lambda)$$

exists uniformly except on a set $\sigma \subset \sigma(S)$, where $E(\sigma) = 0$, and

$$U = \int_{\sigma(S)} f(\lambda) E(d\lambda).$$

Hence, by Lemma 6, U is a scalar type operator.

DEFINITION 6. If $T = S + N$ is the canonical decomposition of the spectral operator T, and E is its resolution of the identity, by $EB(\sigma(T))$ will be meant the space of all E-essentially bounded Borel measurable functions defined on $\sigma(T) = \sigma(S)$. The norm is

$$|f| = E\text{-ess.} \sup_{\lambda \in \sigma(S)} |f(\lambda)| \equiv \inf_{E(\sigma)=I} \sup_{\lambda \in \sigma} |f(\lambda)|.$$

According to what has just been shown we may state:

THEOREM 15. *In the notation of Theorem* 14 *we have* $\mathfrak{A}$ *equivalent to* $EB(\sigma(T))$.

THEOREM 16. *If S is a scalar type operator with resolution of the identity E, and f is an E-essentially bounded Borel function on $\sigma(S)$, then*

$$\sigma(f(S)) = \bigcap_{E(\sigma)=I} \overline{f(\sigma)}.$$

Proof. If $\lambda_0 \notin \overline{f(\sigma)}$, where $E(\sigma) = I$, then

$$h(\lambda) = \begin{cases} (\lambda_0 - f(\lambda))^{-1}, & \lambda \in \sigma, \\ 0 & , \lambda \notin \sigma, \end{cases}$$

is a bounded Borel measurable function and

$$h(S)(\lambda_0 I - f(S)) = I,$$

so that $\lambda_0 \in \rho(f(S))$. Thus $\overline{f(\sigma)} \supset \sigma(f(S))$ if $E(\sigma) = I$, and

$$\bigcap_{E(\sigma)=I} \overline{f(\sigma)} \supset \sigma(f(S)).$$

Conversely, if $\lambda_0 \in \rho(f(S))$ we see from Theorem 15 that $(\lambda_0 - f(\lambda))^{-1}$ is E-essentially bounded on $\sigma(T)$. Hence there is a Borel set σ with $E(\sigma) = I$ and

$$|\lambda_0 - f(\lambda)|^{-1} \leq M, \qquad \lambda \in \sigma.$$

Hence $\lambda_0 \notin \overline{f(\sigma)}$. This shows that

$$\sigma(f(S)) \supset \overline{f(\sigma)} \supset \bigcap_{E(\sigma)=I} \overline{f(\sigma)},$$

and completes the proof.

THEOREM 17. *Let $\mathfrak{A}(\tau)$ be the full algebra generated by a family τ of commuting spectral operators together with their resolutions of the identity. If the Boolean algebra determined by the resolutions of the identity of the operators in τ is bounded, then, as a vector direct sum,*

$$\mathfrak{A}(\tau) = \mathfrak{A}_1 \oplus \mathfrak{R},$$

where $\mathfrak{R}$ is the radical in $\mathfrak{A}(\tau)$ and $\mathfrak{A}_1$ is equivalent to the algebra of continuous functions on the space of maximal ideals in $\mathfrak{A}(\tau)$.

Proof. Note first that if $T, U \in \tau$ have resolutions of the identity $E(T,\cdot)$, $E(U,\cdot)$, respectively, then for every pair σ, μ of Borel sets in the plane the projections $E(T,\sigma)$, $E(U,\mu)$ commute. This follows from a double application of Theorem 5. Thus the various projections $E(T,\sigma)$ determined by Borel sets σ and operators $T \in \tau$ determine a Boolean algebra $\mathfrak{A}_0$, and by assumption there is a constant M with $|E| \leq M$ for $E \in \mathfrak{A}_0$. We shall first show that there is a constant K such that

$$\text{(i)} \qquad \sum_{i=1}^{n} |x^*E_i x| \leq K|x||x^*|, \qquad x \in \mathfrak{X}, \quad x^* \in \mathfrak{X}^*,$$

provided that $E_i \in \mathfrak{A}_0$ and $E_iE_j = 0$ for $i \neq j$. To see this, let $(x^*Ex)_r$ be the real part of x^*Ex. Then, if $E_iE_j = 0$ $(i \neq j)$, we have

$$\sum |(x^*E_i x)_r| = \sum{}' (x^*E_i x)_r - \sum{}'' (x^*E_i x)_r$$

$$= (x^*(\sum{}' E_i)x)_r - (x^*(\sum{}'' E_i)x)_r \leq 2M|x||x^*|,$$

where $\sum'$ $(\sum'')$ represents the sum over those i for which $(x^*E_i x)_r \geq 0$ (< 0). Similarly for the imaginary part of x^*Ex. Thus

$$\sum |x^*E_i x| \leq 4M\, |x||x^*|,$$

which proves (i).

Now consider elements $U \in \mathfrak{A}(\tau)$ of the form

$$\text{(ii)} \qquad U = S + N,$$

where

(iii) $$S = \sum_{i=1}^{n} \alpha_i E_i$$

with

$$0 \neq E_i \in \mathfrak{A}_0, \quad E_i E_j = 0, \quad i \neq j, \quad E_1 + \cdots + E_n = I,$$

and where $N \in \mathfrak{R}$, the radical of $\mathfrak{A}(\tau)$. If $m \in \mathfrak{M}$, the space of maximal ideals in $\mathfrak{A}(\tau)$, then in view of (iii) there is an i with $E_i(m) = 1$, $E_j(m) = 0$ $(j \neq i)$. Thus $\alpha_i = U(m) = S(m)$ and

(iv) $$\sup_i |\alpha_i| = \sup_m |U(m)| \leq |U| \leq |S| + |N|.$$

From (i) and (iii) it is seen that

$$|S| = \sup_{|x| = |x^*| = 1} |\sum \alpha_i x^* E_i x| \leq \sup_i |\alpha_i| K$$

and hence, by (iv),

(v) $$K^{-1}|S| \leq |U| \leq |S| + |N|.$$

The inequality (v) shows that if $U_n = S_n + N_n$ is a convergent sequence of operators, each of the form (ii) with S_n of the form (iii) and $N_n \in \mathfrak{R}$, then $\{S_n\}$ and $\{N_n\}$ are also convergent sequences. Let $\mathfrak{A}_1$ be the algebra of all limits $S_0 = \lim_n S_n$, where S_n has the form (iii). Since for the operator (iii) we have, as shown above,

$$\sup_m |S(m)| \leq |S| \leq \sup_m |S(m)| \cdot K,$$

it is seen that $\mathfrak{A}_1$ is equivalent to a subalgebra C of $C(\mathfrak{M})$, and the Weierstrass theorem[5] shows that $C = C(\mathfrak{M})$. Clearly therefore, $\mathfrak{A}_1 \oplus \mathfrak{R}$ is a direct sum, is contained in $\mathfrak{A}(\tau)$, contains every $E(T, \sigma)$ with $T \in \tau$ and σ a Borel set in the plane, and contains every $T \in \tau$. This last statement, namely that $\tau \subset \mathfrak{A}_1 \oplus \mathfrak{R}$, follows since the canonical reduction $T = S + N$ has the property that $N \in \mathfrak{R}$ and $S \in \mathfrak{A}_1$. To complete the proof it will suffice to show that $\mathfrak{A}_1 \oplus \mathfrak{R}$ is a full algebra; that is, it will suffice to show that if $T \in \mathfrak{A}_1 \oplus \mathfrak{R}$, and $T^{-1} \in B(\mathfrak{X})$,

[5] As proved by M. H. Stone [13] for real algebras $C(\mathfrak{M})$ and by I. Gelfand and G. Silov [9] for complex algebras $C(\mathfrak{M})$.

then $T^{-1} \in \mathfrak{A}_1 \oplus \mathfrak{R}$. Let $T = S + N$ be the canonical form of T; then since $T(m) = S(m) \neq 0$, $m \in \mathfrak{M}$, we see that S^{-1} exists and is in $\mathfrak{A}_1$ because $S^{-1}(\cdot) \in C(\mathfrak{M})$. Thus

$$T(S^{-1} + M) = I,$$

where $M = -T^{-1}NS^{-1} \in \mathfrak{R}$. Thus $\mathfrak{A}_1 \oplus \mathfrak{R}$ is a full algebra containing τ and every projection $E(T, \sigma)$ $(T \in \tau)$, and hence $\mathfrak{A}(\tau) = \mathfrak{A}_1 \oplus \mathfrak{R}$.

THEOREM 18. *Let $\mathfrak{B}$ be the Borel sets in the compact Hausdorff space $\mathfrak{M}$, and let $\mathfrak{A}$ be an algebra of operators on the complex B-space $\mathfrak{X}$ which is equivalent to the algebra $C(\mathfrak{M})$ of continuous functions on $\mathfrak{M}$. Then there is a function A on $\mathfrak{B}$ to $B(\mathfrak{X}^*)$ with the properties:*

(i) *A is a spectral measure in $\mathfrak{X}^*$ of class $(\mathfrak{B}, \mathfrak{X})$;*

(ii) *if $S(f)$ is the element in $\mathfrak{A}$ corresponding to the element f in $C(\mathfrak{M})$ under some homeomorphic isomorphism then, for every $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$,*

$$x^*S(f)x = \int_{\mathfrak{M}} f(m)xA(dm)x^*, \qquad f \in C(\mathfrak{M});$$

(iii) *the adjoint S^* of every S in $\mathfrak{A}$ is a scalar type operator of class $\mathfrak{X}$;*

(iv) *if $\mathfrak{X}$ is reflexive, every S in $\mathfrak{A}$ is a scalar type operator of class $\mathfrak{X}^*$.*

Proof. Let $S(f)$ be the operator in $\mathfrak{A}$ corresponding to the function $f \in C(\mathfrak{M})$ under some homeomorphic isomorphism of $\mathfrak{A}$ onto $C(\mathfrak{M})$. Then for x in $\mathfrak{X}$ and x^* in $\mathfrak{X}^*$ we have $x^*S(f)x$ a linear functional on $C(\mathfrak{M})$ and hence, by the Riesz representation theorem, there is a uniquely determined regular measure $\mu(\cdot, x, x^*)$ such that

$$x^*S(f)x = \int_{\mathfrak{M}} f(m)\mu(dm, x, x^*), \quad f \in C(\mathfrak{M}), \quad x \in \mathfrak{X}, \quad x^* \in \mathfrak{X}^*.$$

Since $\mu(e, x, x^*)$ is uniquely determined by e, x, x^* it is, for each $e \in \mathfrak{B}$, bilinear in x and x^*. Since

$$|\mu(e, x, x^*| \leq \operatorname*{var}_{e} |\mu(e, x, x^*)| = \sup_{|f|=1} |x^*S(f)x| \leq K|x||x^*|,$$

it is seen that $\mu(e, x, x^*)$ is continuous in x and x^*. Hence for fixed e and x^* there is a point $A(e)x^* \in \mathfrak{X}^*$ such that

$$\mu(e, x, x^*) = xA(e)x^*.$$

It follows from the bilinearity and boundedness of μ that $A(e) \in B(\mathfrak{X}^*)$. Thus (ii) is proved and a part of (i) is proved. To complete the proof of (i) we have, for every pair $f, g \in C(\mathfrak{M})$,

$$\int_{\mathfrak{M}} f(m) \int_{\mathfrak{M}} g(\mu) x A(d\mu \cap dm) x^* = \int_{\mathfrak{M}} f(m) \int_{dm} g(\mu) x A(d\mu) x^*$$

$$= \int_{\mathfrak{M}} f(m) g(m) x A(dm) x^* = x^* S(fg) x$$

$$= x^* S(f) S(g) x = \int_{\mathfrak{M}} f(m) S(g) x A(dm) x^*$$

$$= \int_{\mathfrak{M}} f(m) \int_{\mathfrak{M}} g(\mu) x A(d\mu) A(dm) x^* .$$

Thus, since a functional on $C(\mathfrak{M})$ determines the regular measure uniquely, we have

$$A(\sigma \cap \delta) = A(\sigma) A(\delta), \quad \sigma, \delta \in \mathfrak{B},$$

and this completes the proof of (i).

The integral instead of being thought of as a Lebesgue integral in the weak operator topology may be thought of as an integral in the uniform topology as defined immediately preceding Lemma 6. Thus, by Lemma 6, each of the operators

$$S^*(f) = \int_{\mathfrak{M}} f(m) A(dm)$$

is a scalar type operator in $\mathfrak{X}^*$ of class $\mathfrak{X}$, which proves (iii). In case $\mathfrak{X}$ is reflexive, $E(\sigma) = A^*(\sigma)$ is a spectral measure in $\mathfrak{X}$ and hence, by Lemma 6,

$$S(f) = \int_{\mathfrak{M}} f(m) E(dm), \quad f \in C(\mathfrak{M}),$$

is a scalar type operator of class $\mathfrak{X}^*$, which proves (iv) and completes the proof of the theorem.

THEOREM 19. *The adjoint T^* of every operator T in the algebra $\mathfrak{A}(\tau)$ as defined in Theorem 17 is a spectral operator of class $\mathfrak{X}$. If $\mathfrak{X}$ is reflexive, every $T \in \mathfrak{A}(\tau)$ is a spectral operator of class $\mathfrak{X}^*$.*

Proof. This follows immediately from Theorems 8, 17, and 18.

Theorem 19 shows that the sum and product of two spectral operators in a reflexive space $\mathfrak{X}$ will again be spectral operators provided that the Boolean algebra determined by all the projections in both resolutions of the identity is bounded. If $\mathfrak{X}$ is Hilbert space, J. Wermer [**16**] has shown that such is the case. In general, however, the Boolean algebra determined by two bounded Boolean algebras of projections, all of which commute, is not bounded. Also it is not always true that the sum of two spectral operators is a spectral operator. Examples proving both of these statements have been constructed by S. Kakutani [**10**].

Examples of spectral operators other than normal operators on Hilbert space are easy to construct, and some interesting classes have been discussed by J. Schwartz [**12**].[6]

Besides Theorems 8, 13, 14, 19, which are useful in the construction of spectral operators, we shall mention one more which will be needed in the perturbation theory of J. Schwartz.

THEOREM 20. *If T is a compact operator in a reflexive space $\mathfrak{X}$, then T is a spectral operator if and only if the integrals*

$$\text{(i)} \qquad \frac{1}{2\pi i}\int_C T(\lambda)\,d\lambda$$

are bounded as C varies over all admissible contours in the resolvent set. In this case the resolution of the identity is countably additive in the strong operator topology, and the integral (i) *is the value of the resolution of the identity on the domain bounded by C.*

Proof. Let $\lambda_0 = 0$, $\lambda_n \neq 0$ $(n = 1, 2, \cdots)$ be the points in the spectrum of T. Let

$$E(\lambda_n) = \frac{1}{2\pi i}\int_{C_n} T(\lambda)\,d\lambda \qquad (n = 1, 2, \cdots),$$

where C_n is a circle containing λ_n but no other spectral point. Since the Boolean

[6] Other spectral operators occurring in analysis will be found in the forthcoming book *Spectral Theory* by N. Dunford and J. Schwartz. Conditions on the rate of growth of the resolvent which are sufficient to ensure that T be spectral will be found in [**4**].

algebra determined by the $E(\lambda_n)$ is bounded, it may be embedded in a complete[7] Boolean algebra. We may therefore define

$$E(\lambda_0) = I - \bigcup_{n=1}^{\infty} E(\lambda_n)$$

and

$$E(\sigma) = \bigcup_{\lambda_n \in \sigma} E(\lambda_n),$$

σ arbitrary. If $\mathfrak{B}$ is the Boolean algebra of all subsets of the plane, it is clear that the map $\sigma \longrightarrow E(\sigma)$ is a homomorphism of $\mathfrak{B}$ onto a Boolean algebra of projections in $\mathfrak{X}$. From our hypothesis it follows that

(ii) $$|E(\sigma)| \le K, \quad \sigma \in \mathfrak{B}.$$

Now let $\sigma_n \subset \sigma_{n+1} \subset \cdots$ and $\sigma = \bigcup \sigma_n$. Then

(iii) $$E(\sigma) = \bigcup_{\lambda_n \in \sigma} E(\lambda_n) = \bigcup_n \bigcup_{\lambda_m \in \sigma_n} E(\lambda_m) = \bigcup_n E(\sigma_n),$$

and since $E(\sigma_n)x = x \ (n \ge m)$ if $x \in E(\sigma_m)\mathfrak{X}$, we see from (ii) and (iii) that $E(\sigma_n)x \longrightarrow x$ if $x \in E(\sigma)\mathfrak{X}$. Also since

$$E(\sigma')\mathfrak{X} = \bigcap (E(\sigma_n')\mathfrak{X})$$

we have $E(\sigma_n)x = 0$ if $x \in E(\sigma')\mathfrak{X}$. Thus $E(\sigma_n)x \longrightarrow E(\sigma)x$ for every x in $\mathfrak{X}$, and $E(\sigma)$ is countably additive on $\mathfrak{B}$ in the strong operator topology. To complete the proof that T is spectral, it will suffice to show that

(iv) $$\sigma(T, E(\sigma)\mathfrak{X}) \subset \bar{\sigma}, \quad \sigma \in \mathfrak{B}.$$

If $\lambda_0 \notin \bar{\sigma}$, then $E(\sigma)$ has the form (i), from which (iv) follows. If $\lambda_0 \in \bar{\sigma}$, then any spectral point $\lambda_n \notin \bar{\sigma}$ is in $\rho(T, E(\{\lambda_n\}')\mathfrak{X})$ and hence in $\rho(T, E(\sigma)\mathfrak{X})$, which proves (iv).

Finally let σ be an open and closed subset of $\sigma(T)$, and let $A(\sigma)$ be the projection defined by (i), where σ is the intersection of $\sigma(T)$ and the domain bounded by C. Then, since

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[7] Complete relative to the order $A \subset B$ $(AB = A)$. See, for example [6].

$$\sigma(T, A(\sigma)\mathfrak{X}) \subset \sigma, \quad \sigma(T, A(\sigma')\mathfrak{X}) \subset \sigma',$$

Theorems 3 and 4 show that

$$E(\sigma)A(\sigma) = A(\sigma), \quad E(\sigma)A(\sigma') = 0,$$

and hence that

$$E(\sigma) = E(\sigma)(A(\sigma) + A(\sigma')) = A(\sigma).$$

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COMMUTING SPECTRAL MEASURES ON HILBERT SPACE

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1. Introduction. By a "spectral measure" on Hilbert space H we mean a family of bounded operators $E(\sigma)$ on H defined for all Borel sets σ in the plane. We suppose:

(i) If σ_0 denotes the empty set and σ_1 the whole plane, then

$$E(\sigma_0) = 0, \quad E(\sigma_1) = I,$$

where I is the identity.

(ii) For all σ_1, σ_2,

$$E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2);$$

and for disjoint σ_1, σ_2,

$$E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2).$$

(iii) There exists a constant M with $\|E(\sigma)\| \leq M$, all σ. It follows that $E(\sigma)^2 = E(\sigma)$ for each σ, and $E(\sigma_1)E(\sigma_2) = 0$ if σ_1, σ_2 are disjoint.

Mackey has shown in [3], as part of the proof of Theorem 55 of [3], that if $E(\sigma)$ is a spectral measure with the properties just stated, then there exists a bicontinuous operator A such that $A^{-1}E(\sigma)A$ is self-adjoint for every σ. In a special case this result was proved by Lorch in [2]. We shall prove:

THEOREM 1. *Let $E(\sigma)$ and $F(\eta)$ be two commuting spectral measures on H; that is,*

$$E(\sigma)F(\eta) = F(\eta)E(\sigma)$$

for every σ, η. Then there exists a bicontinuous operator A such that $A^{-1}E(\sigma)A$ and $A^{-1}F(\eta)A$ are self-adjoint for every σ, η.

As a corollary of Theorem 1, we shall obtain:

Received March 4, 1953.
Pacific J. Math. 4 (1954), 355-361

THEOREM 2. *If T_1, T_2 are spectral operators on H, in the sense of Dunford* [1], *and $T_1 T_2 = T_2 T_1$, then $T_1 + T_2$ and $T_1 T_2$ are again spectral operators.*

2. Lemmas. We shall use two lemmas in proving Theorem 1.

LEMMA 1. *Let $P_1, P_2, \cdots, P_n$ be operators on Hilbert space with*

$$P_i P_j = 0 \qquad (i \neq j), \qquad P_i^2 = P_i, \qquad \sum_{i=1}^{n} P_i = I.$$

Suppose that, for every set $\delta_1, \delta_2, \cdots, \delta_n$ of zeros and ones,

$$\left\| \sum_{i=1}^{n} \delta_i P_i \right\| \leq M.$$

Then for every x we have

$$\frac{1}{4M^2} \|x\|^2 \leq \sum_{i=1}^{n} \|P_i x\|^2 \leq 4M^2 \|x\|^2$$

This Lemma is proved in [3, p. 147]; we include the proof for completeness.

Proof. We note that

$$\sum_{i=1}^{n} \|P_i x\|^2 = \frac{1}{2^n} \sum \|\epsilon_1 P_1 x + \cdots + \epsilon_n P_n x\|^2,$$

where the sum is taken over all possible sets $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$, where $\epsilon_i = \pm 1$. Hence

$$a_x = \|\epsilon_1' P_1 x + \cdots + \epsilon_n' P_n x\|^2 \leq \sum_{i=1}^{n} \|P_i x\|^2$$

$$\leq \|\epsilon_1 P_1 x + \cdots + \epsilon_n P_n x\|^2 = b_x$$

for some choice of the ϵ_i' and ϵ_i. Now

$$b_x = \left\| \sum_{i=1}^{n} \delta_i^+ P_i x - \sum_{i=1}^{n} \delta_i^- P_i x \right\|^2,$$

where the δ_i^+ and the δ_i^- are 1 or 0.

Hence

$$\sum_{i=1}^{n} ||P_i x||^2 \leq 4M^2 \cdot ||x||^2.$$

Let now $P^+ = \sum P_i$, summed over those i with $\epsilon_i' = 1$; and let $P^- = \sum P_i$, summed over those i with $\epsilon_i' = -1$. Then

$$(P^+ - P^-)^2 = P^+ + P^- = I \text{ and } ||P^+x - P^-x||^2 = a_x.$$

hence

$$||x||^2 = ||(P^+ - P^-)^2 x||^2 \leq ||P^+ - P^-||^2 \cdot ||P^+x - P^-x||^2.$$

Now $||P^+|| \leq M$ and $||P^-|| \leq M$ and so

$$||x||^2 \leq (2M)^2 a_x \leq (2M)^2 \sum_{i=1}^{n} ||P_i x||^2.$$

LEMMA 2. *Let $E(\sigma)$ and $F(\eta)$ be commuting spectral measures on Hilbert space. Then there is a fixed K such that for any set $\sigma_1, \sigma_2, \cdots, \sigma_n$ of disjoint Borel sets, and set $\eta_1, \eta_2, \cdots, \eta_n$ of arbitrary Borel sets,*

$$\left\| \sum_{i=1}^{n} E(\sigma_i) F(\eta_i) \right\| \leq K.$$

Proof. Fix x. By (iii) there exist constants L and M, with $||E(\sigma)|| \leq M$, $||F(\eta)|| \leq L$ for any σ, η. Let σ_{n+1} be the complement of

$$\bigcup_{i=1}^{n} \sigma_i .$$

Then

$$\left\| \sum_{i=1}^{n} E(\sigma_i) F(\eta_i) x \right\|^2 \leq 4M^2 \sum_{\nu=1}^{n+1} \left\| E(\sigma_\nu) \left(\sum_{i=1}^{n} E(\sigma_i) F(\eta_i) x \right) \right\|^2 = C$$

by Lemma 1;

$$C = 4M^2 \sum_{\nu=1}^{n} ||E(\sigma_\nu)F(\eta_\nu)x||^2,$$

since $E(\sigma_\nu)E(\sigma_i) = E(\sigma_\nu \cap \sigma_i)$;

$$C = 4M^2 \sum_{\nu=1}^{n} ||F(\eta_\nu)E(\sigma_\nu)x||^2,$$

by commutativity of the $E(\sigma)$ and $F(\eta)$;

$$C \leq 4M^2 \cdot L^2 \sum_{\nu=1}^{n} ||E(\sigma_\nu)x||^2,$$

since $||F(\eta_\nu)|| \leq L$;

$$C \leq (4M^2)^2 \cdot L^2 ||x||^2,$$

by Lemma 1. Hence

$$\left\| \sum_{i=1}^{n} E(\sigma_i)F(\eta_i) \right\| \leq 4M^2 L.$$

In the proof of Theorem 1 we shall use the method of Mackey in [3], together with Lemmas 1 and 2.

3. Proof of Theorem 1. By a "partition" π of the plane we mean a finite family of Borel sets $\sigma_1, \sigma_2, \cdots, \sigma_n$, mutually disjoint and with union equal to the whole plane. If (x, y) denotes the given scalar product in H, and

$$\pi_1 = (\sigma_i)_{i=1}^{n} \quad \pi_2 = (\eta_j)_{j=1}^{m}$$

are two partitions, set

$$(x, y)_{\pi_1, \pi_2} = \sum_{i=1}^{n} \sum_{j=1}^{m} (E(\sigma_i)F(\eta_j)x, E(\sigma_i)F(\eta_j)y).$$

It is easily verified that the quantity $(x, y)_{\pi_1, \pi_2}$ is a scalar product in H. Further, it follows by Lemma 2 that the operators

$$P_{ij} = E(\sigma_i)F(\eta_j) \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m,)$$

satisfy the hypotheses of Lemma 1.

Hence Lemma 1 yields

$$\frac{1}{4K^2}\ ||x||^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{m} ||E(\sigma_i)F(\eta_j)x||^2 \leq 4K^2||x||^2,$$

where K depends only on $\sup_\sigma ||E(\sigma)||$ and $\sup_\eta ||F(\eta)||$. But

$$\sum_{i=1}^{n} \sum_{j=1}^{m} ||E(\sigma_i)F(\eta_j)x||^2 = (x, x)_{\pi_1,\pi_2} = ||x||^2_{\pi_1,\pi_2}.$$

Finally, each $E(\sigma_i)$ and $F(\eta_j)$ $(i = 1, 2, \cdots n;\ j = 1, 2, \cdots, m)$ is self-adjoint in the scalar product $(x, y)_{\pi_1,\pi_2}$, as is readily verified.

For each pair of vectors $x, y \in H$, now, let S_{xy} be the disk in the complex plane consisting of all z with

$$|z| \leq 4K^2\ ||x|| \cdot ||y||.$$

If S denotes the cartesian product of the disks S_{xy} over all pairs x, y, then S is a compact topological space, by Tychonoff's theorem. Further, as we saw above,

$$||x||^2_{\pi_1,\pi_2} \leq 4K^2\,||x||^2.$$

Hence by Schwarz's inequality, applied to the scalar product $(x, y)_{\pi_1,\pi_2}$, we see that the number $(x, y)_{\pi_1,\pi_2}$ lies in the disk S_{xy} for every pair x, y. Hence there is a point p_{π_1,π_2} in S whose x, y-coordinate is $(x, y)_{\pi_1,\pi_2}$.

Let us now partially order the set of points p_{π_1,π_2} in S by saying that $p_{\pi_1',\pi_2'}$ is "greater than" p_{π_1,π_2} (in symbols $p_{\pi_1',\pi_2'} > p_{\pi_1,\pi_2}$) if π_1' is a refinement of the partion π_1, and π_2' is a refinement of the partition π_2. This ordering makes the set of points p_{π_1,π_2} in S into a directed system. Since S is a compact space, this directed system has a point of accumulation p. Let $(x, y)_p$ denote the (x, y) coordinate of p.

Then given a finite set of vector pairs (x_i, y_i), $i = 1, 2, \cdots, n$, and $\epsilon > 0$, and a pair π_1^0, π_2^0 of partitions, we have

$$|(x_i, y_i)_p - (x_i, y_i)_{\pi_1, \pi_2}| < \epsilon \qquad (i = 1, 2, \cdots, n)$$

for some

$$p_{\pi_1, \pi_2} > p_{\pi_1^0, \pi_2^0} .$$

Since $(x, y)_{\pi_1, \pi_2}$ is a scalar product for all π_1, π_2 it thus follows that $(x, y)_p$ is a scalar product, and since the norm $||x||_{\pi_1, \pi_2}$ is equivalent to the original norm with constants of equivalence independent of π_1, π_2, it follows that

$$||x||_p = \sqrt{(x, x)_p}$$

is also equivalent to the original norm.

Finally, fix a Borel set σ and vectors x, y. Let π_1^0 be the partition defined by σ and its complement, and π_2^0 be arbitrary. Then, if

$$p_{\pi_1, \pi_2} > p_{\pi_1^0, \pi_2^0} ,$$

we have

$$(E(\sigma)x, y)_{\pi_1, \pi_2} = (x, E(\sigma)y)_{\pi_1, \pi_2} ,$$

since π_1 is a refinement of π_1^0, and so σ is a finite union of sets involved in the partition π_1. Thus

$$(E(\sigma)x, y)_p = (x, E(\sigma)y)_p ,$$

and so the $E(\sigma)$ are self-adjoint with respect to the scalar product $(x, y)_p$, and similarly the $F(\eta)$ are self-adjoint with respect to this scalar product.

Since $||x||_p$ is equivalent to the given norm, it now follows that there exists a bi-continuous operator A with $(x, y)_p = (Ax, Ay)$, and hence $AE(\sigma)A^{-1}$ and $AF(\eta)A^{-1}$ are all self-adjoint.

4. Proof of Theorem 2. By Theorem 8 of [**1**], an operator T is spectral if and only if there exist two commuting operators S and N such that N is quasi-nilpotent and S admits a representation:

$$S = \int \lambda E(d\lambda),$$

where $E(d\lambda)$ denotes integration with respect to a certain spectral measure.

Such an S is called in [1] a "scalar type operator."

Now, by hypothesis, T_1 and T_2 are commuting spectral operators. We write

$$T_1 = S_1 + N_1, \quad T_2 = S_2 + N_2,$$

in accordance with the preceding. Then by Theorem 5 of [1] the operators S_1, S_2, N_1, N_2 all commute with one another. We thus have

$$T_1 + T_2 = S_1 + S_2 + Q \quad \text{and} \quad T_1 T_2 = S_1 S_2 + Q',$$

where Q and Q' are quasi-nilpotent, Q commutes with $S_1 + S_2$, and Q' commutes with $S_1 S_2$. By Theorem 8, quoted above, it is thus sufficient to show that $S_1 + S_2$ and $S_1 S_2$ are spectral operators of type 0; that is, of scalar type.

Let $E^1(\sigma)$ and $E^2(\sigma)$ be the spectral measures for S_1 and S_2, respectively. By Theorem 5 of [1] it follows, from the fact that $S_1 S_2 = S_2 S_1$, that $E^1(\sigma)$ and $E^2(\sigma)$ commute with one another for all σ. By our Theorem 1, then, there exists an operator A such that the operators $AE^1(\sigma)A^{-1}$ and $AE^2(\sigma)A^{-1}$ are all self-adjoint. Hence

$$J_1 = AS_1A^{-1} \quad \text{and} \quad J_2 = AS_2A^{-1}$$

are normal operators. Also $J_1 J_2 = J_2 J_1$, since $S_1 S_2 = S_2 S_1$. It follows that $J_1 + J_2$ and $J_1 J_2$ are again normal operators, for they commute with their adjoints as we verify by direct computation, using the fact that J_1 and J_2^* commute and J_2 and J_1^* commute, since J_1 and J_2 commute.

Thus $A(S_1 + S_2)A^{-1}$ and $A(S_1 S_2)A^{-1}$ are normal operators and so of scalar type. But if J is a scalar type operator and A bi-continuous, then, as is easily seen, $A^{-1}JA$ is again a scalar type operator. Hence $S_1 + S_2$ and $S_1 S_2$ are scalar type operators, and all is proved.

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AN EXAMPLE CONCERNING UNIFORM BOUNDEDNESS OF SPECTRAL MEASURES

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1. Introduction. Let $\mathfrak{X} = \{x\}$ be a Banach space with a norm $\|x\|$. A bounded linear operator E which maps $\mathfrak{X}$ into itself is called a *projection* if $E^2 = E$. We do not require that $\|E\| \leq 1$, where

$$\|E\| = \sup_{\|x\| \leq 1} \|Ex\|.$$

Let $\mathfrak{B} = \{\sigma\}$ be a Boolean algebra with a unit element 1. We denote the zero element of $\mathfrak{B}$ by 0, and two fundamental operations in $\mathfrak{B}$ by $\sigma_1 \cup \sigma_2$ and $\sigma_1 \cap \sigma_2$. A family $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ of projections $E(\sigma)$ of $\mathfrak{X}$ into itself is called an $\mathfrak{X}$-*spectral measure* on $\mathfrak{B}$ if the following conditions are satisfied: (i) $E(0) = 0$ (= zero operator), (ii) $E(1) = 1$ (= unit operator), (iii) $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$ for any $\sigma_1, \sigma_2 \in \mathfrak{B}$, (iv) $\sigma_1 \cap \sigma_2 = 0$ implies $E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2)$. An $\mathfrak{X}$-spectral measure $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ is said to be *uniformly bounded* if there exists a constant $K < \infty$ such that $\|E(\sigma)\| \leq K$ for all $\sigma \in \mathfrak{B}$.

Let $\mathfrak{B} = \{\sigma\}$, $\mathfrak{B}' = \{\sigma'\}$ be two Boolean algebras with a unit element, and let $\mathfrak{B}^* = \mathfrak{B} \otimes \mathfrak{B}'$ be the Kronecker product of $\mathfrak{B}$ and $\mathfrak{B}'$. Now $\mathfrak{B}^*$ may be considered as the Boolean algebra of all open-closed subsets σ^* of S^*, where $S^* = S \times S'$ is the topological Cartesian product of two Stone representation spaces S, S' of $\mathfrak{B}$, $\mathfrak{B}'$, respectively. Every element $\sigma^* \in \mathfrak{B}^*$ is expressible in the form:

$$\sigma^* = \bigcup_{i=1}^{n} \sigma_i \times \sigma_i', \tag{1.1}$$

where $\sigma_i \in \mathfrak{B}$, $\sigma_i' \in \mathfrak{B}'$ $(i = 1, \cdots, n)$.

Let $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ and $\{E'(\sigma') \mid \sigma' \in \mathfrak{B}'\}$ be two $\mathfrak{X}$-spectral measures on $\mathfrak{B}$, $\mathfrak{B}'$, respectively, which are commutative with each other; that is,

$$E(\sigma)E'(\sigma') = E'(\sigma')E(\sigma)$$

Received March 4, 1953.
Pacific J. Math. 4 (1954), 363-372

for any $\sigma \in \mathfrak{B}$, $\sigma' \in \mathfrak{B}'$. Let us put

$$(1.2) \qquad F(\sigma^*) = \sum_{i=1}^{n} E(\sigma_i) E'(\sigma_i')$$

if $\sigma^* \in \mathfrak{B}^*$ is of the form (1.1) and if $\sigma_i \times \sigma_i'$ $(i = 1, \cdots, n)$ are disjoint. Then it is easy to see that $F(\sigma^*)$ is uniquely determined (although the expression (1.1) with disjoint $\sigma_i \times \sigma_i'$ is not necessarily unique), and $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$ is an $\mathfrak{X}$-spectral measure on $\mathfrak{B}^*$; $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$ is called the *direct product $\mathfrak{X}$-spectral measure of* $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ *and* $\{E'(\sigma') \mid \sigma' \in \mathfrak{B}'\}$.

It was asked by N. Dunford [2] whether the uniform boundedness of $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$ and $\{E'(\sigma') \mid \sigma' \in \mathfrak{B}'\}$ implies that of $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$. This question was answered in the affirmative by J. Wermer [5] in case $\mathfrak{X}$ is a Hilbert space. The main purpose of this note is to show that the answer is negative if $\mathfrak{X}$ is a general Banach space; that is, we want to prove the following proposition:

PROPOSITION. *There exists a Banach space $\mathfrak{X}$ and a commutative pair of uniformly bounded $\mathfrak{X}$-spectral measures for which the direct product $\mathfrak{X}$-spectral measure is not uniformly bounded.*

Such an example will be given in §3. In our example, the Banach space $\mathfrak{X}$ is given as a cross product space $C(S) \circledast C(S')$ of two Banach spaces of continuous functions which will be defined in §2. This Banach space is not reflexive and hence it remains open to decide whether the answer to the question is positive or negative in case $\mathfrak{X}$ is a reflexive Banach space.

2. The Banach space $C(S) \circledast C(S')$. Let $S = \{s\}$, $S' = \{s'\}$ be two compact Hausdorff spaces. Let $C(S)$, $C(S')$ be the Banach spaces of all complex-valued continuous functions $y(s)$, $z(s')$ defined on S, S' with the norms

$$\|y\|_\infty = \max_{s \in S} |y(s)|, \qquad \|z\|_\infty = \max_{s' \in S'} |z(s')|.$$

Let

$$S^* = S \times S' = \{s^* = (s, s') \mid s \in S, s' \in S'\}$$

be the topological Cartesian product of S and S', and let $C(S^*)$ be the Banach space of all complex-valued continuous functions

$$x(s^*) = x(s, s')$$

defined on S^* with the norm

$$||x||_\infty = \max_{s^* \in S^*} |x(s^*)|.$$

Now $C(S)$, $C(S')$ may be considered as closed linear subspaces of $C(S^*)$ by identifying $y(s) \in C(S)$, $z(s') \in C(S')$ with $x(s, s') \in C(S^*)$ defined by

$$x(s, s') = y(s), \quad x(s, s') = z(s'),$$

respectively.

Consider $C(S^*)$ as a normed ring with the norm $||x||_\infty$. Then $C(S)$ and $C(S')$ are closed subrings of $C(S^*)$. Let $C(S) \otimes C(S')$ be the subring of $C(S^*)$ algebraically generated by $C(S)$ and $C(S')$; that is, the set of all functions $x(s, s') \in C(S^*)$ of the form:

$$x(s, s') = \sum_{i=1}^{n} y_i(s) z_i(s'), \tag{2.1}$$

where $y_i(s) \in C(S)$, $z_i(s') \in C(S')$ $(i = 1, \cdots, n)$. From the Stone-Weierstrass theorem it follows that $C(S) \otimes C(S')$ is dense in $C(S^*)$.

Let us now introduce a new norm on $C(S) \otimes C(S')$ defined by

$$|||x||| = \inf \sum_{i=1}^{n} ||y_i||_\infty \cdot ||z_i||_\infty, \tag{2.2}$$

where inf is taken for all possible representations of $x(s, s') \in C(S) \otimes C(S')$ in the form (2.1).

It is easy to see that $|||x|||$ is a norm on $C(S) \otimes C(S')$ and satisfies

$$||x||_\infty \leq |||x|||$$

for all $x(s, s') \in C(S) \otimes C(S')$. Let $C(S) \circledast C(S')$ be the completion of $C(S) \otimes C(S')$ with respect to the norm $|||x|||$. The completion $C(S) \circledast C(S')$ is obtained from $C(S) \otimes C(S')$ by means of Cauchy sequences in $C(S) \otimes C(S')$ with respect to the norm $|||x|||$. Since a Cauchy sequence with respect to $|||x|||$ is a Cauchy sequence with respect to $||x||_\infty$, we may consider $C(S) \circledast C(S')$ as a subset of $C(S^*)$:

LEMMA 1. *Let $C(S) \circledast C(S')$ be the set of all functions $x_0(s^*) \in C(S^*)$ for which there exists a sequence $\{x_n(s^*) \mid n = 1, 2, \cdots\}$ of functions from*

$C(S)\otimes C(S')$ with the following properties:

(i) $\lim_{n\to\infty} \|x_n - x_0\|_\infty = 0$, *that is* $\lim_{n\to\infty} x_n(s^*) = x_0^*(s)$ *uniformly on* S^*;

(ii) $\lim_{m,n\to\infty} |||x_m - x_n||| = 0$, *that is,* $\{x_n \mid n = 1, 2, \cdots\}$

is a Cauchy sequence with respect to the norm $|||x|||$.

If we put

$$|||x_0||| = \lim_{n\to\infty} |||x_n|||,$$

then $C(S) \circledast C(S')$ *is a Banach space with respect to the norm* $|||x|||$ *and contains* $C(S)\otimes C(S')$ *as a dense subset.*

The proof is easy and so it is omitted. It is interesting to observe that $C(S) \circledast C(S')$ is a normed ring with respect to the norm $|||x|||$.

$C(S) \circledast C(S')$ is called the *minimal cross product Banach space* of $C(S)$ and $C(S')$. It is easy to see that the minimal cross product Banach space $\mathfrak{Y} \circledast \mathfrak{Z}$ of any two Banach spaces $\mathfrak{Y}$ and $\mathfrak{Z}$ can be defined in a similar way. $\mathfrak{Y} \circledast \mathfrak{Z}$ is one of the cross product Banach spaces defined and discussed by R. Schatten and J. von Neumann [**3**; **4**].

3. Construction of an example. Let us now consider the case when both S and S' are Cantor sets. Let $S = S'$ be the set of all real numbers s of the form

$$(3.1) \qquad s = 2\left\{\frac{\epsilon_1(s)}{3} + \frac{\epsilon_2(s)}{3^2} + \cdots + \frac{\epsilon_n(s)}{3^n} + \cdots\right\},$$

where $\epsilon_n(s) = 0$ or 1 $(n = 1, 2, \cdots)$. Let $\mathfrak{B} = \{\sigma\}$ be the Boolean algebra of all open-closed subsets σ of S.

Let $S^* = S \times S$ be the Cartesian product of S with itself, and let $\mathfrak{B}^* = \{\sigma^*\}$ be the Boolean algebra of all open-closed subsets σ^* of S^*. It is clear that $\mathfrak{B}^* = \mathfrak{B}\otimes\mathfrak{B}$; that is, $\mathfrak{B}^*$ consists of all subsets σ^* of S^* which are expressible in the form (1.1), where $\sigma_i, \sigma_i' \in \mathfrak{B}$ $(i = 1, \cdots, n)$.

For each $\sigma \in \mathfrak{B}$, let $\phi_\sigma(s)$ be the characteristic function of σ, and put

$$E(\sigma)x(s, s') = \phi_\sigma(s)x(s, s'), \quad E'(\sigma)x(s, s') = \phi_\sigma(s')\,x(s, s').$$

It is clear that $E(\sigma)$, $E'(\sigma)$ are projections of $\mathfrak{X} = C(S) \circledast C(S')$ into itself, and that $\{E(\sigma) \mid \sigma \in \mathfrak{B}\}$, $\{E'(\sigma) \mid \sigma \in \mathfrak{B}\}$ are $\mathfrak{X}$-spectral measures on $\mathfrak{B}$. Both of these spectral measures are uniformly bounded since $E(\sigma)$, $E'(\sigma)$ have norm 1 for any $\sigma \in \mathfrak{B}$ with $\sigma \neq 0$. Since

$$E(\sigma)E'(\sigma') = E'(\sigma')E(\sigma)$$

for any $\sigma, \sigma' \in \mathfrak{B}$, we can consider the direct product $\mathfrak{X}$-spectral measure $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$, defined on $\mathfrak{B}^* = \mathfrak{B} \otimes \mathfrak{B}$. We shall show that $\{F(\sigma^*) \mid \sigma^* \in \mathfrak{B}^*\}$ is not uniformly bounded.

Let us define a sequence of functions $\{\rho_n(s^*) \mid n = 0, 1, 2, \cdots\}$ defined on $S^* = S \times S$ as follows: $\rho_0(s^*) \equiv 1$ on S^*, and

$$\rho_n(s^*) \equiv \rho_n(s, s') = (-1)^{\sum_{k=1}^{n} \epsilon_k(s)\epsilon_k(s')}, \tag{3.2}$$

where $\epsilon_k(s)$ is the kth coefficient in the expansion (3.1) of s. It is easy to see that $\rho_n(s^*)$ takes only the values ± 1 and belongs to $C(S) \otimes C(S')$ for $n = 0, 1, 2, \cdots$. Let us put

$$\sigma_n^* = \{s^* \mid \rho_n(s^*) = 1\} \qquad (n = 0, 1, 2, \cdots).$$

Then $\sigma_n^* \in \mathfrak{B}^*$ for $n = 0, 1, 2, \cdots$, and it is easy to see that

$$\rho_n = (2F(\sigma_n^*) - I)\rho_0 \qquad (n = 0, 1, 2, \cdots).$$

Thus, in order to prove the proposition of § 1, it suffices to prove the following lemma:

LEMMA 2. *Let S be the Cantor set. Let $\{\rho_n(s^*) \mid n = 1, 2, \cdots\}$ be a sequence of functions defined on $S^* = S \times S$ by (3.2). Then*

$$\lim_{n \to \infty} |||\rho_n||| = \infty,$$

where the norm $|||\rho_n|||$ of ρ_n is defined by (2.2).

In order to prove this lemma, let us put

$$\tau(s) = \frac{\epsilon_1(s)}{2} + \frac{\epsilon_2(s)}{2^2} + \cdots + \frac{\epsilon_n(s)}{2^n} + \cdots. \tag{3.3}$$

Then $t = \tau(s)$ is a mapping of S onto the closed unit interval

$$I = \{t \mid 0 \le t \le 1\}$$

which is one-to-one except for a countable set. Let

$$\mu(\sigma) = m(\tau(\sigma))$$

be a measure defined on $B = \{\sigma\}$ which corresponds to the Lebesgue measure m on I. Let us consider the L^2-space $L^2(S;\mu)$ on S with respect to the measure μ, where the norm is given by

$$\|y\|_2 = \left\{\int_S |y(s)|^2 \mu(ds)\right\}^{1/2}. \tag{3.4}$$

Let $\sigma_i^{(n)}$ be the open-closed subset of S consisting of all $s \in S$ such that

$$\frac{\epsilon_1(s)}{2} + \cdots + \frac{\epsilon_n(s)}{2^n} = \frac{i-1}{2^n} \qquad (i = 1, \cdots, 2^n). \tag{3.5}$$

We observe that

$$\mu(\sigma_i^{(n)}) = 2^{-n} \qquad (i = 1, \cdots, 2^n)$$

and that $\rho_n(s, s')$ is constant ($= \epsilon_{ij}^{(n)} = \pm 1$) on each $\sigma_i^{(n)} \times \sigma_j^{(n)}$ $(i, j = 1, \cdots, 2^n)$. Further, if we put

$$\rho_j^{(n)}(s) = \rho_n(s, s') \tag{3.6}$$

for $s \in S$ and $s' \in \sigma_j^{(n)}$ $(j = 1, \cdots, 2^n)$, that is, $\rho_j^{(n)}(s) = \epsilon_{ij}^{(n)}$ if $s \in \sigma_i^{(n)}$, then the functions $\rho_j^{(n)}(s)$ $(j = 1, \cdots, 2^n)$ form an ortho-normal set in $L^2(S;\mu)$. Consequently, by Bessel's inequality,

$$\begin{aligned} &\int_S \left| \int_S \rho_n(s, s') y(s) \mu(ds) \right|^2 \mu(ds') \\ &= \frac{1}{2^n} \sum_{j=1}^{2^n} \left| \int_S \rho_j^{(n)}(s) y(s) \mu(ds) \right|^2 \\ &\le \frac{1}{2^n} \|y\|_2^2 \end{aligned} \tag{3.7}$$

for any $y(s) \in L^2(S;\mu)$. From this it follows that

$$(3.8)\quad \left| \int_S \int_S \rho_n(s,s')y(s)z(s')\mu(ds)\mu(ds') \right|^2$$

$$\leqq \left\{ \int_S \left| \int_S \rho_n(s,s')y(s)\mu(ds) \right| \cdot |z(s')|\,\mu(ds') \right\}^2$$

$$\leqq \int_S \left| \int_S \rho_n(s,s')y(s)\mu(ds) \right|^2 \mu(ds') \cdot \int_S |z(s')|^2\,\mu(ds')$$

$$\leqq \frac{1}{2^n} \cdot \|y\|_2^2 \cdot \|z\|_2^2$$

$$\leqq \frac{1}{2^n} \|y\|_\infty^2 \cdot \|z\|_\infty^2$$

for any $y(s)$, $z(s) \in C(S)$. From (3.8) it follows further that

$$(3.9)\quad \left| \int_S \int_S \rho_n(s,s')x(s,s')\mu(ds)\mu(ds') \right| \leqq \sqrt{\frac{1}{2^n}} \cdot |||x|||$$

for any $x(s,s') \in C(S) \otimes C(S')$. Since

$$\rho_n(s,s') \in C(S) \otimes C(S') \text{ and } (\rho_n(s,s'))^2 = 1$$

on $S \times S'$, we obtain, by setting $x(s,s') = \rho_n(s,s')$ in (3.9), that

$$(3.10)\qquad |||\rho_n||| \geq \sqrt{2^n} \qquad (n = 1, 2, \cdots),$$

and hence $\lim_{n\to\infty} |||\rho_n||| = \infty$.

4. Remarks. Let us consider the bounded linear operators T, T' defined on $C(S) \circledast C(S')$ by

$$(4.1)\qquad Tx(s,s') = f(s)x(s,s'),$$

$$(4.2)\qquad T'x(s,s') = f(s')x(s,s'),$$

where $f(s)$ is a continuous function defined on S by

$$(4.3)\qquad f(s) = 3\left\{ \frac{\epsilon_1(s)}{4} + \frac{\epsilon_2(s)}{4^2} + \cdots + \frac{\epsilon_n(s)}{4^n} + \cdots \right\}.$$

It is easy to see that T, T' are spectral operators of scalar type and are given by

$$T = \int_S f(s)E(ds), \tag{4.4}$$

$$T' = \int_S f(s')E'(ds'), \tag{4.5}$$

where $\{E(\sigma)\,|\,\sigma \in \mathfrak{B}\}$ and $\{E'(\sigma)\,|\,\sigma \in \mathfrak{B}\}$ are a commutative pair of uniformly bounded spectral measures defined in § 3.

It is possible to show that $T + T'$ is not a spectral operator of scalar type. In order to show this we first observe that the range S^{**} of $f(s) + f(s')$ on $S^* = S \times S'$ is a totally disconnected set. Let $\mathfrak{B}_0^*$ be the Boolean algebra of all open-closed subsets σ^* of S^* os the form:

$$\sigma^* = \{s^* = (s, s')\,|\,f(s) + f(s') \in \sigma^{**}\},$$

where σ^{**} is an open-closed subset of S^{**}. It suffices to show that the family of projections $\{F(\sigma^*)\,|\,\sigma^* \in \mathfrak{B}_0^*\}$ is not uniformly bounded.

For each n, let $\{\eta_i^{(n)}\,|\,i = 1, 2, \cdots\}$ be a sequence of period 2^n; thus

$$\eta_{i+2^n}^{(n)} = \eta_i^{(n)} \qquad (i = 1, 2, \cdots).$$

Further, let the sequence consist only of $+1$ and -1 such that $(\eta_i^{(n)}, \cdots, \eta_{i+n-1}^{(n)})$ runs through all 2^n different sequences of length n consisting of $+1$ and -1 as i runs through $1, \cdots, 2^n$. The existence of such a sequence was proved by N. G. de Bruijn [**1**]. Let us put

$$\pi_n(s^*) = \pi_n(s, s') = \eta_{i+j-1}^{(n)}, \tag{4.6}$$

if $s \in \sigma_i^{(n)}$, $s' \in \sigma_j^{(n)}$ $(i, j = 1, \cdots, 2^n)$. Then $\{\pi_n(s^*)\,|\,n = 1, 2, \cdots\}$ is a sequence of functions from $C(S) \otimes C(S')$ taking only the values $+1$ and -1 such that the set

$$\sigma_n^* = \{s^*\,|\,\pi_n(s^*) = +1\} \in \mathfrak{B}_0^* \qquad \text{for } n = 1, 2, \cdots.$$

Thus, by the same reason as in § 3, it suffices to show that

$$\lim_{n \to \infty} |||\pi_n||| = \infty.$$

Let us put

$$\pi_j^{(n)}(s) = \pi_n(s, s')$$

if $s' \in \sigma_j^{(n)}$. Then $\{\pi_j^{(n)}(s) \mid j = 1, \cdots, 2^n\}$ is a set of functions from $L^2(S;\mu)$ such that

$$\{\pi_i^{(n)}(s), \cdots, \pi_{i+n-1}^{(n)}(s)\}$$

is an orthonormal system for $i = 1, \cdots, 2^n - n + 1$. This follows from the fact that

$$j + 1 \leqq k \leqq j + n - 1$$

implies

$$(4.7) \qquad \int_S \pi_j^{(n)}(s)\, \pi_k^{(n)}(s)\, \mu(ds)$$

$$= \frac{1}{2^n} \sum_{i=1}^{2^n} \eta_{i+j-1}^{(n)} \cdot \eta_{i+k-1}^{(n)} = 0.$$

(The last equality holds because

$$\eta_{i+j-1}^{(n)} \cdot \eta_{i+k-1}^{(n)} = +1$$

happens 2^{n-1} times and

$$\eta_{i+j-1}^{(n)} \cdot \eta_{i+k-1}^{(n)} = -1$$

happens 2^{n-1} times as i runs through $1, \cdots, 2^n$.)

Thus, for any $y \in L^2(S;\mu)$, Bessel's inequality

$$(4.8) \qquad \sum_{j=i}^{i+n-1} \left| \int_S \pi_j^{(n)}(s) y(s) \mu(ds) \right|^2 \leqq \|y\|_2^2$$

holds for $i = 1, \cdots, 2^n - n + 1$, and hence

$$(4.9)\qquad \int_S \left| \int_S \pi_n(s, s')y(s)\mu(ds)\right|^2 \mu(ds')$$

$$= \frac{1}{2^n} \sum_{j=1}^{2^n} \left| \int_S \pi_j^{(n)}(s)y(s)\mu(ds)\right|^2$$

$$\leqq \frac{1}{2^n}\left(\left[\frac{2^n}{n}\right] + 1\right) \|y\|_2^2$$

$$\leqq \left(\frac{1}{n} + \frac{1}{2^n}\right) \|y\|_2^2 \leqq \frac{2}{n} \|y\|_2^2.$$

From this follows, exactly as in § 3, that

$$(4.10)\qquad \left| \int_S \int_S \pi_n(s, s')x(s,s')\mu(ds)\mu(ds')\right| \leqq \sqrt{\frac{2}{n}}\ |||x|||$$

for any $x(s, s') \in C(S) \otimes C(S')$, and hence

$$(4.11)\qquad |||\pi_n||| \geq \sqrt{\frac{n}{2}}$$

for $n = 1, 2, \cdots$.

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UNBOUNDED SPECTRAL OPERATORS

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1. Introduction. Our purpose in the present paper is to study the structure and operational calculus of unbounded spectral operators. Bounded spectral operators have been introduced and studied by N. Dunford in [2] and [3], and the present paper is an investigation in the unbounded case of certain of the results of [3]. Interest in the abstract theory of unbounded spectral operators arises from important results of J. Schwartz [7], who has shown that the members of a large class of differential operators on a finite interval determine unbounded spectral operators in Hilbert space.

Let $\mathfrak{B}$ denote the Borel subsets of the complex plane, and let $\mathfrak{X}$ be a complex Banach space. We shall call a mapping E from $\mathfrak{B}$ to projection operators in $\mathfrak{X}$ a *resolution of the identity* if it is a homomorphism. That is,

$$E(e)E(f) = E(ef), \qquad E(e) \cup E(f) = E(e \cup f), \qquad e, f \in \mathfrak{B}$$

$$E(e') = I - E(e), \qquad E(\phi) = 0, \qquad E(p) = I, \qquad e \in \mathfrak{B};$$

$E(e)$ is bounded,

$$|E(e)| \leq M, \qquad e \in \mathfrak{B};$$

and[1] the vector-valued set function $E(e)x$ is countably additive. Here ϕ is the void set, p the plane, and e' the complement of e in p.

A closed operator T will be called a *spectral operator* if there is a resolution of the identity E such that:

(1) The domain $D(T)$ of T contains the dense subspace $\mathfrak{X}_0 = \{x \mid x = E(\sigma)x, \sigma \in \mathfrak{B}, \sigma \text{ bounded}\}$.

(2) If $\sigma \in \mathfrak{B}$, $E(\sigma)D(T) \subset D(T)$ and $E(\sigma)Tx = TE(\sigma)x$, $x \in D(T)$.

[1] The last condition is somewhat more restrictive than in [3].

Received March 4, 1953. This paper was prepared under Office of Naval Research contract number onr 609(04). The author is grateful to Professor Dunford for suggesting this investigation.

Pacific J. Math. 4 (1954), 373-392

(3) $\sigma(T, E(\sigma)\mathfrak{X}) \subset \bar{\sigma}$ where $\sigma(T, E(\sigma)\mathfrak{X})$ is the spectrum of T in the subspace $E(\sigma)\mathfrak{X}$.

If σ is a bounded set, then T is a bounded spectral operator in the subspace $E(\sigma)\mathfrak{X}$, and in this subspace its structure and operational calculus are known from [3]. The idea of the present paper is to determine the properties of T in $\mathfrak{X}$ from those of the sequence of approximating bounded spectral operators $TE(\sigma_n)$, where $\{\sigma_n\}$ is an increasing sequence of bounded sets for which

$$E\left(\bigcup_{n=1}^{\infty} \sigma_n\right) = I.$$

We outline briefly the main results:

The simplest type of spectral operator S is that of *scalar type*:

$$Sx = \lim_{n \to \infty} \int_{e_n} \lambda E(d\lambda)x,$$

where this limit exists and

$$e_n = \{\lambda \mid |\lambda| \leq n\}.$$

With each spectral operator T we can construct an associated scalar type operator S from its resolution of the identity. One of the principal results of the bounded case is the characterization theorem [3, Theorem 8] that T is a bounded spectral operator if and only if $T = S + N$, where S is a bounded scalar type operator and N is a generalized nilpotent operator commuting with S. In the unbounded case the relation of T to S is not so simple, as we shall show by examples. The operator $N = T - S$ (with suitably defined domain) may be bounded but not a generalized nilpotent or even unbounded with spectrum covering the plane. We give a sufficient condition (Theorem 4.1) that $T = S + N$ shall be a spectral operator.

If S is a spectral operator of scalar type, it has an operational calculus exactly analogous to that of an unbounded normal operator in Hilbert space (which is an example of a spectral operator). To each Borel measurable function f on $\sigma(S)$ we can assign a densely defined closed operator $f(S)$ which is also a spectral operator of scalar type, the operators corresponding to f and $|f|$ having the same domain. In case $T = S + N$ is a general spectral operator we can, by the formula

$$f(T)x = \lim_{p\to\infty} \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{e_p} f^{(n)}(\lambda)E(d\lambda)x,$$

assign a densely defined operator $f(T)$ to each function analytic and single-valued in the complement of a set θ for which $E(\theta) = 0$. (Here $\{e_n\}$ is an increasing sequence of compact sets on each of which f is analytic and with $E(\bigcup_{n=1}^{\infty} e_n) = I$.) However, as we shall show by an example, this operator need not be a spectral operator without other restrictions. If f is a rational function, $f(T)$ is always a spectral operator. Conditions are given to ensure that $f(T)$ is bounded. A result of the calculus is the theorem that a closed operator T with nonempty resolvent set is a spectral operator if and only if $(\lambda I - T)^{-1}$ is a bounded spectral operator for some $\lambda \notin \sigma(T)$. In case T is of the form $T = S + N$, where N is a generalized nilpotent, we obtain quite an extensive operational calculus of spectral operators. In order that $f(T)$ shall be a spectral operator it is sufficient that the singularities of $f(\lambda)$ in the finite plane (with the possible exception of a finite set of poles on $\sigma(T)$) shall not get arbitrarily close to $\sigma(T)$.

2. **Closed extensions.** In this first section we establish the existence of a closed extension of certain densely defined operators. This result will be the main tool of the paper and it will be convenient to formulate it under rather general conditions. We shall suppose throughout this section the existence of a resolution of the identity E.

DEFINITION 2.1. Let Q be an operator defined on a dense subspace $D_0(Q)$ of $\mathfrak{X}$. Let there be associated with Q a class $\mathfrak{A}$ of Borel sets satisfying:

(a) $\mathfrak{A}$ is closed under finite unions and contains any Borel subset of one of its members;

(b) If $e \in \mathfrak{A}$, then $E(e)\mathfrak{X} \subseteq D_0(Q)$ and Q is bounded in $E(e)\mathfrak{X}$;

(c) $E(e)QE(e) = QE(e)$, $e \in \mathfrak{A}$;

(d) $\mathfrak{A}$ contains an increasing sequence $\{e_n\}$ such that $E(\bigcup_{n=1}^{\infty} e_n) = I$.

Under these conditions we say Q satisfies *condition* (α) and write

$$\mathfrak{X}_{\mathfrak{A}} = \{x \mid x = E(e)x \text{ for some } e \in \mathfrak{A}\}.$$

An important case occurs when $\mathfrak{A}$ consists of all bounded Borel sets. We shall be interested in finding a particular closed extension of Q. The construction will be based on two lemmas.

LEMMA 2.1. *Let* $\{d_n\}$ *and* $\{e_n\}$ *be two increasing sequences of sets from* $\mathfrak{A}(Q)$ *for which*

$$E\left(\bigcup_{n=1}^{\infty} d_n\right) = E\left(\bigcup_{n=1}^{\infty} e_n\right) = I.$$

If $x \in \mathfrak{X}$, *and* $\lim_{n \to \infty} QE(d_n)x$ *exists, then*

$$\lim_{n \to \infty} QE(e_n)x = \lim_{n \to \infty} QE(d_n)x.$$

Proof. Given $\epsilon > 0$, let m_0 be chosen so that if $m > m_0$ then

$$|QE(d_m - d_{m_0})x| < \frac{\epsilon}{3M}.$$

Now, as $E(\bigcup_{m=0}^{\infty} e_n) = I$ and Q is bounded in $E(d_{m_0})\mathfrak{X}$, we can find an n_0 such that, if $n > n_0$,

$$|QE(d_{m_0} - e_n)x| < \frac{\epsilon}{3}.$$

For any such fixed $n > n_0$ we can, for the same reasons, find an $m_1 > m_0$ so that

$$|QE(e_n - d_{m_1})x| < \frac{\epsilon}{3}.$$

Now, since

$$E(e_n) - E(d_{m_0}) = E(e_n - d_{m_1}) + E(e_n)E(d_{m_1} - d_{m_0}) - E(d_{m_0} - e_n),$$

it follows that

$$|QE(e_n)x - QE(d_{m_0})x| < \epsilon.$$

DEFINITION 2.2. Let $\{e_n\}$ be any increasing sequence of sets from $\mathfrak{A}(Q)$ for which $E(\bigcup_{m=1}^{\infty} e_n) = I$. We define

$$D(Q) = \{x \mid \lim_{n \to \infty} QE(e_n)x \text{ exists}\},$$

and set $Qx = \lim_{n \to \infty} QE(e_n)x$ for $x \in D(Q)$.

LEMMA 2.2. *The operator Q with domain $D(Q)$ is closed and is the minimal closed extension of Q on $\mathfrak{X}_{\mathfrak{A}}$. Further, if $x \in D(Q)$, and $e \in \mathfrak{B}$, then $E(e)x \in D(Q)$ and $E(e)Qx = QE(e)x$. Also, Q, with domain $E(e)D(Q)$, is the minimal closed extension in $E(e)\mathfrak{X}$ of Q on $\mathfrak{X}_{\mathfrak{A}_1}$, $\mathfrak{A}_1 = \{ e\sigma \mid \sigma \in \mathfrak{A} \}$.*

Proof. Clearly, first, if $e \in \mathfrak{A}(Q)$ and $x \in D(Q)$, then $QE(e)x = E(e)Qx$ since we can suppose e a member of the sequence $\{e_n\}$. Now let $x_n \in D(Q)$ $(n = 1, 2, \cdots)$ and

$$x_0 = \lim_{n \to \infty} x_n, \quad y_0 = \lim_{n \to \infty} Qx_n.$$

For any m,

$$E(e_m)y_0 = \lim_{n \to \infty} E(e_m)Qx_n,$$

and

$$QE(e_m)x_0 = \lim_{n \to \infty} QE(e_m)x_n$$

as Q is bounded in $E(e_m)\mathfrak{X}$. But since

$$QE(e_m)x_n = E(e_m)Qx_n,$$

we have

$$\lim_{n \to \infty} QE(e_m)x_0 = \lim_{n \to \infty} E(e_m)y_0 = y_0.$$

Thus $x_0 \in D(Q)$ and $Qx_0 = y_0$. Clearly the extension is minimal. Finally let $x \in D(Q)$, $e \in \mathfrak{B}$. Then

$$E(e)x = \lim_{n \to \infty} E(ee_n)x$$

and

$$QE(ee_n) = E(e)QE(e_n)x$$

converges to $E(e)Qx$. The last statement follows easily.

We will also need:

LEMMA 2.3. *Let $\{e_n\}$ be an increasing sequence of sets from $\mathfrak{A}$ for which*

$$E\left(\bigcup_{n=1}^{\infty} e_n\right) = I.$$

If, for each n, $\lambda \in \rho(Q, E(e_n)\mathfrak{X})$ *and*

$$\lim_{n\to\infty} (\lambda I - Q)^{-1} E(e_n)x$$

exists for each $x \in \mathfrak{X}$, *then* $\lambda \in \rho(Q)$.

Proof. Clearly $\lambda I - Q$ is a closed one-to-one mapping of

$$D(\lambda I - Q) = D(Q)$$

into $\mathfrak{X}$. We must show it is onto. Let $x \in \mathfrak{X}$ and

$$y_n = (\lambda I - Q)^{-1} E(e_n)x.$$

Then $\lim_{n\to\infty} y_n = y$ exists by hypothesis, and

$$\lim_{n\to\infty} (\lambda I - Q)y_n = \lim_{n\to\infty} E(e_n)x = x.$$

Hence $y \in D(Q)$ and $(\lambda I - Q)y = x$.

We note that if T is a spectral operator and T_0 is the closed operator obtained by taking for $\mathfrak{A}$ the class of bounded Borel sets and defining $Qx = Tx$, $x \in \mathfrak{X}_{\mathfrak{A}}$, then $T = T_0$. *Thus a spectral operator has no proper closed extension which is a spectral operator.*

3. Scalar type spectral operators. We begin by studying the simplest type of spectral operators, those which can be constructed from a resolution of the identity E by integrating scalar functions. The integral we use for bounded functions over bounded sets is that introduced by Dunford [3, Lemma 6]. We particularly recall the relations

$$(3.1) \qquad \frac{1}{v(E)} \inf_{\lambda\in e} |f(\lambda)| \le \left| \int_e f(\lambda)E(d\lambda) \right| \le v(E) \sup_{\lambda\in e} |f(\lambda)|$$

and

$$(3.2) \qquad \int_e f(\lambda)g(\lambda)E(d\lambda) = \int_e f(\lambda)E(d\lambda) \int_e g(\mu)E(d\mu),$$

where e is a bounded Borel set, $v(E) = 4M$, and f and g are bounded Borel measurable functions.[1] We denote by $\mathfrak{M}$ the set of Borel measurable functions f each of which is finite-valued in the complement of a set ϕ_f for which $E(\phi_f) = 0$.

If $f \in \mathfrak{M}$, we let $\mathfrak{A}$ be the class of bounded Borel sets on which $|f(\lambda)|$ is bounded and take

$$e_n = \{\lambda \mid |\lambda| \le n, \ |f(\lambda)| \le n\} \qquad (n = 1, 2, \cdots).$$

We define

$$f(S)x = \lim_{n \to \infty} \int_{e_n} f(\lambda) E(d\lambda)x$$

on the set $D(f(S))$ of x for which this limit exists. Lemma 2.2 shows that $f(S)$ is a closed operator, and Lemma 2.1 that we would have obtained the same result by using any other increasing sequence $\{\sigma_n\}$ from $\mathfrak{A}$ for which

$$E\left(\bigcup_{n=1}^{\infty} \sigma_n\right) = I.$$

We shall denote by S the operator obtained by taking $f(\lambda) = \lambda$ and call it the *scalar operator associated with* E (or if E is the resolution of the identity of a spectral operator T, we call S the *scalar operator associated with* T). Now S is a generalization of an unbounded normal operator in Hilbert space.[2] The method we have used to construct the operators $f(S)$ is an extension of the method of forming direct sums of Hilbert spaces (see [6, p. 43]).

THEOREM 3.1. *Concerning the operator* $f(S)$ *we have*:

(1) *if* $f \in \mathfrak{M}$, *then* $D(f(S)) = D(|f|(S))$;

(2) *if* $f, g \in \mathfrak{M}$ *and* $|f(\lambda)| \le K|g(\lambda)|$, *then* $D(g(S)) \subset D(f(S))$;

(3) $g(S)$ *is bounded if and only if* g *is essentially bounded with respect to* $\{E(e)\}$;

(4) *if* $f \in \mathfrak{M}$ *and* $g(S)$ *is bounded, then* $g(S)D(f(S)) \subset D(f(S))$.

Proof. We note that (3) follows from formula (3.1). To prove (1), let

[1] The first half of (3.1) does not appear explicitly in [3] but follows from the second half and (3.2).

[2] "Maximal normal operator" in the terminology of Stone [8].

$\epsilon > 0$ be given, and let

$$\mu = \{\lambda \mid |f(\lambda)| < \epsilon\}.$$

We define $s(\lambda)$ to be $|f(\lambda)|\,[f(\lambda)]^{-1}$ for $\lambda \notin \mu$, and zero for $\lambda \in \mu$. Then if $x \in D(f(S))$, for any n we have

$$\int_{e_n} |f(\lambda)| E(d\lambda)x = s(S) \int_{e_n - \mu} f(\lambda) E(d\lambda)x + \int_{e_n \mu} |f(\lambda)| E(d\lambda)x.$$

But $|s(S)| \leq v(E)$, and the last term is in norm not greater than $\epsilon\, v(E)$. It follows that the sequence

$$\left\{\int_{e_n} |f(\lambda)| E(d\lambda)x\right\}$$

is a Cauchy sequence if

$$\left\{\int_{e_n} f(\lambda) E(d\lambda)x\right\}$$

also is one. Thus $D(f(S)) \subset D(|f|(S))$. The converse inclusion and (2) are proved similarly. Finally (4) follows from (3.2), since

$$\int_{e_n} f(\lambda) E(d\lambda) g(S)x = \int_{e_n} f(\lambda) g(\lambda) E(d\lambda)x = g(S) \int_{e_n} f(\lambda) E(d\lambda)x.$$

THEOREM 3.2. *Let f and $g \in \mathfrak{M}$.*

(1) *If $x \in D(f(S)) \cap D(g(S))$, then $x \in D((f+g)(S))$ and $[f(S) + g(S)]x = (f+g)(S)x$.*

(2) *If $x \in D(g(S))$ and $g(S)x \in D(f(S))$, then $x \in D((fg)(S))$ and $f(S)g(S)x = (fg)(S)x$.*

Proof. (1) is clear. For (2), let $\mathfrak{A}$ consist of the bounded Borel sets on which both $f(\lambda)$ and $g(\lambda)$ are bounded, and let

$$e_n = \{\lambda \mid |f(\lambda)|,\ |g(\lambda)| \text{ and } |\lambda| \leq n\}.$$

Then, for any n,

$$\int_{e_n} f(\lambda)E(d\lambda)g(S)x = \lim_{m\to\infty} \int_{e_n} f(\lambda)E(d\lambda) \int_{e_m} g(\mu)E(d\mu)$$

$$= \int_{e_n} f(\lambda)g(\lambda)E(d\lambda)x,$$

since $\int_{e_n} f(\lambda)E(d\lambda)$ is a bounded operator. Thus $f(S)g(S)x = (fg)(S)x$.

For the next theorem we will need a lemma which it will be convenient later to have formulated for a general spectral operator.

LEMMA 3.1. *If T is a spectral operator $E(\sigma(T)) = I$, and if $\{e_n\}$ is an increasing sequence of bounded Borel sets for which*

$$E\left(\bigcup_{n=1}^{\infty} e_n\right) = I,$$

then

$$\sigma(T) = \overline{\bigcup_{n=1}^{\infty} \sigma(T, E(e_n)\mathfrak{X})}.$$

Proof. The argument follows that of [3, Theorem 1]. Let

$$\mu = \overline{\bigcup_{n=1}^{\infty} \sigma(T, E(e_n)\mathfrak{X})}.$$

Clearly $\mu \subset \sigma(T)$. If σ is a closed subset of μ', then, for each n, $\sigma(T, E(\sigma e_n)\mathfrak{X})$ is a subset of both σ and $\sigma(T, E(e_n)\mathfrak{X})$. Thus

$$E(\sigma e_n) = 0, \quad E(\sigma) = 0, \quad \text{and} \quad E(\mu') = 0.$$

Hence $E(\mu) = I$ and $\mu = \sigma(T)$.

THEOREM 3.3. *If $f \in \mathfrak{M}$, then $f(S)$ is a spectral operator whose resolution of the identity is given by*

$$E_f(e) = E(f^{-1}(e)),$$

and spectrum by

$$\sigma(f(S)) = \bigcap_{E(e)=I} \overline{f(e)}.$$

Proof. Let σ be a fixed Borel set. If $\lambda_0 \notin \overline{\sigma}$ then

$$g(\lambda) = (\lambda_0 - f(\lambda))^{-1} \psi_{f^{-1}(\sigma)}$$

is bounded, and the equations

$$g(S)(\lambda_0 I - f(S))x = x, \quad x \in E(\sigma)D(f(S)),$$

$$(\lambda_0 I - f(S))g(S)x = x, \quad x \in E(\sigma)\mathfrak{X},$$

show $\lambda_0 I - f(S)$ is a closed one-to-one map of $E(\sigma)D(f(S))$ onto $E(\sigma)\mathfrak{X}$. Thus $\sigma(f(S), E_f(\sigma)\mathfrak{X}) \subset \overline{\sigma}$.

Now let

$$e_n = \{\lambda \mid |\lambda| \leq n, \ |f(\lambda)| \leq n\}.$$

By [**3**, Theorem 16],

$$\sigma(f(S), E(e_n)\mathfrak{X}) = \bigcap_{E(e)=E(e_n)} \overline{f(e)} = \mu_n.$$

Now, by Lemma 3.1,

$$\sigma(f(S)) = \overline{\bigcup_{n=1}^{\infty} \mu_n}.$$

Let

$$\mu = \bigcap_{E(e)=I} \overline{f(e)}.$$

Clearly $\mu_n \subset \mu$ for each n. If

$$\lambda \notin \overline{\bigcup_{n=1}^{\infty} \mu_n},$$

we can pick a $\delta > 0$ and for each n a Borel set $\sigma_n \subset e_n$ such that

$$E(\sigma_n) = E(e_n) \text{ and } \operatorname{dist.}(\lambda, \overline{f(\sigma_n)}) > \delta.$$

Now if

$$\sigma_0 = \bigcup_{n=1}^{\infty} \sigma_n ,$$

then $E(\sigma_0) = I$ and $\lambda \notin \overline{f(\sigma_0)}$, and thus $\lambda \notin \mu$. Hence

$$\mu = \overline{\bigcup_{n=1}^{\infty} \mu_n} = \sigma(f(S)).$$

4. The relation of T to its scalar operator. One of Dunford's principal results for bounded spectral operators is the characterization theorem [3, Theorem 8] that T is a bounded spectral operator if and only if $T = S + N$, where

$$S = \int \lambda E(d\lambda)$$

is the associated scalar type operator and N is a generalized nilpotent operator commuting with T. The absence of such a theorem in the unbounded case greatly complicates the theory. While in each subspace $E(\sigma)\mathfrak{X}$, σ bounded, $N = T - S$ will be a generalized nilpotent, the natural closed extension provided by Lemma 2.2 of N on $\mathfrak{X}_{\mathfrak{A}}$, ($\mathfrak{A}$ the class of bounded Borel sets) may be bounded but not a generalized nilpotent, or even unbounded. We now construct two examples which exhibit these possibilities.

EXAMPLE 1. For each n, let $\mathfrak{H}_n$ be n-dimensional unitary space and let $\mathfrak{H}$ be the space of sequences $\{x_n\}$, where

$$x_n = (\xi_{1n}, \xi_{2n}, \cdots, \xi_{nn}) \in \mathfrak{H}_n ,$$

$$|x_n| = \left(\sum_{i=1}^{n} |\xi_{in}|^2\right)^{1/2}, \quad |x| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}.$$

Then $\mathfrak{H}$ is a Hilbert space. We denote by $E(n)$ the orthogonal projection mapping $\mathfrak{H}$ onto $\mathfrak{H}_n$. The Boolean algebra E of projections

$$E(\sigma) = \sum_{n \in \sigma} E(n),$$

where σ is any subset of the positive integers, is a resolution of the identity of the self-adjoint operator S which we define in $\mathfrak{H}_n$ by

$$Sx_n = (n\xi_{1n}, \cdots, n\xi_{nn})$$

and extend by Lemma 2.2 to

$$D(S) = \left\{ x \mid \sum_{n=1}^{\infty} |Sx_n|^2 < \infty \right\}.$$

The operator N we define in $\mathfrak{H}_n$ by

$$Nx_n = (0, n\xi_{1n}, n\xi_{2n}, \cdots, n\xi_{n-1\, n}).$$

The extension to $\mathfrak{H}$ yields an operator of norm one which is nilpotent of order n on $\mathfrak{H}_n$. We shall show that the operator

$$T = S + N, \qquad (D(T) = D(S))$$

is a spectral operator. Let σ be any subset of the positive integers and $\alpha \notin \sigma$. If $n \in \sigma$, the operator

$$R_\alpha(T, \mathfrak{H}_n) = \sum_{i=0}^{n-1} \frac{N^i E(n)}{(\alpha - n)^{i+1}}$$

is the resolvent operator of T in the subspace $\mathfrak{H}_n$. Because of the quadratic nature of the norm in Hilbert space, α will be in the resolvent set of T in $E(\sigma)\mathfrak{H}$ if and only if $|R_\alpha(T, \mathfrak{H}_n)|$ is uniformly bounded for all n in σ. But this is satisfied; in fact,

$$\lim_{n \to \infty} |R_\alpha(T, \mathfrak{H}_n)| = 0,$$

where n is not restricted to σ. For, given $1 > \epsilon > 0$, we can pick an n_0 so large that

$$|\alpha - n|^{-1} < \frac{\epsilon}{2} \qquad \text{for } n > n_0.$$

Then, if $n > n_0$,

$$|R_\alpha(T, \mathfrak{H}_n)| \le \sum_{i=0}^{n-1} \frac{1}{|\alpha - n|^{i+1}} < \epsilon.$$

Thus $\sigma(T, E(\sigma)\mathfrak{H}) \subset \sigma$, and T is a spectral operator. To show that N is not a generalized nilpotent, let $x = \{x_i\}$, where

$$x_i = (2^{-i/2}, 0, 0, \cdots, 0).$$

Then $|x| = 1$, but

$$|N^n x|^{1/n} = \frac{1}{\sqrt{2}}.$$

The transformation N is of a type studied by H. Hamburger [**4**].

EXAMPLE 2. In this case let $\mathfrak{H}_n$ be two-dimensional unitary space for each n, and form $\mathfrak{H}$ as the Hilbert space of sequences $\{x_n\}$ with $x_n = (\xi_{1n}, \xi_{2n}) \in \mathfrak{H}_n$ as before. In $\mathfrak{H}_n$ we define

$$Sx_n = (n\xi_{1n}, n\xi_{2n}),$$

$$Nx_n = (0, n\xi_{1n}),$$

and $T = S + N$. Then

$$D(T) = \left\{ x \mid \sum_{n=1}^{\infty} |Tx_n|^2 < \infty \right\},$$

with similar expressions for $D(S)$ and $D(N)$. As $D(S) \subset D(N)$, we have $D(T) = D(S)$. Now N has the entire plane as its spectrum since, clearly, $0 \in \sigma(N)$, and, if $\beta \neq 0$, the formula

$$R_\beta(N, \mathfrak{H}_n)x_n = \left(\frac{\xi_{1n}}{\beta^2}, \frac{n\xi_{1n}}{\beta^2} + \frac{\xi_{2n}}{\beta} \right)$$

shows that $|R_\beta(N, \mathfrak{H}_n)|$ is unbounded with n. However, T is a spectral operator. If σ is a set of integers and $\alpha \notin \sigma$ then, for $n \in \sigma$,

$$R_\alpha(T, \mathfrak{H}_n)x_n = \left(\frac{\xi_{1n}}{\alpha - n}, \frac{n\xi_{1n}}{(\alpha - n)^2} + \frac{\xi_{2n}}{(\alpha - n)} \right)$$

Thus $|R_\alpha(T, \mathfrak{H}_n)|$ is bounded, $n \in \sigma$.

The last example shows the degree of pathology that may arise. It is interesting that we do have the following result which covers the case of Example 1.

THEOREM 4.1. *Let S be an unbounded scalar type operator, and let N be a bounded operator which commutes with the resolution of the identity for S and is a generalized nilpotent on each of the subspaces $E(\sigma)\mathfrak{X}$, σ bounded. Then $T = S + N$ is a spectral operator with the same resolution of the identity.*

Proof. The relation $\sigma(T, E(\sigma)\mathfrak{X}) \subset \bar{\sigma}$ is clearly satisfied for all bounded Borel sets. Let σ be an unbounded Borel set and let

$$e_n = \{\lambda \mid |\lambda| \leq n\}.$$

By [3, Lemma 3], the resolvent of T in $E(\sigma e_n)\mathfrak{X}$ is given by

$$(\lambda I - T)^{-1} = \sum_{i=0}^{\infty} N^i \int_{\sigma e_n} \frac{E(d\mu)}{(\lambda - \mu)^{i+1}}.$$

We conclude the proof by showing that

$$\lim_{n \to \infty} (\lambda I - T)^{-1} E(\sigma e_n) x$$

exists for each $x \in E(\sigma)\mathfrak{X}$ and applying Lemma 2.3 in that subspace. We show in fact that the series

$$\sum_{i=0}^{\infty} N^i \int_{\sigma} \frac{E(d\mu)}{(\lambda - \mu)^{i+1}}$$

converges. For given $0 < \epsilon < 1$, we may pick n_0 so large that

$$2|N| < \epsilon \operatorname{dist}(\lambda, e'_{n_0}), \quad 2v(E) < \operatorname{dist}(\lambda, e'_{n_0}),$$

and pick an $n_1 > n_0$ such that for any m and n with $m > n > n_1$,

$$\left| \sum_{i=n}^{m} N^i \int_{\sigma e_{n_0}} \frac{E(d\mu)}{(\lambda - \mu)^{i+1}} \right| < \frac{\epsilon}{2}.$$

Then, using (3.1), we get

$$\left| \sum_{i=n}^{m} N^i \int_\sigma \frac{E(d\mu)}{(\lambda-\mu)^{i+1}} \right| < \frac{\epsilon}{2} + \sum_{i=n}^{m} |N|^i \left| \int_{\sigma e'_{n_0}} \frac{E(d\mu)}{(\lambda-\mu)^{i+1}} \right|$$

$$< \frac{\epsilon}{2} + \frac{v(E)}{\operatorname{dist}(\lambda, e'_{n_0})} \sum_{i=n}^{m} \frac{\epsilon^i}{2^i} < \epsilon.$$

5. Operational calculus for a general spectral operator. When T is a bounded operator and f is a function analytic on $\sigma(T)$, it is well known [**1**; **9**] that a comprehensive operational calculus is obtained by defining

$$f(T) = \frac{1}{2\pi i} \int_C f(\lambda)(\lambda I - T)^{-1} d\lambda, \tag{5.1}$$

where C is a bounded positively oriented contour containing $\sigma(T)$ and excluding the singularities of f. Also,

$$\sigma(f(T)) = f(\sigma(T)). \tag{5.2}$$

Moreover, in the case that $T(=S+N)$ is a bounded spectral operator, Dunford has shown [3, Theorem 9] that the operator $f(T)$ may be expressed in terms of the values of f and its derivatives on $\sigma(T)$ by the formula

$$f(T) = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{\sigma(T)} f^{(n)}(\lambda) E(d\lambda), \tag{5.3}$$

the series converging absolutely in the uniform operator topology. We shall make formula (5.3) the basis of an operational calculus in the unbounded case.

Given an unbounded spectral operator T, we denote by $\mathfrak{R}$ the class of functions f each analytic and single-valued in the complement of a closed set θ_f for which $E(\theta_f) = 0$. If for $f \in \mathfrak{R}$ we take

$$e_n = \left\{ \lambda \mid |\lambda| \le n, \ \operatorname{dist}(\lambda, \theta_f) \ge \frac{1}{n} \right\},$$

then $\{e_n\}$ is an increasing sequence of closed sets for which

$$E\left(\bigcup_{n=1}^{\infty} e_n \right) = I,$$

and on each of which f is analytic. Moreover, $T = S + N$ is a bounded spectral operator in $E(e_n)\mathfrak{X}$. Defining

$$f(T)x = \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{N^i}{i!} \int_{e_n} f^{(i)}(\lambda)E(d\lambda)x$$

on the set $D(f(T))$ of x for which this limit exists, we obtain via Lemma 2.2 a closed densely defined operator. The class $\mathfrak{R}$ is closed under sums and products, and by an argument exactly analogous to that of Theorem 3.2 we obtain:

THEOREM 5.1. *Let f and $g \in \mathfrak{R}$.*

(1) *If $x \in D(f(T)) \cap D(g(T))$, then $x \in D((f+g)(T))$ and $(f(T)+g(T))x = (f+g)(T)x$.*

(2) *If $x \in D(g(T))$ and $g(T)x \in D(f(T))$, then $x \in D((fg)(T))$ and $f(T)g(T)x = (fg)(T)x$.*

As we show now by an example, the operator $f(T)$ need not be a spectral operator. Let T be the operator of Example 2 whose spectrum is the set of positive integers. Taking

$$f(\lambda) = \sqrt{2}\ \operatorname{cosec} \pi\left(\lambda + \frac{1}{4}\right),$$

we see that the spectrum of $f(T)$ in $E(\sigma)\mathfrak{H}$ for σ any finite subset of $\sigma(T)$ is the range of $f(\lambda)$ on σ, that is, lies in the pair of points ± 1. By Lemma 3.1, this must be true also of the closed operator $f(T)$ on $D(f(T))$ if it is a spectral operator. However, $0 \in \sigma(f(T))$ since, for $x_n \in \mathfrak{H}_n$,

$$[f(T)]^{-1}x_n = \frac{(-1)^n}{2}(\xi_{1n}, n\pi\xi_{1n} + \xi_{2n}),$$

showing that the norm of $[f(T)]^{-1}$ in $\mathfrak{H}_n$ is unbounded with n. In fact, $\sigma(f(T))$ is the whole plane.

In connection with Example 1, it is worth noting that there are bounded operators which are spectral operators on each of an increasing sequence $E(e_n)\mathfrak{X}$ of subspaces for which

$$E\left(\bigcup_{n=1}^{\infty} e_n\right) = I,$$

without being spectral operators on $\mathfrak{X}$. Such an operator in the case of Example 1 is given by $S^{-1} + N$, where

$$e_n = \{p \mid 1 \leq p \leq n\}.$$

We now give conditions under which $f(T)$ is a spectral operator.

THEOREM 5.2. *Let T be a spectral operator, and let f be analytic on $\sigma(T)$ with the exception of a finite set $\theta = (p_1, p_2, \cdots, p_k)$ of poles for which $E(\theta) = 0$, and let f be either analytic at infinity or have a pole there. Then $f(T)$ is a spectral operator with resolution of the identity*

$$E_f(e) = E(f^{-1}(e)) \tag{5.3}$$

and spectrum

$$\sigma(f(T)) = \overline{f(\sigma(T))}. \tag{5.4}$$

For the proof we shall need the following lemma:

LEMMA 5.1. *Let f and T satisfy the conditions of Theorem 5.2. Then $\sigma(f(T)) \subset \overline{f(\sigma(T))}$.*

Proof. Clearly we can suppose that $\overline{f(\sigma(T))}$ is not the entire plane. Let $\lambda_0 \notin \overline{f(\sigma(T))}$, and define the function $g(\lambda)$ to be $[\lambda_0 - f(\lambda)]^{-1}$ where f is analytic and zero at the poles of f. Then g is analytic on $\sigma(T)$ and at infinity. To show that $g(T)$ is a bounded operator, we can suppose that $\sigma(T)$ is not the whole plane, since otherwise g is constant. Now A.E. Taylor [**10**] has shown that if T is a closed operator whose spectrum does not cover the plane, and g is a function analytic on $\sigma(T)$ and at infinity, then there is an unbounded Cauchy domain D such that $\sigma(T) \subset D$, $\bar{D}$ is contained in the domain of g, and an operational calculus is established by defining

$$g[T] = g(\infty)I + \int_K g(\lambda)(\lambda I - T)^{-1} d\lambda,$$

where K is the positively oriented bounded contour forming the boundary of D. The operator $g[T]$ is bounded, and, in the case T is bounded, $g[T] = g(T)$, the operator of (5.1). Now, recalling the equivalence of (5.1) and (5.3) when T is a bounded spectral operator, we let

$$e_n = \sigma(T) \cap \left\{\lambda \mid |\lambda| \leq n,\ \text{dist}(\lambda, \theta) \geq \frac{1}{n}\right\},$$

and note that

$$g[T] = \sum_{i=0}^{\infty} \frac{N^i}{i!} \int_{e_n} g^{(i)}(\lambda) E(d\lambda)$$

in $E(e_n)\mathfrak{X}$. Thus, in $\mathfrak{X}$,

$$g[T] = \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{N^i}{i!} \int_{e_n} g^{(i)}(\lambda) E(d\lambda) = g(T).$$

Moreover, $g(T) = [\lambda_0 I - f(T)]^{-1}$ in $E(e_n)\mathfrak{X}$. Thus, by Lemma 2.3, $\lambda_0 \notin \sigma(f(T))$.

Proof of Theorem 5.2. Let σ be a fixed Borel set. Then

$$\sigma(T, E(f^{-1}(\sigma))\mathfrak{X}) \subset \overline{f^{-1}(\sigma)}.$$

We now apply either (5.2) or the preceding lemma in the subspace $E(f^{-1}(\sigma))\mathfrak{X}$, depending on whether or not $f^{-1}(\sigma)$ is a bounded set, to conclude that

$$\sigma(f(T), E(f^{-1}(\sigma))\mathfrak{X}) \subset f(\overline{f^{-1}(\sigma)}) \subset \bar{\sigma}.$$

That $\sigma(f(T)) = \overline{f(\sigma(T))}$ follows from (5.2) and Lemma 3.1.

COROLLARY. *Any polynomial in a spectral operator is a spectral operator. A closed operator T is a spectral operator if and only if, for some $\lambda_0 \notin \sigma(T)$, $(\lambda_0 I - T)^{-1}$ is a bounded spectral operator.*

Proof. The first statement is clear, as is the necessity of the second. For the sufficiency we note that

$$T = f((\lambda_0 I - T)^{-1}), \qquad \text{where } f(\lambda) = \lambda_0 - \frac{1}{\lambda}.$$

If we restrict N to be a generalized nilpotent we obtain a broad operational calculus of spectral operators. All we need require of an analytic function f is that its singularities in the finite plane (with the exception of a finite set of poles as before) shall not be arbitrarily close to $\sigma(T)$.

THEOREM 5.3. *Let T be a spectral operator and $T = S + N$, where N is a generalized nilpotent. Let f be a function for which there exists a constant $r > 0$ such that f is analytic* (*with the possible exception of a finite set*

$\theta = (p_1, \cdots, p_k)$ *of poles for which* $E(\theta) = 0$) *in the open set*

$$\mu_f = \{\lambda \mid \text{dist}(\lambda, \sigma(T)) < r\}.$$

Then $f(T)$ *is a spectral operator whose resolution of the identity and spectrum are given by* (5.3) *and* (5.4). *The class of such functions is closed under sums and products. If* f *if bounded on* μ_f, *then* $f(T)$ *is bounded.*

The proof proceeds exactly as before once we have:

LEMMA 5.2. *If* f *satisfies the conditions of Theorem* 5.3, *then*

$$\sigma(f(T)) \subset \overline{f(\sigma(T))}.$$

Proof. Let f and r be given and $\lambda_0 \notin \overline{f(\sigma(T))}$. Again we define $g(\lambda)$ to be $(\lambda_0 - f(\lambda))^{-1}$ where f is analytic and zero at the poles of f. Then as $\lambda_0 \notin \overline{f(\sigma(T))}$ there is a constant $s > 0$ such that g is analytic and bounded in

$$\mu_g = \{\lambda \mid \text{dist}(\lambda, \sigma(T)) < 2s\}.$$

The formula

$$g^{(n)}(\lambda) = \frac{n!}{2\pi i} \int_C \frac{g(\mu)}{(\mu - \lambda)^{n+1}} \, d\mu, \qquad \lambda \in \sigma(T),$$

where C is a circle of radius s, shows that if $|g(\lambda)| < K$ on μ_g, then

$$\left| \frac{g^{(n)}(\lambda)}{n!} \right| < Ks^{-n}, \qquad \lambda \in \sigma(T).$$

Since

$$\lim_{n \to \infty} |N^n|^{1/n} = 0,$$

the series

$$g(T) = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{\sigma(T)} g^{(n)}(\lambda) E(d\lambda)$$

converges in the uniform operator topology. Moreover, if

$$e_n = \sigma(T) \cap \left\{\lambda \mid |\lambda| \leq n, \ \text{dist}(\lambda, \theta) \geq \frac{1}{n}\right\},$$

$g(T)$ is the resolvent of $f(T)$ on $E(e_n)\mathfrak{X}$. Application of Lemma 2.3 shows that $\lambda_0 \notin \sigma(f(T))$.

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WEAK AND STRONG LIMITS OF SPECTRAL OPERATORS

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The present paper is a contribution to the theory of spectral operators in Banach spaces developed by N. Dunford in [8] and [9]. A bounded operator S is a *spectral operator of scalar type* if, roughly speaking, it has a representation

$$S = \int_{\sigma(S)} \lambda E(d\lambda)$$

where $E(\cdot)$ is a resolution of the identity similar to that possessed by a normal operator in Hilbert space. The initial problem we are concerned with is to find conditions under which a weak or strong limit of scalar type spectral operators is again in this class. The results are then applied to the study of certain weakly closed algebras of spectral operators.

Section 1 contains a brief summary of definitions and results from [8] and [9]. In §2 conditions are found under which a strong limit of scalar type spectral operators is a scalar type spectral operator, the principal restriction imposed in the limiting operators being on the nature of their spectra. The operators need not commute.

Suppose that the underlying space $\mathfrak{X}$ is reflexive. If $\mathfrak{A}$ is an algebra generated by a bounded Boolean algebra $\mathfrak{B}$ of projections, then by a theorem of Dunford [9], each operator in $\mathfrak{A}$ is a scalar type spectral operator. We show (Theorem 4.1) that every operator in the weak closure $\mathfrak{W}$ of $\mathfrak{A}$ is a scalar type operator, and characterize $\mathfrak{W}$ as the algebra generated (in the uniform topology) by the strong closure of $\mathfrak{B}$. The principal tool used is the equivalence (due to Dunford [7]) of strong closure and lattice completeness for bounded Boolean algebras of projections. We give a new proof of this theorem.

The paper concludes with a characterization of the weakly closed algebra generated by a single scalar type spectral operator with real spectrum. Our proof of this theorem gives a more direct proof of the corresponding result of Segal [22] for Hilbert space.

Received February 8, 1954. The research contained in this paper was done under contract onr 609(04) with the Office of Naval Research.

Pacific J. Math. 4 (1954), 393-413

1. Preliminaries. In this section we collect certain definitions and results, principally taken from [9].

Two projections in a Banach space $\mathfrak{X}$ are said to be ordered in their *natural order*, $E_1 \leq E_2$ if $E_1 E_2 = E_2 E_1 = E_1$. This is equivalent to the conditions

$$E_1 \mathfrak{X} \subseteq E_2 \mathfrak{X} \text{ and } (I - E_1)\mathfrak{X} \supseteq (I - E_2)\mathfrak{X}.$$

The natural order partially orders the set of all projections in $\mathfrak{X}$, and any pair of commuting projections E_1 and E_2 has a least upper bound

$$E_1 \vee E_2 = E_1 + E_2 - E_1 E_2$$

and greatest lower bound

$$E_1 \wedge E_2 = E_1 E_2.$$

If $\{E_\alpha\}$ is an arbitrary set of projections and $\mathfrak{X}$ admits the direct sum decomposition $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$ where

$$\mathfrak{M} = \overline{\text{sp}}\{\cup_\alpha E_\alpha \mathfrak{X}\}, \quad \mathfrak{N} = \cap_\alpha (I - E_\alpha)\ \mathfrak{X},$$

then the projection with range $\mathfrak{M}$ defined by this decomposition is denoted by $\vee_\alpha E_\alpha$ and is the least upper bound of the set $\{E_\alpha\}$. Correspondingly the greatest lower bound $\wedge_\alpha E_\alpha$ with range $\mathfrak{M}_1$ is defined by the decomposition $\mathfrak{X} = \mathfrak{M}_1 \oplus \mathfrak{N}_1$ where

$$\mathfrak{M}_1 = \cap_\alpha E_\alpha \mathfrak{X}, \quad \mathfrak{N}_1 = \overline{\text{sp}}\{\cup_\alpha (I - E_\alpha)\mathfrak{X}\},$$

if it exists.

Throughout much of this paper we will be concerned with Boolean algebras of projections; that is, sets of commuting projections containing 0 and the identity I which are Boolean algebras under the operations $E_1 \vee E_2$ and $E_1 \wedge E_2$. A Boolean algebra $\mathfrak{B}$ of projections is *bounded* if there is a constant M such that $|E| \leq M$ for $E \in \mathfrak{B}$. $\mathfrak{B}$ is *complete* if it contains $\vee_\alpha E_\alpha$ and $\wedge_\alpha E_\alpha$ for every subset $\{E_\alpha\} \subseteq \mathfrak{B}$. We remark that $\mathfrak{B}$ may be complete *as a lattice* but not complete as a Boolean algebra of projections in $\mathfrak{X}$ in the present sense.

Let $\mathfrak{F}$ be a σ-field of subsets of a set Ω. A homomorphic map $E(\cdot)$ of $\mathfrak{F}$ onto a bounded Boolean algebra of projections in $\mathfrak{X}$ will be called a *spectral measure*. Thus

$$(1.1)\qquad \left.\begin{array}{l} E(\sigma\cup\delta)=E(\sigma)\vee E(\delta),\ \ E(\sigma\cap\delta)=E(\sigma)\wedge E(\delta), \\ E(\Omega)=I,\ \ E(\sigma')=I-E(\sigma),\ \ |E(\sigma)|\le M \end{array}\right\}\ \sigma,\delta\in\mathfrak{F}.$$

The set function $x^*E(\cdot)x$, $x\in\mathfrak{X}$, $x^*\in\mathfrak{X}^*$, satisfies (see proof of [**9**, Theorem 17])

$$(1.2)\qquad \operatorname{var} x^*E(\cdot)x \le 4M|x||x^*|.$$

The spectral measure $E(\cdot)$ is *countably additive* if $x^*E(\cdot)x$ is countably additive for each $x\in\mathfrak{X}$, $x^*\in\mathfrak{X}^*$. Countable additivity of $E(\cdot)$ implies that the vector valued set functions $E(\cdot)x$ are countably additive for $x\in\mathfrak{X}$ [**8**, page 579].

If $F(\cdot)$ is a spectral measure in the conjugate space $\mathfrak{X}^*$ of $\mathfrak{X}$ we say $F(\cdot)$ is $(\mathfrak{X})$-*countably additive* if $F(\cdot)x^*x$ is countably additive for all $x^*\in\mathfrak{X}^*$ and $x\in\mathfrak{X}$.

We will need a notion of integration of scalar functions with respect to a spectral measure [**9**, Lemma 6]. Let $E(\cdot)$ on $(\Omega,\mathfrak{F})$ be either a countably additive spectral measure in $\mathfrak{X}$ or an $(\mathfrak{X})$-countably additive spectral measure in $\mathfrak{X}^*$. Then for f an essentially bounded measurable function on Ω, the integral $\int_\Omega f(\omega)E(d\omega)$ is defined as the limit

$$(1.3)\qquad \int_\Omega f(\omega)E(d\omega)=\lim_{n\to\infty}\int_\Omega f_n(\omega)E(d\omega)$$

in the uniform operator topology, where the functions

$$f_n(\omega)=\sum \alpha_{in}\,k_{\sigma_{in}}(\omega),\qquad n=1,2,\cdots,$$

form a sequence of finite linear combinations of characteristic functions of disjoint sets $\sigma_{in}\in\mathfrak{F}$ converging uniformly to f on Ω and

$$\int_\Omega f_n(\omega)E(d\omega)=\sum \alpha_{in}\,E(\sigma_{in}).$$

This integral satisfies

$$(1.4)\qquad \frac{1}{4M}\operatorname{ess}_\Omega\inf|f(\omega)|\le\left|\int_\Omega f(\omega)E(d\omega)\right|\le 4M\operatorname{ess}_\Omega\sup|f(\omega)|.$$

A countably additive spectral measure on the Borel sets of the complex plane is called a *resolution of the identity*. A bounded operator T in $\mathfrak{X}$ is called a *spectral operator of scalar type* if there is a resolution of the identity $E(\cdot)$ such that

$$\left.\begin{aligned} &E(\mu)T = TE(\mu) \\ &\sigma(T; E(\mu)X) \subseteq \overline{\mu} \end{aligned}\right\} \mu \in \text{Borel sets}$$

and

$$T = \int_{\sigma(T)} \lambda E(d\lambda).$$

Here $\sigma(T; E(\mu)\mathfrak{X})$ is the spectrum of the restriction of T to the range of $E(\mu)$. In exactly the same way we have the notions of an $(\mathfrak{X})$-*countably additive resolution of the identity* and a *scalar type spectral operator of class* $(\mathfrak{X})$ in $\mathfrak{X}^*$. In either case $E(\cdot)$ is unique and $E(\sigma(T)) = I$. Moreover if $F(\cdot)$ is a countably additive ($(\mathfrak{X})$-countably additive) spectral measure on $(\Omega, \mathfrak{F})$, the operator

$$S(f) = \int_{\Omega} f(\omega) F(d\omega)$$

defined by (1.3) is a spectral operator of scalar type (scalar type and class $(\mathfrak{X})$) whose resolution of the identity $E(\cdot\,; S(f))$ is given by

$$E(\sigma; S(f)) = F(f^{-1}(\sigma)), \quad \sigma \in \text{Borel sets}.$$

Finally we will need the following specialization of a theorem of Dunford [9, Theorem 17].

1.1 THEOREM. *Let $\mathfrak{B}$ be a bounded Boolean algebra of projections in a reflexive space and let $\mathfrak{A}$ be the algebra generated by $\mathfrak{B}$ in the uniform operator topology. If $\mathfrak{M}$ denotes the compact Hausdorff space of maximal ideals in $\mathfrak{A}$, then $\mathfrak{A}$ is equivalent to $C(\mathfrak{M})$ under a topological and algebraic isomorphism S. There is a spectral measure $E(\cdot)$ defined on the Baire sets of $\mathfrak{M}$ such that if $f \in C(\mathfrak{M})$, then*

$$S(f) = \int_{\mathfrak{M}} f(m) E(dm).$$

1.2 REMARKS. By the discussion above each operator in $\mathfrak{A}$ is a scalar type spectral operator. It is easily seen that $\mathfrak{M}$ may be identified with the Stone representation space of the Boolean algebra $\mathfrak{B}$. Since $\mathfrak{M}$ is totally disconnected, the class $\mathfrak{F}$ of Baire sets of $\mathfrak{M}$ is generated by the open and closed sets. Thus $\{E(\sigma) \mid \sigma \in \mathfrak{F}\}$ is an extension of $\mathfrak{B}$ and lies in the strong closure of $\mathfrak{B}$ by the strong countable additivity of $E(\cdot)x$, $x \in \mathfrak{X}$.

2. Strong limits of operators with restricted spectra. In this section we determine conditions under which a limit in the strong operator topology of scalar type spectral operators is again in this class. The principal restriction imposed on the limiting operators is on the distribution of their spectra. For application in later sections our principal result (Theorem 2.3) is stated in terms of Moore-Smith convergence, or convergence of nets in the terminology of Kelley [**15**]. We recall that the strong operator topology for $B(\mathfrak{X})$ is generated by neighborhoods of the form

$$N(T_0; x_1, \cdots, x_n, \epsilon) = \{T \mid |(T - T_0)x_i| < \epsilon, i = 1, \cdots, n\},$$

the weak operator topology by neighborhoods of the form

$$N(T_0; x_1, \cdots, x_n, x_1^*, \cdots, x_n^*, \epsilon) = \{T \mid |x_i^*(T - T_0)x_i| < \epsilon, i = 1, \cdots, n\}.$$

A net $\{T_\alpha\}$, $\alpha \in A$, converges strongly to $T \in B(\mathfrak{X})$ if

$$\lim_\alpha T_\alpha x = Tx, \qquad x \in \mathfrak{X}.$$

It converges weakly to T if

$$\lim_\alpha x^* T_\alpha x = x^* Tx, \qquad x \in \mathfrak{X},\ x^* \in \mathfrak{X}^*.$$

If V is an unbounded closed subset of the complex plane we denote by $C_\infty(V)$ the B-space of complex valued continuous functions on V which vanish at infinity (A function f vanishes at infinity on V if given $\epsilon > 0$ there is a number K with $|f(\lambda)| < \epsilon$ if $\lambda \in V$, $|\lambda| > K$.) If V is bounded we let $C_\infty(V) = C(V)$.

2.1. Definition. A closed nowhere dense set V in the complex plane will be called an *R-set* if the set of functions

$$\left\{ f \mid f(\lambda) = \frac{1}{\mu - \lambda}, \ \mu \notin V \right\}$$

is a fundamental in $C_\infty(V)$.

It is easily shown that V is an R-set if and only if rational functions are dense in $C_\infty(V)$. To approximate a rational function q in $C_\infty(V)$ by linear combinations of functions of the prescribed type when V is unbounded, one approximates by Riemann sums the integral in the representation

$$q(\lambda) = \frac{1}{2\pi i} \int_C \frac{q(\mu)}{\mu - \lambda}\, d\mu, \qquad \lambda \in V.$$

Here C is a clockwise contour consisting of small circles exterior to V which contain the poles of q. We leave to the reader the fact that the approximation may be made uniform on V. The case where V is bounded is treated by a similar argument.

The characterization of R-sets is apparently an unsolved problem of approximation theory. It is known that not every closed nowhere dense set is an R-set. The most important example of an R-set is, of course, the real line. That R-sets form an extensive class of sets is shown by the following lemma.

2.2. LEMMA. *In order for a closed nowhere dense set V to be an R-set it is sufficient either that V has plane measure zero or that V does not separate the plane.*

The case that V is bounded follows from important theorems of approximation theory. By a theorem of Lavrentieff [**16**] (see also Mergelyan [**18**]) polynomials are dense in $C(V)$ if V does not separate the plane. Hartogs and Rosenthal [**12**] have shown that rational functions are dense in $C(V)$ if V has plane measure zero. If V is unbounded let $V_1 = V \cup \{\infty\}$ have the usual topology as a subset of the complex sphere. If $\beta \notin V$ the mapping Φ defined by $\Phi(\lambda) = (\beta - \lambda)^{-1}$ maps V_1 homeomorphically onto a closed and bounded nowhere dense set W containing zero. If $f \in C_\infty(V)$ then $\phi(z) = f(\Phi^{-1}(z))$ is in $C(W)$ and vanishes at zero. Moreover ϕ is rational if and only if f is rational, and W does not separate the plane or has measure zero if and only if V has the same property.

We now suppose that $\{T_\alpha\}$, $\alpha \in A$, is a net of bounded scalar type spectral operators with $\lim_\alpha T_\alpha x = Tx$, $x \in \mathfrak{X}$, $T \in B(\mathfrak{X})$. The operators T_α need not commute or be uniformly bounded in norm. We examine the spectral properties of T under two assumptions.

(A). If $E_\alpha(\cdot)$ denotes the resolution of the identity for T_α, then there is a constant M such that $|E_\alpha(\sigma)| \leq M$, $\alpha \in A$, $\sigma \in$ Borel sets.

(B). There is a fixed closed (possibly unbounded) R-set V with $\sigma(T_\alpha) \subseteq V$, $\alpha \in A$.

2.3. THEOREM. *If* $T \in B(\mathfrak{X})$ *is the strong limit of a net* $\{T_\alpha\}$ *of scalar type spectral operators satisfying conditions* (A) *and* (B), *then* T^* *is a scalar type spectral operator in* $\mathfrak{X}^*$ *of class* $(\mathfrak{X})$. *If* $\mathfrak{X}$ *is reflexive,* T *is a scalar type spectral operator in* $\mathfrak{X}$.

It should be remarked that for applications in later sections we will need only the case that V is the real line. The method of proof is a straightforward extension of that used by Stone in [23] to prove the spectral theorem in Hilbert space. The proof will require two lemmas.

2.4. LEMMA. *If* $\lambda \notin V$ *then* $\lambda \in \rho(T)$,

$$|R(\lambda;\, T)| \leq 4Md(\lambda, V)^{-1}$$

(*where* $d(\lambda, V) =$ dist. (λ, V)), *and*

$$\lim_\alpha R(\lambda;\, T_\alpha)x = R(\lambda; T)x, \qquad x \in \mathfrak{X}.$$

Proof. Since

$$R(\lambda, T_\alpha) = \int_{\sigma(T_\alpha)} (\lambda - \mu)^{-1} E_\sigma(d\mu)$$

we have from (1.4)

$$|(\lambda I - T_\alpha)x| \geq \frac{|x|\, d(\lambda, V)}{4M}, \qquad \alpha \in A,\ x \in \mathfrak{X},$$

from which it follows that

$$|(\lambda I - T)x| \geq \frac{|x|\, d(\lambda, V)}{4M}, \qquad x \in \mathfrak{X}.$$

The last conclusion follows from the identity

$$R(\lambda;\, T_\alpha)x - R(\lambda; T)x = R(\lambda; T_\alpha)(T - T_\alpha)R(\lambda; T)x.$$

2.5. LEMMA. *Given* $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, *there is a unique measure* $\rho(\,\cdot\,; x^*, x)$, *bilinear in* x *and* x^*, *which satisfies*

$$x^* R(\lambda; T)x = \int_V \frac{1}{\lambda - \mu} \rho(d\mu; x^*, x), \quad \lambda \notin V.$$

Moreover

$$\text{var}\,(\rho(\cdot\,; x^* x)) \leq 4M|x|\,|x^*|.$$

Proof. By (1.2)

$$\text{var}\, x^* E_\sigma(\cdot)x \leq 4M|x|\,|x^*|.$$

Thus the set of measures $\{x^* E_\alpha(\cdot)x\}$, $\alpha \in A$, is a net in the closed sphere S about the origin of radius $4M|x|\,|x^*|$ in the space $R(V)$ of regular measures on the Borel sets of V. Since $R(V)$ is the conjugate space of $C_\infty(V)$, the set S is compact in the w^*-topology [**1**]; that is, the topology generated by neighborhoods of the form

$$N(\theta_0; f_1, \cdots, f_n, \epsilon) =$$

$$\{\theta \mid \theta \in R(V), \ |\int_V f_i(\lambda)\theta(d\lambda) - \int_V f_i(\lambda)\theta_0(d\lambda)| < \epsilon, i = 1, \cdots, n\},$$

where $f_1, \cdots, f_n \in C_\infty(V)$. It follows [**15**] that the net $\{x^* E_\alpha(\cdot)x\}$ has a cluster point $\rho(\cdot\,; x^*, x)$; that is, given $\alpha_0 \in A$, every neighborhood of ρ contains a measure $x^* E_\alpha(\cdot)x$ for some $\alpha \geq \alpha_0$. In particular if $\lambda \notin V$, $\epsilon > 0$, and $\alpha_0 \in A$, then

$$\left| \int_V \frac{1}{\lambda - \mu} x^* E_\alpha(d\mu)x - \int_V \frac{1}{\lambda - \mu} \rho(d\mu; x^*, x) \right|$$

$$= |x^* R(\lambda; T_\alpha)x - \int_V \frac{1}{\lambda - \mu} \rho(d\mu; x^*, x)| < \epsilon$$

for some $\alpha \geq \alpha_0$. By Lemma 2.4,

$$\lim_\alpha x^* R(\lambda; T_\alpha)x = x^* R(\lambda; T)x.$$

Thus

$$x^* R(\lambda; T)x = \int_V \frac{1}{\lambda - \mu} \rho(d\mu; x^*, x), \qquad \lambda \notin V, \ x^* \in \mathfrak{X}^*, \ x \in \mathfrak{X}.$$

The uniqueness of $\rho(\cdot\,; x^*, x)$ and its bilinearity in x and x^* follow from the fact V is an R-set.

To continue the proof of the theorem, we now extend the measure $\rho(\cdot\,; x^*, x)$ on V to all Borel sets of the plane in the obvious way. Since for any Borel set e,

$$|\rho(e; x^*, x)| \leq 4M\,|x^*|\,|x|,$$

there is a unique operator $A(e)$ in X^* satisfying $|A(e)| \leq 4M$ and

$$\rho(e; x^*, x) = A(e)x^*x, \qquad x^* \in \mathfrak{X}^*,\ x \in \mathfrak{X}.$$

It will now be shown that the family $\{A(\cdot)\}$ is a resolution of the identity for T^*. Let $\nu \notin V$. Then

$$R(\nu; T^*)\,A(e_0)x^*x = \int_{e_0} \frac{1}{\nu-\mu}\,A(d\mu)x^*x \tag{2.1}$$

for each Borel set e_0; for the equation

$$\int_V \frac{R(\nu; T^*)}{\lambda-\mu}\,A(d\mu)x^*x = R(\nu; T^*)R(\lambda; T^*)x^*x = \int_V \frac{\theta(d\mu)}{\lambda-\mu}, \tag{2.2}$$

where

$$\theta(e) = \int_e \frac{A(d\mu)x^*x}{\nu-\mu},$$

is valid for every $\lambda \notin V$, $\lambda \neq \nu$. Since the corresponding functions $(\lambda-\mu)^{-1}$ are fundamental in $C_\infty(V)$, formula (2.1) follows from equating the measures in (2.2). However,

$$R(\nu; T^*)A(e_0)x^*x = \int_V \frac{1}{\nu-\mu}\,A(d\mu)A(e_0)x^*x,$$

and the same uniqueness argument for the measure yields

$$A(e_0)A(e_1) = A(e_0 \cap e_1)$$

for arbitrary Borel sets e_0 and e_1. Hence $A(e)$ is a projection. In view of the countable additivity of $A(\cdot)x^*x$, it remains to show that $\sigma(T^*) = I$ and $\sigma(T^*, A(e)\mathfrak{X}) \subseteq \bar{e}$ for arbitrary e. The second statement follows from formula (2.1) since $R(\nu; T^*)A(e)x^*x$ has a unique analytic continuation to all of e', because $\sigma(T^*) \cap \bar{e}$ is nowhere dense. To prove the first statement let e_0 be any compact subset of $V - \sigma(T^*)$. Since

$$R(\lambda;\, T^*)A(e_0) = A(e_0)R(\lambda;\, T^*) \quad \text{for} \quad \lambda \in \rho(T^*),$$

$A(e_0)$ commutes with T^*. Thus

$$\sigma(T^*,\, A(e_0)X^*) \subseteq \sigma(T^*).$$

But again $R(\lambda;\, T^*)A(e_0)x^*x$ has a unique analytic continuation to e_0', from which it follows that $A(e_0) = 0$, and hence $A(V - \sigma(T^*)) = 0$ as $V - \sigma(T^*)$ is the union of an ascending sequence of compact sets. Finally if C is any contour enclosing the bounded set $\sigma(T^*)$,

$$x^*x = \int_{\sigma(T^*)} \left\{ \frac{1}{2\pi i} \int_C \frac{1}{\lambda - \mu}\, d\lambda \right\} A(d\mu)x^*x = A(\sigma(T^*))x^*x,$$

showing $A(\sigma(T^*)) = I$. If $\mathfrak{X}$ is reflexive the projections $E(e) = A^*(e)$ form a resolution of the identity for T in $\mathfrak{X}$. This completes the proof.

2.6. THEOREM. *Let a net $\{T_\alpha\}$, $\alpha \in A$, of bounded scalar type spectral operators satisfying conditions* (A) *and* (B) *converge strongly to a bounded scalar type spectral operator T. Let h be a bounded Borel function on V with set K of discontinuities. If $E(\overline{K}) = 0$ where $E(\cdot)$ is the resolution of the identity for T, then $h(T_\alpha)$ converges strongly to $h(T)$.*

Proof. We consider first the case that $h \in C_\infty(V)$. By Lemma 2.4, $R(\lambda; T)$, $\lambda \notin V$, is the strong limit of $R(\lambda; T_\alpha)$, and hence $\lim_\alpha g(T_\alpha) = g(T)$ strongly for g in a dense subset of $C_\infty(V)$. If $|h - g| < \epsilon$, then

$$|h(T_\alpha)x - h(T)x| \leq |\int_V (h(\lambda)) - g(\lambda)E_\alpha(d\lambda)x|$$

$$+ |\int_V (h(\lambda) - g(\lambda))\, E(d\lambda)x| + |g(T_\alpha)x - g(T)x|$$

$$\leq 8M\, \epsilon\, |x| + |g(T_\alpha)x - g(T)x|, \qquad x \in \mathfrak{X},$$

from which the conclusion follows for $h \in C_\infty(V)$. In the case h is a bounded Borel function whose set K of discontinuities satisfies $E(\overline{K}) = 0$, choose $g \in C_\infty(V)$ such that $g(\lambda) = 0$, $\lambda \in K$, and $g(\lambda) > 0$ for $\lambda \in V - \overline{K}$. The function $g\,h$ is in $C_\infty(V)$. Moreover, the range of $g(T)$ is dense in X; for given $x \in \mathfrak{X}$ and $\epsilon > 0$ there is a closed subset σ of V disjoint from $\overline{K}$ such that

$|x - E(\sigma)x| < \epsilon$. Then

$$E(\sigma)x = g(T)y \text{ where } y = \int_\sigma \frac{E(d\lambda)x}{g(\lambda)}.$$

Now if $x \in \mathfrak{X}$,

$$\begin{aligned}
&|h(T_\alpha)g(T)x - h(T)g(T)x| \\
&\quad \leq |h(T_\alpha)g(T)x - h(T_\alpha)g(T_\alpha)x| + |(hg)(T_\alpha)x - (hg)(T)x| \\
&\quad \leq 4M \operatorname*{ess\,sup}_{\lambda \in V} |h(\lambda)| \cdot |g(T)x - g(T_\alpha)x| + |(hg)(T_\alpha) - (hg)(T)x|.
\end{aligned}$$

By the previous case $g(T_\alpha)$ and $(hg)(T_\alpha)$ converge strongly to $g(T)$ and $(hg)(T)$. Thus $\lim_\alpha h(T_\alpha)y = h(T)y$ for y in a dense set. Since the $h(T_\alpha)$ are uniformly bounded, $h(T_\alpha)$ converges strongly to $h(T)$.

Theorem 2.6 generalizes a theorem of Kaplansky [**14**] for the case that the T_α are self adjoint operators on Hilbert space and $\overline{K} \cap \sigma(T) = 0$. The present theorem contains a result of Rellich [**21**] that if $\{T_n\}$ is a sequence of self adjoint operators converging strongly to T, then

$$\lim_{n \to \infty} E_n((-\infty, \lambda])x = E((-\infty, \lambda])x, \quad x \in \mathfrak{X}$$

for each λ not in the point spectrum of T.

3. Bounded Boolean algebras of projections. It is natural to ask when a Boolean algebra $\mathfrak{B}$ of projections may be embedded in a complete Boolean algebra of projections. Under the assumptions that $\mathfrak{X}$ is reflexive and $\mathfrak{B}$ is bounded, Dunford in [**7**] constructs the projection $\vee_\alpha E_\alpha$ corresponding to any subset $\{E_\alpha\} \subseteq \mathfrak{B}$, and states the theorem that the least complete Boolean algebra of projections containing $\mathfrak{B}$ is the closure of $\mathfrak{B}$ in the strong operator topology. In this section we will give a proof of Dunford's theorem by showing first that the strong closure of $\mathfrak{B}$ (denoted by $\overline{\mathfrak{B}}^s$) is complete. It will then be required to show that a complete bounded Boolean algebra of projections is strongly closed. Actually we will show it contains every projection in the weakly (equivalently, strongly) closed algebra which it generates. This stronger result will be needed in § 4.

The proofs will require the following lemma on monotone nets of projections.

3.1. LEMMA. *Let $\{E_\alpha\}$, $\alpha \in A$, be a net of projections in a reflexive space*

$\mathfrak{X}$ *satisfying* $|E_\alpha| \le M$, $\alpha \in A$. *If* $E_\alpha \le E_\beta$ *whenever* $\alpha \le \beta$, *then* $\lim_\alpha E_\alpha$ *exists in the strong operator topology and* $\lim_\alpha E_\alpha = \bigvee_\alpha E_\alpha$. *Correspondingly if* $E_\beta \le E_\alpha$ *whenever* $\alpha \le \beta$, *then* $\lim_\alpha E_\alpha = \bigwedge_\alpha E_\alpha$ *in the strong operator topology.*

This result is due to Lorch [**17**] for the case of monotone sequences. A proof of the general case has been given by J. Y. Barry [**3**].

3.2. THEOREM. *If* $\mathfrak{B}$ *is a bounded Boolean algebra of projections in a reflexive space, then* $\overline{\mathfrak{B}}^s$ *is a complete bounded Boolean algebra of projections containing* $\mathfrak{B}$.

Proof. Clearly $|E| \le M$ if $E \in \overline{\mathfrak{B}}^s$. If E, E_1, F and F_1 are in $\mathfrak{B}$, $|(E - E_1)x| < \epsilon$, and $|(F - F_1)x| < \epsilon$, then

$$|(EF - E_1F_1)x| \le |(EF - EF_1)x| + |(EF_1 - E_1F_1)x| \le 2M\epsilon.$$

Thus the mapping $[E, F] \longrightarrow EF$ is a continuous map of $\mathfrak{B} \times \mathfrak{B} \longrightarrow B(\mathfrak{X})$ in the strong operator topology. Thus $\overline{\mathfrak{B}}^s$ is a bounded Boolean algebra of projections. If $\mathfrak{B}_0$ is any subset of $\overline{\mathfrak{B}}^s$, let Σ be the family of all finite subsets of $\mathfrak{B}_0$, directed by inclusion. If $\sigma = \{E_1, \cdots, E_n\} \subseteq \mathfrak{B}_0$ let $E_\sigma = E_1 \vee E_2 \vee \cdots \vee E_n$. The net $\{E_\sigma\}$, $\sigma \in \Sigma$, is monotone in the natural order of projections. By Lemma 3.1, we have

$$\lim_\sigma E_\sigma = \bigvee_\sigma E_\sigma \in \overline{\mathfrak{B}}^s.$$

The next lemma is an extension of a result of Dixmier [**5**] for Hilbert space (see also Michael [**19**]). The proof is similar, but we give it for completeness.

3.3. LEMMA. *If* $\mathfrak{X}$ *is B-space, a convex subset of* $B(\mathfrak{X})$ *has the same closure in the weak operator topology as it does in the strong operator topology.*

Proof. Under either the weak or the strong operator topology $B(\mathfrak{X})$ is a locally convex linear topological space. In view of the separation theorem for convex sets [**4**] it is enough to show that these two spaces have the same continuous linear functionals; or, since the strong topology is stronger than the weak, that a functional continuous in the strong topology is continuous in the weak topology. If θ is continuous in the strong topology, there is a finite subset $\{x_1, \cdots, x_n\}$ of $\mathfrak{X}$ and an $\epsilon > 0$, such that $|Tx_i| < \epsilon$, $i = 1, \cdots, n$, $T \in B(\mathfrak{X})$, implies $|\theta(T)| < 1$. Let $\mathfrak{Z}$ be the Banach space of n-tuples $\zeta = [z_1, \cdots, z_n]$, $z_i \in \mathfrak{X}$ with norm $|\zeta| = \max_{1 \le i \le n} |z_i|$. If Φ is the mapping of $B(\mathfrak{X})$ into

$\mathfrak{Z}$ defined by $\Phi(T)=[Tx_1,\cdots,Tx_n]$, it is easily seen that the functional f_0 on $\Phi(B(\mathfrak{X}))$ defined by $f_0(\zeta)=\theta(T)$ is well defined and continuous. If f is a continuous extension of f_0 to all of $\mathfrak{Z}$, then f has the form

$$f([z_1,\cdots,z_n])=\sum_{i=1}^{n} x_i^* z_i$$

where $x_i^* \in \mathfrak{X}^*$. Consequently $\theta(T)=f(\Phi(T))$ has the form

$$\theta(T)=\sum_{i=1}^{n} x_i^* Tx_i .$$

It follows that θ is continuous in the weak operator topology.

3.4. THEOREM.[1] *A complete bounded Boolean algebra of projections in a reflexive space contains every projection in the weakly closed algebra it generates.*

Proof. Let $\mathfrak{A}$ be the algebra generated by $\mathfrak{B}$ in the uniform operator topology, and let $\overline{\mathfrak{A}}^w$ be the closure of $\mathfrak{A}$ in the weak operator topology. Since $\overline{\mathfrak{A}}^w$ is an algebra, it is the weakly closed algebra generated by $\mathfrak{B}$. Moreover, $\overline{\mathfrak{A}}^w=\overline{\mathfrak{A}}^s$ by Lemma 3.3. Let $F^2=F$, $F\in\overline{\mathfrak{A}}^s$. The proof that $F\in\mathfrak{B}$ will be made by showing that to each pair (y,z) where $y\in\mathfrak{M}=F\,\mathfrak{X}$ and $z\in\mathfrak{N}=(I-F)\mathfrak{X}$ there can be associated a projection $E_{yz}\in\mathfrak{B}$ such that $E_{yz}\,y=y=F\,y$, and $E_{yz}\,z=0=Fz$. For if this is granted, the projection

$$(3.1)\qquad E=\wedge_{z\in\mathfrak{N}}\vee_{y\in\mathfrak{M}} E_{yz}$$

is in $\mathfrak{B}$ since $\mathfrak{B}$ is complete. If $x_0\in\mathfrak{X}$, $x_0=y_0+z_0$, $y_0\in\mathfrak{M}$, $z_0\in\mathfrak{N}$, then $\vee_{y\in\mathfrak{M}} E_{yz}y_0=y_0$ for each $z\in\mathfrak{N}$, and $\vee_{y\in\mathfrak{M}} E_{yz_0}z_0=0$. Thus $Ey_0=y_0$, $Ez_0=0$ and $E=F$.

We now construct the projections E_{yz}. It should be remarked that the construction uses only the fact that $\mathfrak{B}$ is σ-complete. Let y and z be fixed elements of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Then since $F\in\overline{\mathfrak{A}}^s$, elements $A_n\in\mathfrak{A}$ may be selected such that

$$(3.2)\qquad |y-A_n y|<1/2^n,\quad |A_n z|<1/2^n,\qquad n=1,2,\cdots,$$

[1]This theorem does not answer the question: if a sequence $\{E_n\}\subseteq\mathfrak{B}$ converges weakly to a projection F, does $\{E_n\}$ converge strongly to F?

and each A_n is a finite linear combination of disjoint projections $\{E_1, \cdots, E_{s_n}\}$ in $\mathfrak{B}$. It is now convenient to use the fact (Theorem 1.1) that $\mathfrak{A}$ is equivalent to $C(\mathfrak{M})$ where $\mathfrak{M}$ is the space of maximal ideals in $\mathfrak{A}$. Thus $A_n = S(f_n)$ where f_n is a finite linear combination of characteristic functions of disjoint open and closed sets, and $A_n = \int_{\mathfrak{M}} f_n(m)\, E(dm)$, where the integral is that of § 1. Let ϵ be an arbitrary positive number and

$$\sigma_{n\epsilon} = \{ m \mid |f_n(m)| > \epsilon \}.$$

If $E(\sigma_{n\epsilon})$ is the projection corresponding to $\sigma_{n\epsilon}$, the remainder of the proof consists of showing we may take

$$E_{yz} = \bigvee_{\epsilon > 0} \bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} E(\sigma_{i\epsilon}).$$

Let

$$E_{n\epsilon} = \bigvee_{i=n}^{\infty} E(\sigma_{i\epsilon}),\quad E_{0\epsilon} = \bigwedge_{n=1}^{\infty} E_{n\epsilon}.$$

Since the sequence $\{E_{n\epsilon}\}$ is monotone decreasing,

$$\lim_{n\to\infty} E_{n\epsilon}\, y = E_{0\epsilon}\, y$$

by Lemma 3.1. Defining

$$A_{n\epsilon} = \int_{\sigma_{n\epsilon}} f_n(m)\, E(dm)$$

we have

$$|E_{n\epsilon} y - y| \le |E_{n\epsilon}(y - A_n y)| + |A_{n\epsilon} y - A_n y| + |A_n y - y|$$

$$< \frac{(M+1)}{2^n} + \left| \int_{\sigma'_{n\epsilon}} f_n(m)\, E(dm) y \right| \le \frac{(M+1)}{2^n} + 4M\epsilon |y|.$$

Thus

$$(3.3) \qquad |E_{0\epsilon} y - y| \le 4M\epsilon |y|.$$

Now

$$E_{n\epsilon} z = \lim_{p \to \infty} \bigvee_{i=n}^{p} E(\sigma_{i\epsilon}) z$$

by Lemma 3.1. But

$$|\bigvee_{i=1}^{p} E(\sigma_{i\epsilon}) z| = |\{E(\sigma_{n\epsilon}) + E(\sigma_{(n+1)\epsilon})(I - E(\sigma_{n\epsilon}))$$

$$+ \cdots + E(\sigma_{p\epsilon})(I - E(\bigcup_{j=n}^{p-1} \sigma_{j\epsilon}))\} z| \leq M \sum_{i=n}^{p} |E(\sigma_{i\epsilon}) z|.$$

Since by (1.4)

$$|A_{i\epsilon} z| = |A_i E(\sigma_{i\epsilon}) z| \geq \frac{\epsilon |E(\sigma_{i\epsilon}) z|}{4M}$$

we have

$$|E(\sigma_{i\epsilon}) z| \leq \frac{4M^2}{\epsilon 2^i}, \quad i \geq n.$$

Thus

$$|E_{n\epsilon} z| < \frac{4M^3}{\epsilon 2^{n-1}}$$

and

$$E_{0\epsilon} z = 0 \tag{3.4}$$

by Lemma 3.1. If $0 < \delta < \epsilon$ then $E_{0\epsilon} \leq E_{0\delta}$; so if $\epsilon_n = 1/n$ and

$$E_0 = \bigvee_{n=1}^{\infty} E_{0\epsilon_n} = \lim_{n \to \infty} E_{0\epsilon_n},$$

we have $E_0 y = y$ and $E_0 z = 0$ by (3.3), (3.4) and Lemma 3.1. Thus we may take $E_{yz} = E_0$. This completes the proof.

3.5. COROLLARY. *A bounded Boolean algebra of projections in a reflexive space is complete if and only if it is strongly closed.*

4. Weakly closed algebras. The theorems of §§ 2 and 3 enable us to prove the following result.

4.1. THEOREM. *Let $\mathfrak{B}$ be a bounded Boolean algebra of projections in a reflexive space, and let $\mathfrak{L}$ be the weakly closed algebra generated by $\mathfrak{B}$. Then $\mathfrak{B}$ is generated in the uniform topology by $\overline{\mathfrak{B}}^s$. Each operator in $\mathfrak{B}$ is a scalar type spectral operator whose resolution of the identity has its range in $\overline{\mathfrak{B}}^s$.*

Proof. Let A be an element of $\mathfrak{B}$. Then since $\mathfrak{B} = \overline{\mathfrak{A}}^w = \overline{\mathfrak{A}}^s$ there is a net $\{A_\alpha\} \subseteq \mathfrak{A}$ such that $Ax = \lim_\alpha A_\alpha x$, $x \in \mathfrak{X}$. Let $A_\alpha = S(f_\alpha)$, $f_\alpha \in C(\mathfrak{M})$. Now $f_\alpha = g_\alpha + ih_\alpha$ where g_α and h_α are real, and $A_\alpha = B_\alpha + iC_\alpha$, $B_\alpha = S(g_\alpha)$, $C_\alpha = S(h_\alpha)$. Moreover,

$$B_\alpha x - B_\beta x = \int_{\mathfrak{M}} r_{\alpha\beta}(m) E(dm)(A_\alpha x - A_\beta x)$$

where

$$r_{\alpha\beta}(m) = \begin{cases} \dfrac{g_\alpha(m) - g_\beta(m)}{f_\alpha(m) - f_\beta(m)}, & f_\alpha(m) \neq f_\beta(m) \\ 0, & f_\alpha(m) = f_\beta(m). \end{cases}$$

Since $|r_{\alpha\beta}(m)| \leq 1$,

$$\left| \int_{\mathfrak{M}} r_{\alpha\beta}(m) E(dm) \right| \leq 4M$$

Thus $\{B_\alpha x\}$ is a Cauchy net for each $x \in \mathfrak{X}$. The operator B defined by

$$Bx = \lim_\alpha B_\alpha x, \qquad x \in \mathfrak{X}$$

is in $\mathfrak{L}$ since the inequality

$$|Bx| \leq 4M \overline{\lim_\alpha} |A_\alpha x| = 4M |Ax|$$

shows B is bounded. Similarly the net $\{C_\alpha\}$ converges strongly to a bounded operator $C \in \mathfrak{B}$, and $A = B + iC$.

By Theorem 2.3 B and C are scalar type spectral operators. To show that A is a scalar type spectral operator it is sufficient to prove that the resolutions of the identity of B and C generate a bounded Boolean algebra.[2] It will be

[2] Cf. [9, Theorem 19]. It is not known whether in a reflexive space the sum of two commuting scalar type spectral operators is a scalar type spectral operator. An example to the contrary has been given by Kakutani [13] in a non reflexive space.

shown that $E(\cdot\,;B)$ and $E(\cdot\,;C)$ have their range in $\overline{\mathfrak{B}}^s$. Let σ be any bounded closed subset of the real line and let $\{\phi_n\}$ be a monotone decreasing sequence of continuous functions with

$$\lim_{n\to\infty} \phi_n(\lambda) = k_\sigma(\lambda), \qquad -\infty < \lambda < \infty.$$

by Theorem 2.6 $\phi_n(B) \in \mathfrak{B}$ for each n. But $\phi_n(B)$ converges weakly to $E(\sigma;B)$, and thus $E(\sigma;B) \in \overline{\mathfrak{B}}$. But then $E(\sigma;B) \in \overline{\mathfrak{B}}^s$ by Theorem 3.4. The assertion that the range of $E(\cdot\,;B)$ is in $\overline{\mathfrak{B}}^s$ now follows from the countable additivity of $E(\cdot\,;B)x$, $x \in \mathfrak{X}$. The operator C is treated in the same way. Theorem 1.1 may be applied to the bounded Boolean algebra $\overline{\mathfrak{B}}^s$ to complete the proof.

4.2. COROLLARY. *In a reflexive B-space the uniformly closed algebra generated by a complete bounded Boolean algebra of projections is weakly closed.*

4.3. REMARK. The use of Theorem 3.4 in the proof of Theorem 4.1 to show $E(\cdot\,;B) \subseteq \overline{\mathfrak{B}}^s$ can be avoided if $\mathfrak{X}$ is separable. In this case $\sigma(B)$ contains at most denumerably many eigenvalues, and Theorem 2.6 shows

$$\lim_{\alpha} E((-\infty,\lambda];B_\alpha)x = E((-\infty,\lambda];B)x, \qquad x \in \mathfrak{X}$$

for a dense set of numbers λ.

4.4. DEFINITION. A scalar type spectral operator will be said to be *real* if $\sigma(A)$ is real.

Our next objective is to characterize the weakly closed algebra generated by a single real scalar type operator and the identity. We will require certain preliminary material.

4.5. DEFINITION. A compact Hausdorff space Ω is *extremely disconnected* if the closure of every open set is open. A positive regular Borel measure μ on Ω is *normal* if it vanishes on sets of the first category. An extremely disconnected compact Hausdorff space Ω is *hyperstonian* if it has sufficiently many normal measures that the union of their supports is dense in Ω.

Stone has shown [**26**] that the representation space of a complete Boolean algebra is characterized by the property of being extremely disconnected. It can be shown [**10**] that corresponding to each Borel set e of an extremely disconnected space there is a unique open and closed set σ such that the symmetric difference $(e-\sigma) \cup (\sigma - e)$ is of the first category. The notion of a hyperstonian space is due to Dixmier [6] who has proved that a compact Hausdorff space is the space of maximal ideals for a commutative W^*-algebra on Hilbert space if and only if it is hyperstonian. A hyperstonian space Ω is of *countable type* if each mutually disjoint family of open and closed subsets of Ω is at most countable. By a theorem of Dixmier [6] each hyperstonian space contains a family Ω_i, $i \in I$ of mutually disjoint open and closed subsets whose union is dense, with the property that each Ω_i is of countable type.

Now let $\mathfrak{B}$ be a complete bounded Boolean algebra of projections in a reflexive space with representation space $\mathfrak{M}$. If $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, the measure $x^*E(\cdot)x$ on $\mathfrak{M}$ vanishes on sets of the first category. Thus the positive measure $v(\cdot\,; x^*, x)$ defined by

$$v(\sigma\,; x^*, x) = \underset{\sigma}{\text{tot. var.}}\ x^*E(\sigma)x$$

is normal. Clearly the union of the carriers of such measures is dense in $\mathfrak{M}$. Following Segal [22] we call a projection $E \in \mathfrak{B}$ *countably decomposable* if each mutually disjoint family of projections in $\mathfrak{B}$ bounded by E is at most countable. Thus we have proved:

4.6. THEOREM. *A complete bounded Boolean algebra of projections in a reflexive space contains a family of mutually disjoint countably decomposable projections whose least upper bound is the identity.*

4.7. DEFINITION. Let A be a real scalar type operator. We denote by $\mathfrak{W}(A)$ the weakly closed algebra generated by A and I. An operator B is an *extended bounded Baire function* of A if for every countably decomposable projection E in $\mathfrak{W}(A)$, B commutes with E and the contraction B_E of B to $E\mathfrak{X}$ is a bounded Baire function (in the usual sense) of the contraction A_E of A.

The concept of an extended bounded Baire function is due to Segal [22]. One verifies easily that the contraction of A to $E\mathfrak{X}$ is a real scalar type spectral operator.

4.8. THEOREM. *The algebra $\mathfrak{W}(A)$ generated in the weak operator topology by a real scalar type operator A and I consists of all extended bounded Baire functions of A.*

Note that since A is real $\mathfrak{W}(A)$ is also the weakly closed algebra generated by the resolution of the identity $E(\cdot\,; A)$ of A. (Cf. the discussion in the proof of Theorem 4.1.) Let $\mathfrak{B}$ be the Boolean algebra of projections in $\mathfrak{W}(A)$. Then $E(\cdot\,; A)$ is strongly dense in $\mathfrak{B}$ by Theorem 4.1. Clearly each extended bounded

Baire function of A lies in $\mathfrak{C}(A)$ since it lies in the uniformly closed span of $\mathfrak{B}$ by (1.3). Conversely, let $\mathfrak{B} \in \mathfrak{W}(A)$ and let $E \in \mathfrak{C}(A)$ be countably decomposable. Since B_E is in $\mathfrak{C}(A_E)$ it is sufficient for the rest of the proof to suppose the identity I is countably decomposable. We next show that the algebra $\mathfrak{A}$ generated in the uniform operator topology by $E(\cdot\,;A)$ consists of all bounded Baire functions of A. If

$$f = \sum_{i=1}^{n} \alpha_i k_{\mu_i}$$

where the sets μ_i are disjoint Baire sets, then

$$f(A) = \sum_{i=1}^{n} \alpha_i E(\mu_i).$$

If $E(\mu_j)x = x$, $|x| = 1$, then $|f(A)x| = |\alpha_j|$. Thus the inequality

$$E(\cdot\,, A) - \text{ess sup } |f(\lambda)| = |f(A)| \leq 4M(E(\cdot\,, A) - \text{ess sup } |f(\lambda)|)$$

is established for finitely valued functions. From this follows a result of Dunford [**9**; Theorem 15] that $\mathfrak{A}$ is equivalent to the algebra of all $E(\cdot\,, A)$-essentially bounded Baire functions on $\sigma(A)$. But each bounded Baire function is a uniform limit of finitely valued functions. It remains to show $\mathfrak{A}$ is weakly closed. However, this follows from Corollary 4.2 and the next lemma (which is valid for arbitrary Boolean algebras).

4.9. LEMMA. *A σ-complete bounded Boolean algebra of projections in a reflexive space with the property that each mutually disjoint subset is countable is complete.*

If $\mathfrak{B}$ is not complete it contains a monotone net whose least upper bound does not belong to $\mathfrak{B}$. By transfinite induction one may construct from the net a family of mutually disjoint projections of cardinality $> \aleph_0$.

Theorem 4.8 is due to von Neumann [**27**] for the case of a self adjoint operator on a separable Hilbert space. The generalization to the case of an arbitrary Hilbert space is proved by Segal [**22**] as a corollary of his treatment of multiplicity theory. Our proof of Theorem 4.8, via Corollary 4.2, yields a more direct proof of Segal's theorem.

It is important to know when an algebra $\mathfrak{W}$ is $\mathfrak{W}(A)$ for some $A \in \mathfrak{W}$. An answer is given by the following theorem.

4.10. THEOREM. *For a bounded Boolean algebra $\mathfrak{B}$ of projections in a reflexive space the following conditions are equivalent.*

(a). *$\mathfrak{B}$ is separable in the strong operator topology.*

(b). *$\mathfrak{B}$ is separable in the weak operator topology.*

(c). *$\mathfrak{B}$ is generated in the weak operator topology by an element $A \in \mathfrak{B}$ and the identity.*

Clearly (c) implies (b). Let $\{A_n\}$ be weakly dense in $\mathfrak{B}$. Since each A_n may be approximated in the uniform topology by a linear combination of projections, there is a sequence $\{E_m\} \subseteq \mathfrak{B}$ which generates $\mathfrak{B}$ in the weak topology. By Theorems 4.1 and 3.4 the countable Boolean algebra $\mathfrak{B}_0$ generated by $\{E_m\}$ is strongly dense in $\mathfrak{B}$, proving (a).

The proof that (a) implies (c) follows a well known argument. Let $\mathfrak{A}_0$ be the algebra generated by $\mathfrak{B}_0$ in the uniform operator topology and let $\mathfrak{M}_0$ be its space of maximal ideals. By a theorem of Gelfand [**11**] $\mathfrak{M}_0$ is separable metric. Since $\mathfrak{M}_0$ is totally disconnected, it is homeomorphic to a subset of the Cantor discontinuum [**2**; p. 121], and thus $C(\mathfrak{M}_0)$ contains a real function h which distinguishes points in $\mathfrak{M}_0$. By the Stone-Weierstrass theorem [**25**] and Theorem 1.1, $A = S(h)$ and I generate $\mathfrak{A}_0$. But $\mathfrak{A}_0$ is weakly dense in $\mathfrak{B}$.

When $\mathfrak{X}$ is separable every subset of $B(\mathfrak{X})$ is separable in the strong operator topology. This fact for Hilbert space is due to von Neumann [**27**]. However, the proof in [**20**, p. 12] extends in a natural way to any Banach space.

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PERTURBATIONS OF SPECTRAL OPERATORS, AND APPLICATIONS

I. BOUNDED PERTURBATIONS

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1. Introduction. A principal theorem on self-adjoint boundary-value problems is the existence of a complete orthonormal set of eigenfunctions. This corresponds to the diagonal reduction of a hermitian matrix, and to the spectral theorem for self-adjoint operators in Hilbert space. How much remains true if we drop the fundamental condition of self-adjointness? Infinite dimensional examples show that, in general, we cannot expect even the existence of a single eigenvector.

Nevertheless, there does exist a class of operators which behave in a "regular" fashion from this spectral theoretic point of view, namely, the spectral operators introduced in [**4**, p. 560]. The paper [**4**], while extensively developing the theory of these operators, still leaves open a very significant question. Are many (or any) of the nonsymmetric integral, differential, and so on, operators arising in the more "classical" branches of analysis spectral? The main result of the present paper is a positive answer to the foregoing question.

The principal indication that a positive answer is to be expected comes from a classical series of papers [**1**; **2**; **3**; **11**; **13**;], in which it is demonstrated that for certain general types of boundary-value problems involving nonsymmetric linear differential operators, expansions in eigenfunctions exist and converge in much the same way as ordinary Fourier series. The method in all of these papers is "analytic;" that is, it operates with asymptotic estimates of the solutions of the various differential equations and of the partial sums of the various series arising. The method in the present paper is abstract, and is phrased in terms of Banach spaces, linear operators, and so on. This has the advantage of greater simplicity in proof, and greater generality in applications. For instance, we shall be able to prove results on certain types of partial differential operators which appear difficult to prove by an analytic method.

The general idea of our abstract method is the following. Let T be a spectral

Received March 4, 1953. The research contained in this paper was done under contract onr 609 (04) with the Office of Naval Research.

Pacific J. Math. 4 (1954), 415-458

operator. Let B be an operator which is, in some sense, small relative to T. Then $T+B$ will be a spectral operator. A less stringent restriction on B will yield a weaker conclusion on the spectral nature of $T+B$. In particular, there are many cases in which it can be asserted that the set of generalized eigenvectors of $T+B$ spans our Banach space, but not that $T+B$ is spectral.

2. Preliminaries. Let $\mathfrak{X}$ be a (complex) reflexive Banach space. A bounded operator in $\mathfrak{X}$ is an everywhere-defined continuous linear mapping of $\mathfrak{X}$ into itself. An unbounded operator is a linear mapping of a dense linear subspace of $\mathfrak{X}$ into $\mathfrak{X}$. The set on which the operator T is defined is its domain, denoted by $\mathfrak{D}(T)$. The open set of λ in the complex plane, for which

$$(T-\lambda I)^{-1} = (T-\lambda)^{-1}$$

is everywhere defined and bounded, is the resolvent of T. Its closed complement, which is bounded for bounded operators, is the spectrum $\sigma(T)$ of T.

DEFINITION 1. An operator T is *regular* if its spectrum $\sigma(T)$ is not the entire complex plane, and if $(T-\lambda)^{-1}$ is compact for some $\lambda \notin \sigma(T)$.

REMARK. Except in the trivial case where $\mathfrak{X}$ is finite dimensional, a regular T cannot be bounded. For, if T is bounded,

$$I = (T-\lambda)(T-\lambda)^{-1}$$

is compact; and this implies immediately that $\mathfrak{X}$ is finite dimensional.

LEMMA 1. *If T is regular, then:*

(a) *Its spectrum is a denumerable set of points with no finite limit point.*

(b) *$(T-\lambda)^{-1}$ is compact for every $\lambda \notin \sigma(T)$.*

(c) *Every $\lambda_0 \in \sigma(T)$ is a pole of finite order $\nu(\lambda_0)$ of the resolvent $R_\lambda = (T-\lambda)^{-1}$. If a vector f satisfies*

$$(T-\lambda_0)^k f = 0,$$

then f satisfies

$$(T-\lambda_0)^{\nu(\lambda_0)} f = 0.$$

The set of all such vectors f makes up a finite dimensional linear space, called the space of generalized eigenvectors of T corresponding to the eigenvalue λ_0. If $E(\lambda_0)$ is the idempotent function of T corresponding to the analytic function *which is one on* λ_0 *and zero elsewhere on the spectrum of* T, *then* $E(\lambda_0)$ *projects* $\mathfrak{X}$ *onto the space of generalized eigenvectors corresponding to* λ_0.

Proof. We can suppose, without loss of generality, that $0 \notin \sigma(T)$, and that T^{-1} is compact. If we then make use of the identity

$$\lambda(T^{-1} - \lambda)^{-1} T^{-1} = (\lambda^{-1} - T)^{-1},$$

parts (a), (b), and the first statement in (c), of our result follow readily from the corresponding statements in the ordinary Fredholm theory of compact operators. (For this theory, see, for instance, [7, Chap. VII.].) We have

$$(T - \lambda_0)^k f = 0$$

if and only if

$$(\lambda_0^{-1} - T^{-1})^k f = 0,$$

so that the second and third parts of the lemma also follow by a simple application of the corresponding result for compact operators.

To prove the last part of the lemma, we may argue as follows: If C is a small closed curve surrounding the point λ_0 and traversed once in the positive sense, then by definition

$$E(\lambda_0) = \frac{1}{2\pi i} \int_C (\lambda - T)^{-1} d\lambda$$

$$= \frac{1}{2\pi i} \int_C T^{-1}(T^{-1} - \lambda)^{-1} \lambda^{-1} d\lambda$$

$$= \frac{1}{2\pi i} \int_{C'} \mu^{-1} T^{-1}(\mu - T^{-1})^{-1} d\mu,$$

where C' is a small curve surrounding λ_0^{-1}, and traversed in the positive sense. This last integral can easily be evaluated in terms of the functional calculus for bounded operators (cf [4]), and turns out to be the idempotent analytic function $\overline{E}(\lambda_0^{-1})$ of T^{-1} corresponding to the analytic function which is one on λ_0^{-1} and zero elsewhere on $\sigma(T^{-1})$, and now the desired result for T follows readily from the corresponding result for T^{-1}.

REMARK. It is to be noted that we have actually proved a little more than is stated in Lemma 1. We have, in fact, proved that the points of $\sigma(T)$ and the nonzero points of $\sigma(T^{-1})$ are in one-to-one correspondence through the map

$$\lambda \leftrightarrow \lambda^{-1},$$

and, that if we call $E(\lambda_0)(\bar{E}(\lambda_0))$ the spectral measure of the point λ_0 corresponding to the operator T (the operator T^{-1}), then

$$E(\lambda_0) = \bar{E}(\lambda_0^{-1}).$$

This result is, of course, merely a particular case of the "unbounded" analogue of the general "Spectral Mapping Theorem" of Dunford [**4**].

Now, by [**6**, Theorem 20], it follows that if S is a compact spectral operator, and $E(e)$ is its spectral resolution, then $E(\lambda_0)$ is the projection associated above with the point λ_0 (for $\lambda_0 \in \sigma(S)$; for $\lambda_0 \notin \sigma(S)$, $E(\lambda_0) = 0$). Conversely if S is a compact operator, and $E(\lambda_0)$ is the spectral measure of the point λ_0, then S is spectral if and only if there is a uniform bound for all sums $\sum_{i=1}^{k} E(\lambda_i)$ taken over finite subsets $\lambda_1, \lambda_i, \cdots, \lambda_k$ of $\sigma(S)$; that is, if and only if the various projections $E(\lambda_0)$, $\lambda_0 \in \sigma(T)$, generate a uniformly bounded Boolean algebra of projections. We can carry this result over to unbounded operators in a trivial way, making use of the following:

LEMMA 2. *Let T be a regular unbounded operator.*

(a) *If $(\lambda_0 - T)^{-1}$ is spectral for some $\lambda_0 \notin \sigma(T)$, then $(\lambda - T)^{-1}$ is spectral for all $\lambda \notin \sigma(T)$. In this case we say that T is an unbounded spectral operator.*

(b) *The regular operator T is spectral if and only if the spectral measures $E(\lambda_0)$ of the various points $\lambda_0 \in \sigma(T)$ generate a uniformly bounded Boolean algebra.*

Proof. Suppose that T^{-1} is spectral. Then the spectral measures $\bar{E}(\lambda_0)$ of the points $\lambda_0 \in \sigma(T^{-1})$ generate a uniformly bounded Boolean algebra. Since

$$\bar{E}(\lambda_0^{-1}) = E(\lambda_0),$$

the projections $E(\lambda_0)$ generate a uniformly bounded Boolean algebra. The converse argument to this argument evidently goes through. Moreover, since the spectral measure $E_1(\lambda_0)$ corresponding to the operator $T + c$ is evidently $E(\lambda_0 - c)$, it is evident that T has the property of part (b) if and only if $T + c$ does. But this immediately implies part (a).

3. Bounded perturbations. We now come to the main point of the paper.

THEOREM 1. *Let T be a regular spectral operator, and suppose that λ_n is an enumeration of its spectrum. Let d_n denote the distance from λ_n to the rest of* *the spectrum. Suppose that for all but a finite number of n, $E(\lambda_n)$ projects onto a one dimensional subspace; suppose that*

$$\sum_{i=1}^{\infty} E(\lambda_i) = I.^{1}$$

Let B be a bounded operator.

(a) *If* $\sum_{n=1}^{\infty} d_n^{-1} < \infty$, *then* $T + B$ *is spectral.*

(b) *If* $\mathfrak{X}$ *is Hilbert space and* T *is normal, and* $\sum_{n=1}^{\infty} d_n^{-2} < \infty$, *then* $T + B$ *is spectral.*[2]

Proof. Put $R_\lambda = (\lambda - T)^{-1}$ for $\lambda \notin \sigma(T)$. Then we have

$$(1) \qquad (\lambda - T - B)^{-1} = (I - R_\lambda B)^{-1} R_\lambda,$$

whenever $(I - R_\lambda B)^{-1}$ exists. Now, by Lemma 3 below, there exists a constant $K > 0$ such that

$$|R_\lambda| \le K[\text{dist}(\lambda, \sigma(T))]^{-1}.$$

Hence no λ at a greater distance than KB from the spectrum of T is in the spectrum of $T + B$, since, for such λ, $|R_\lambda B| < 1$. It follows also that $T + B$ is regular.

From (1) it follows that

$$\overline{R}_\lambda = (\lambda - T - B)^{-1} = \{I + R_\lambda B(I - R_\lambda B)^{-1}\} R_\lambda$$

$$= R_\lambda + R_\lambda B(I - R_\lambda B)^{-1} R_\lambda.$$

That is,

$$\overline{R}_\lambda - R_\lambda = R_\lambda B(I - R_\lambda B)^{-1} R_\lambda.$$

Let C_n be a circle about λ_n of radius $d_{n/2}$. Then, for $\lambda \in C_n$, we have $|R_\lambda| \le 2Kd_n^{-1}$, and thus when n is large enough to ensure $2Kd_n^{-1} < 1$, we have

$$|(I - R_\lambda B)^{-1}| \le (1 - 2Kd_n^{-1})^{-1}.$$

Since $d_n \longrightarrow \infty$, we may replace this estimate, at least for all but a finite number

[1] The series $\sum_{i=1}^{\infty} E(\lambda_i)$ converges in the strong operator topology.

[2] Of course, $T + B$ is also regular. This is proved in the course of the following argument; but c.f. also Lemma 17 below.

of C_n, by

$$|(I - R_\lambda B)^{-1}| \leq 2.$$

It then follows that

$$|\overline{R}_\lambda - R_\lambda| \leq 8K^2 |B| d_n^{-2}.$$

If we integrate this inequality around C_n in the positive sense, we obtain the inequality

$$|E(\lambda_n) - E_n| \leq 8K^2 |B| d_n^{-1},$$

where $E(\lambda_n)$ is the spectral measure of λ_n corresponding to the operator T, and where E_n is the sum of the spectral measures $E'(\lambda)$ corresponding to $T + B$ of the points λ of the $\sigma(T + B)$ lying within C_n.

Lemma 4 below then implies that for n sufficiently large, E_n has a one-dimensional range. It follows immediately that there must be exactly one point λ'_n of $\sigma(T + B)$ in C_n, and that $E_n = E'(\lambda'_n)$. That is,

$$|E(\lambda_n) - E'(\lambda'_n)| \leq 8K^2 |B| d_n^{-1}$$

for all but a finite number of n. From the above, case (a) of our theorem follows immediately.

To prove case (b), we have only to refine our estimates slightly. We have, from (1),

$$\overline{R}_\lambda = \{I + R_\lambda B + (R_\lambda B)^2 (I - R_\lambda B)^{-1}\} R_\lambda.$$

We then obtain the expression

$$\overline{R}_\lambda - R_\lambda - R_\lambda B R_\lambda = (R_\lambda B)^2 (I - R_\lambda B)^{-1} R_\lambda.$$

so that for $\lambda \in C_n$, and n sufficiently large,

$$|\overline{R}_\lambda - R_\lambda - R_\lambda B R_\lambda| \leq 16K^3 |B|^2 d_n^{-3}.$$

The question now is, what is the integrated form of this inequality? The only problem is to find

$$F_n = \frac{1}{2\pi i} \int_{C_n} R_\lambda B R_\lambda d\lambda,$$

and this is easily done.

Indeed, R_λ has the Laurent expansion

$$R_\lambda = (\lambda - \lambda_n)^{-1}E(\lambda_n) + R^0(\lambda_n) + c_1(\lambda - \lambda_n) + \cdots$$

around λ_n. In this expression $R^0(\lambda_n)$ is a "partial resolvent" of T; that is, we have

$$R^0(\lambda_n) = \lim_{\lambda \to \lambda_n} (I - E(\lambda_n))R_\lambda .$$

Thus, $R^0(\lambda_n)$ is that analytic function of T which corresponds to the analytic function $f(z)$ which is equal to $(z - \lambda_n)^{-1}$ everywhere on $\sigma(T)$ but in the immediate neighborhood of λ_n, where we put $f(z) = 0$. In terms of this Laurent expansion, we readily find that

$$F_n = E(\lambda_n)BR^0(\lambda_n) + R^0(\lambda_n)BE(\lambda_n).$$

Having majorized

$$|E'(\lambda_n') - E(\lambda_n) - F_n|$$

by the terms $16K^3|B|^2 d_n^{-2}$ of an absolutely convergent series, we have only to prove that a uniform bound exists for finite sums $\sum_{i=1}^{R} F_{n_i}$ of the terms F_n. Since a term of the form $E(\lambda_n)BR^0(\lambda_n)$ can be treated as an adjoint of a term of the form $R^0(\lambda_n)BE(\lambda_n)$, we have only to show that a uniform bound exists for finite sums

$$\sum_{i=1}^{R} R^0(\lambda_{n_i})BE(\lambda_{n_i})$$

of these latter terms. It follows from Lemma 3′ below that a constant K' exists such that

$$|R^0(\lambda_n)| \leq K' d_n^{-1}.$$

Thus

$$\left| \sum_{i=1}^{l} R^0(\lambda_{n_i})BE(\lambda_{n_i})f \right| \leq |B|K' \sum_{i=1}^{l} d_{n_i}^{-1}|E(\lambda_{n_i})f|$$

$$\leq |B| K' \left\{ \sum_{i=1}^{l} d_{n_i}^{2} \right\}^{1/2} \left\{ \sum_{i=1}^{l} |E(\lambda_{n_i}) f|^2 \right\}^{1/2} \leq |B| K' \left\{ \sum_{i=1}^{\infty} d_i^{-2} \right\} |f|,$$

since the normality of T implies that the projections $E(\lambda_i)$ are orthogonal perpendicular projections in the Hilbert space $\mathfrak{X}$. Thus both parts of our theorem are proved.

Before continuing with the main line of our discussion, we shall state and prove the lemmas referred to in the foregoing proof.

Lemma 3, below, depends on the functional calculus for our unbounded operators; before proceeding to the proof of this lemma, we must discuss the functional calculus. We consider a regular unbounded operator S with a denumerable spectrum $\{\lambda_n\}$. We shall allow a finite set $\lambda_1, \lambda_2, \cdots, \lambda_N$ of the eigenvalues to be multiple poles of the resolvent, but shall require that all the remaining eigenvalues are simple poles of the resolvent. In addition, we require that

$$\sum_{i=1}^{\infty} E(\lambda_i) = I.$$

In this situation, we can set up the functional calculus for T by setting

$$f(T) = \sum_{i=1}^{N} \sum_{j=0}^{\nu(\lambda_i)} \frac{f^{(j)}(\lambda_i)}{j!} (T - \lambda_i)^j E(\lambda_i) + \sum_{j=N+1}^{\infty} f(\lambda_i) E(\lambda_i)$$

for every function f which is uniformly bounded on the spectrum $\sigma(S)$ and which belongs to the class $C^{\nu(\lambda_i)}$ near the spectral point $\lambda_i (1 \leq i \leq N)$. It may be remarked that, here and in all that follows, the finite number of multiple poles $\lambda_1, \lambda_2, \cdots, \lambda_N$ of the resolvent function $(\lambda - S)^{-1}$ contribute only a finite number of terms, whose influence on any of our arguments it will be trivial to determine by inspection. Thus, to avoid notational complications, we shall assume, without loss of generality, that all the λ_i are simple poles of the resolvent; that is, that $N = 0$. In this case, our proposed expression for the functional calculus is

$$f(T) = \sum_{i=1}^{\infty} f(\lambda_i) E(\lambda_i),$$

where $f(\lambda)$ is any function uniformly bounded on the spectrum.

Functional calculi of this sort are discussed in [**6**], in a much more general situation. In particular, it follows from [**6**, Lemma 6] that the series defining $f(T)$ converges in the strong topology, and that there exists an absolute constant $K = K(T)$ such that we have

$$|f(T)| \leq K \cdot \max_{\lambda \in \sigma(T)} |f(\lambda)|.$$

From this fact, we have:

LEMMA 3. *If S is a regular spectral operator all but a finite set of whose eigenvalues λ_n are simple poles of the resolvent, and S also satisfies*

$$\sum_{i=1}^{\infty} E(\lambda_i) = I,$$

then there exists an absolute constant K such that

$$|(\lambda - S)^{-1}| \leq K \operatorname{dist}(\lambda, \sigma(S))^{-1}$$

for all λ not within a fixed radius ϵ of any multiple pole of the resolvent.

Lemma 3′ involves the operator $R^0(\lambda_n)$ defined as the constant term in the Laurent expansion

$$(\lambda - S)^{-1} = \frac{E(\lambda_n)}{\lambda - \lambda_n} + R^0(\lambda_n) + \cdots$$

of the resolvent function around λ_n. Since

$$(\lambda - S)^{-1} = \frac{E(\lambda_n)}{\lambda - \lambda_n} + \sum_{i \neq n} (\lambda - \lambda_i)^{-1} E(\lambda_i),$$

it is evident that

$$R^0(\lambda_n) = \sum_{i \neq n} (\lambda_n - \lambda_i)^{-1} E(\lambda_i). \tag{2}$$

We obtain, as an immediate consequence of this formula:

LEMMA 3′. *If S is a regular spectral operator having the properties described in Lemma 3, then there exists an absolute constant K' such that if $\lambda_n \in \sigma(S)$ and*

$$d_n = \min_{i \neq n} \operatorname{dist}(\lambda_n, \lambda_i),$$

then for the operator $R^0(\lambda_n)$ defined by formula (2) we have

$$|R^0(\lambda_n)| \leq K' d_n^{-1}.$$

LEMMA 4.[3] *Let E be a projection of $\mathfrak{X}$ onto an n-dimensional space. E' is a projection in $\mathfrak{X}$ satisfying*

$$|E - E'| = \frac{1}{2}|E|^{-1},$$

then E' also projects $\mathfrak{X}$ onto an n-dimensional space.

Proof. We have

$$|E - EE'| < \frac{1}{2}|E||E|^{-1} < 1$$

and

$$|E'| \leq |E| + \frac{1}{2}|E|^{-1} < 2|E|,$$

so that

$$|E'E - E'| < 2|E| \cdot \frac{1}{2}|E|^{-1} = 1.$$

If we then consider EE' as a mapping of $E(\mathfrak{X})$ into itself, it follows that EE' has an inverse. Thus E' maps $\mathfrak{X}$ onto a space of dimension n at least. Applying the same argument to $E'E$, we see that E' maps $\mathfrak{X}$ onto a space of dimension n at most. It follows that the dimension of $E'(\mathfrak{X})$ is exactly n.

Part (b) of Theorem 1 is capable of some improvement. Inspection of the proof of this result reveals that the only thing essential is that the spectral measures $E(\lambda_i)$ should be orthogonal projections. But, by a theorem of Lorch and Mackey (proved in [**17**]), any uniformly bounded Boolean algebra $\{E\}$ of projections in Hilbert space can be reduced to a Boolean algebra of orthogonal projections by an inner automorphism

$$E \longrightarrow D^{-1}ED,$$

where D is a bounded operator in Hilbert space with a bounded inverse. Since such an inner automorphism evidently preserves all operator theoretic properties of the sort involved in our proof, we may state:

Corollary 1b′.[4] *If T is a regular spectral operator in Hilbert space, if all but*

[3] A similar lemma is found in [**18**, remark after Corollary 2.5].

[4] This improvement of Theorem 1b was pointed out to the author in conversation with N. Dunford.

a finite number of its eigenvalues λ_n *are simple poles of the resolvent and correspond to one-dimensional eigenspaces, if*

$$\sum_{i=1}^{\infty} E(\lambda_i) = I,$$

and if, putting

$$d_n = \min_{i \neq n} \operatorname{dist}(\lambda_n, \lambda_i),$$

we have $\sum d_n^{-2} < \infty$, *then* $T + B$ *is a spectral operator for any bounded* B.

4. Two counterexamples. It would be useful to be able to prove Theorem 1 without the restriction to simple eigenvalues. Unfortunately, the appropriate generalization is not true, even if the eigenvalues are restricted to be simple poles of the resolvent, and even if the eigenvalues go to infinity very rapidly The following example shows this to be the case:

EXAMPLE 1. We take two infinite sequences ϕ_n^+ and ϕ_n^- of vectors to be, together, an orthonormal basis for Hilbert space $\mathfrak{X}$. We let T be the self-adjoint unbounded operator defined by

$$T\phi_n^+ = n!\phi_n^+, \quad T\phi_n^- = n!\phi_n^-.$$

Then $\lambda_n = n!$ is a simple pole of the resolvent, but a double eigenvalue. We then let B be the compact operator defined by

$$B\phi_n^+ = (n!)^{-1}\phi_n^+, \quad B\phi_n^- = \{(n-1)!\}^{-1}\phi_n^+.$$

It may be noted that if we realize $\mathfrak{X}$ as a space of L_2 functions, taking

$$\phi_n^+ = \cos 2\pi nx, \quad \phi_n^- = \sin 2\pi nx,$$

say, then B is an operator defined as an integral transform with an analytic kernel. At any rate, this peturbation breaks up the double eigenvalue $n!$ into two single eigenvalues $n!$ and $n! + (n!)^{-1}$, with the corresponding eigenfunctions $n\phi_n^+ - \phi_n^-$ and ϕ_n^+. A brief calculation shows that the corresponding projections $E(n!)$ are defined by

$$E(n!)\phi_j^{\pm} = 0 \text{ for } n \neq j,$$

$$E(n!)\phi_n^{+} = 0,$$

$$E(n!)\phi_n^{-} = \phi_n^{-} - n\phi_n^{+}.$$

Thus, the spectral measures of the points in the spectrum of $T + B$ are not uniformly bounded, so that $T + B$ is surely not spectral.

This example also indicates that the spectral property of $T + B$ fails because we do not group the two projections arising out of the double eigenvalues of T together in forming our spectral sums. We shall see later that this is very typical behavior.

In view of the importance for our proof of the property described in Lemma 3, we shall give an example which shows it to fail if we allow regular operators with an infinity of double poles of the resolvent. This is:

EXAMPLE 2. We introduce an orthonormal basis for Hilbert space $\mathfrak{X}$ consisting of two infinite sequences of vectors ϕ_n^{+}, ϕ_n^{-}, as in Example 1. We let T be the smallest closed operator satisfying

$$T\phi_n^{+} = n^2\phi_n^{+} + n\phi_n^{-}, \quad T\phi_n^{-} = n^2\phi_n^{-}.$$

Then $\sigma(T)$ is the set of points n^2, and $(\lambda - T)^{-1}$ is defined by

$$(\lambda - T)^{-1}\phi_n^{+} = (n^2 - \lambda)^{-1}(\phi_n^{+} - n(n^2 - \lambda)^{-1}\phi_n^{-})$$

$$(\lambda - T)^{-1}\phi_n^{-} = (n^2 - \lambda)^{-1}\phi_n^{-}.$$

Hence T is regular. If we put $k_n = n^2 - n^{1/2}$, then

$$d(k_n, \sigma(T)) = n^{1/2}$$

for all large n, while

$$(k_n - T)^{-1}\phi_n^{+} = n^{-1/2}\phi_n^{+} - \phi_n^{-}$$

has norm at least 1.

5. Basic properties of ordinary differential operators. We wish ultimately to apply our abstract theory to the study of linear differential operators. We shall take our formal differential operators to have the form

$$\tau = \sum_{i=0}^{n} a_i(x)\left(\frac{d}{dx}\right)^i, \tag{3}$$

where

$$a_n(x) \equiv 1, \quad a_{n-1}(x) \equiv 0,$$

and where the coefficient function $a_i(x)$ belongs to the class $C^\infty[0,1]$. The restriction on the coefficients a_n and a_{n-1} is not as severe as might at first appear, since any operator τ of the form (3) in which $a_n(x) \neq 0$ and $a_n(x)$ is real can be reduced to one of the restricted form we have chosen by an elementary transformation.

In connection with the study of the n-th order differential operator τ, it is convenient to introduce the Banach space $A^n = A^n[0, 1]$ consisting of those functions f in C^{n-1} such that $f^{(n-1)}(x)$ is absolutely continuous and such that $f^{(n)} \in L_2[0, 1]$. We introduce the norm in A^n by the definition

$$|f| = \left\{\int_0^1 |f^{(n)}(x)|^2 dx\right\}^{1/2} + \max_{0 \le x \le 1} \max_{0 \le i \le n-1} |f^{(i)}(x)|.$$

A fundamental formula in the study of τ is then the Green's formula, which we can obtain readily by partial integration:

$$\int_0^1 \tau f(x)\overline{g(x)}\,dx - \int_0^1 f(x)\overline{\tau^* g(x)}\,dx = F_1(f, g) - F_0(f, g). \tag{4}$$

Here, f and g are arbitrary elements of $A^n[0, 1]$, τ is the formal differential operator

$$\tau = \sum_{i=0}^{n} a_i(x)\left(\frac{d}{dx}\right)^i,$$

and τ^* is the formal differential operator

$$\tau^* = \sum_{i=0}^{n} b_i(x)\left(\frac{d}{dx}\right)^i,$$

where

$$b_i(x) = \sum_{j=i}^{n} (-1)^j \binom{j}{i}\left(\frac{d}{dx}\right)^{j-i} \overline{a_j(x)}.$$

The operator τ^* is called the formal, or Lagrange, adjoint of τ. The bilinear forms $F_1(f, g)$ and $F_0(f, g)$ are given by the formulas

$$F_1(f, g) = \sum_{i,j=0}^{n-1} \alpha_{ij} f^{(i)}(1)\overline{g^{(j)}(1)},$$

$$F_0(f, g) = \sum_{i,j=0}^{n-1} \beta_{ij} f^{(i)}(0)\overline{g^{(j)}(0)},$$

where the coefficients α_{ij} and β_{ij} are calculated readily from the functions $a_i(x)$. We can see, in particular, that

$$\beta_{ij} = \alpha_{ij} = 0 \text{ for } i + j \geq n - 1,$$

$$\beta_{n-1-k,k} = -\alpha_{n-1-k,k} = (-1)^k.$$

Thus, the matrices β_{ij} and α_{ij} are nonsingular subdiagonal matrices, and hence define nonsingular bilinear forms.

If a formal differential operator τ is given, we set up a corresponding unbounded operator T_0 in the Hilbert space $L_2[0, 1]$ as follows:

(a) $\mathfrak{D}(T_0)$ is the set of all C^n functions f defined in $[0,1]$ and vanishing outside some compact subset of the interior of $[0, 1]$.

(b) If $f \in \mathfrak{D}(T_0)$, $T_0 f$ is defined simply as τf.

Our principal analytic problem at this point is to determine the adjoint of T_0. The solution is contained in the following:

LEMMA 5. *The adjoint T_0^* of the operator T_0 is the operator T_1 defined as follows:*

(a) *Its domain is $A^{(n)}$.*

(b) *If $f \in A^{(n)}$, $T_0^* f = \tau^* f$.*

Proof. It follows immediately from Green's formula that $T_1 \subseteq T_0^*$. To prove the opposite inclusion, we proceed by stages.

(a) Consider first an element $z \in L_2$ such that $T_0^* z = 0$. That is, $(z, T_0 y) = 0$ for every $T_0 y$ in the range of T_0. We shall show that $z \in C^n$. Let Σ be the n-dimensional space of solutions of $\tau^*\sigma = 0$. We shall show that if $f \in L_2$ is orthogonal to Σ, then $(f, z) = 0$. Since Σ is finite dimensional and hence closed, we shall be able to conclude that $z \in \Sigma$, which will give us the desired result $z \in C^n$.[5] We begin by proving the somewhat weaker statement contained in:

SUBLEMMA 5. *If*

(a) f *is orthogonal to* Σ,

(b) $f \in C^n$,

(c) $f(x) \equiv 0$ *outside some compact subset of* $(0, 1)$,

then f *is orthogonal to* z.

Proof. We know by the standard theory of ordinary differential equations that the equation $\tau\hat{f} = f$ has a unique solution $\hat{f} \in C^n$ which satisfies the boundary conditions

$$0 = \hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(n-1)}(0).$$

If we can only verify that $\hat{f}(x) \equiv 0$ outside some closed subinterval of $(0, 1)$, we will know that $\hat{f} \in \mathfrak{D}(T_0)$, so that $f = T_0\hat{f}$, and therefore $(f, z) = 0$. Now $\hat{f}$ is, in some interval $[0, \epsilon]$, the unique solution of the equation $\tau\hat{f} = 0$ satisfying the boundary conditions

$$0 = \hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(n-1)}(0) = 0.$$

Hence $\hat{f}(x) \equiv 0$ in $[0, \epsilon]$. We could apply the same argument to an interval $[1 - \epsilon, 1]$, if only we knew that

$$0 = \hat{f}(1) = \hat{f}'(1) = \cdots = \hat{f}^{(n-1)}(1),$$

and it is this which we propose to verify. This we can do as follows: let $\sigma \in \Sigma$. Then we have, from Green's formula,

$$0 = \int_0^1 \tau\hat{f}(x)\sigma(x)\,dx - \int_0^1 \hat{f}(x)\overline{\tau^*\sigma(x)}\,dx$$

$$= F_1(\hat{f}, \sigma) - F_0(\hat{f}, \sigma) = F_1(\hat{f}, \sigma).$$

That is, $F_1(\hat{f}, \sigma) = 0$ for every $\sigma \in \Sigma$. Since there exists a $\sigma \in \Sigma$ with any preassigned values

$$\sigma(1), \sigma'(1), \cdots, \sigma^{n-1}(1),$$

[5] It may be noted that the method of proof of this lemma is actually that adapted to proving the following result:

THEOREM. *Let a distribution* δ *satisfy an ordinary linear differential equation with* C^∞ *coefficients. Then* δ *is itself a* C^∞ *function.*

In connection with this proof, see [9, Theorem 1.1], where the same result is proved by a different method.

it follows that

$$\hat{f}(1) = \hat{f}'(1) = \cdots = \hat{f}^{(n-1)}(1) = 0,$$

by the nonsingularity of the form $F_1(\hat{f}, \sigma)$. This concludes the proof of the sublemma.

Now we must show that hypotheses (b) and (c) of the sublemma can be dropped without invalidating the conclusion. Indeed, let f be a function which is orthogonal to Σ. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be an orthonormal basis for Σ. Then, by approximating σ_i sufficiently closely by a C^n function ϕ_i which vanishes outside a compact subinterval of $(0, 1)$, we can ensure that the matrix $(\phi_i, \sigma_j) = m_{ij}$ is nonsingular. Now, let f be approximated by a sequence f_k of C^n functions which vanish outside a closed subinterval of $(0, 1)$. Then, if $\hat{m}_{ij}$ is the inverse matrix of m_{ij},

$$\hat{f}_k = f_k - \sum_{j=1}^{n} \sum_{l=1}^{n} (f_k, \sigma_j)\hat{m}_{jl}\phi_l$$

is a sequence of C^n functions orthogonal to Σ which vanish outside a compact subinterval of $(0, 1)$, and such that $\lim_{k\to\infty} f_k = f$. Since, by the sublemma, $(\hat{f}_k, z) = 0$, we are able to conclude that $(f, z) = 0$.

To complete the proof of Lemma 5 it still remains to consider the case $T_0^* z = g$, where $g \neq 0$ and $g \in L_2$, and to show that $z \in A^n$. We know by the standard theory of ordinary differential equations that there exists a solution $z_1 \in A^n$ of the equation $\tau^* z_1 = g$. Now, as remarked at the beginning of the proof of Lemma 5, $z_1 \in \mathfrak{D}(T_0^*)$. Hence

$$T_0^*(z - z_1) = 0.$$

By what we have already proved, $z - z_1 \in C^n$ and

$$\tau^*(z - z_1) = 0.$$

Hence it follows that $z \in A^n$ and that

$$\tau^* z = \tau^* z_1 = g = T_0^* z.$$

Thus the proof of Lemma 5 is complete.

Lemma 5 has as a consequence an interesting topological property of our formal differential operators, expressed in:

LEMMA 6. *Suppose that f_m is a sequence of elements of A^n, and that f_m and τf_m converge (weakly) in the topology of $L_2[0,1]$. Then f_m converges (weakly) in the topology of $A^n[0,1]$ (and conversely).*

Proof. Let us introduce a norm in $\mathfrak{D}(T_0^*)$ in two ways:

$$|f|_1 = \left\{\int_0^1 |f(x)|^2\,dx\right\}^{1/2} + \left\{\int_0^1 |\tau^* f(x)|^2\,dx\right\}^{1/2},$$

$$|f|_2 = |f|_1 + \max_{0\le x\le 1}\ \max_{0\le i\le n-1} |f^{(i)}(x)|.$$

Then, since T_0^* is closed, $\mathfrak{D}(T_0^*)$ is complete in the first norm. On the other hand, it follows from this that $\mathfrak{D}(T_0^*)$ is complete in the second norm. Since $|f|_2 \le |f|_1$, it follows from a well-known principle in the theory of Banach spaces [7, Theorem 11.7] that $|f|_1$ and $|f|_2$ are equivalent. On the other hand, it is evident on inspection that $|f|_2$ and the norm introduced for A^n determine the same topology. Hence it follows that $|f|_1$ determines the same topology in A^n as the norm of A^n, and this proves our lemma.

On the basis of these two lemmas we can proceed systematically to set up the exact operator theory of differential operators. We first make:

DEFINITION 2. Let τ be a formal differential operator of order n, and let

$$(5)\qquad A_j(f) = \sum_{i=0}^{n-1} A_{ji} f^{(i)}(0) + \sum_{i=0}^{n-1} \hat{A}_{ji} f^{(i)}(1) = 0, \qquad (j = 1 \cdots k)$$

be a set k linear boundary conditions. Then we define an operator T in $L_2[0,1]$ by putting:

$$\text{(a)}\ \mathfrak{D}(T) = \left\{ f \in A^n \,\middle|\, \sum_{i=0}^{n-1} A_{ji} f^{(i)}(0) + \sum_{i=0}^{n-1} \hat{A}_{ij} f^{(i)}(1) = 0,\ j = 1, \cdots, k \right\}.$$

(b) If $f \in \mathfrak{D}(T)$, $Tf = \tau f$.

Then T is said to be the differential operator determined by the formal operator τ and the boundary conditions [3]. Any such operator is called a *differential operator.*

LEMMA 7. *Any differential operator T is a closed operator in Hilbert space with a dense domain. Moreover, the range of T is closed.*

Proof. Let $f_n \longrightarrow f$, $Tf_n \longrightarrow g$. Then, by Lemma 6, we have $f \in A^n$, $f_n \longrightarrow f$ in the topology of A^n. It is then evident that f satisfies the boundary conditions which define T, so that $f \in \mathfrak{D}(T)$. Moreover, if T is defined by the formal operator τ, we have $\tau f_n \longrightarrow \tau f$ in the topology of L_2, so that $Tf = \tau f = g$; thus T is closed.

Let T_1 be the differential operator defined by the formal operator τ and the boundary conditions

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = f(1) = f'(1) = \cdots = f^{(n-1)}(1) = 0.$$

Then T is an extension of T_1. Now, it is clear that the differential operator T defined by the boundary conditions $A_j(f) = 0$ will remain the same if we drop from our list of conditions all A_j which are linear combinations of A_k with $k < j$. Hence, without loss of generality, we can suppose that the vectors

$$[A_{j0} \cdots A_{j(n-1)} \hat{A}_{j0} \cdots \hat{A}_{j(n-1)}]$$

form a linearly independent set. Thus, we can find a finite set of functions $\phi_1, \phi_2, \cdots, \phi_k \in \mathfrak{D}(T)$ such that

$$A_j(\phi_i) = \delta_{ji}.$$

It follows that

$$\mathfrak{D}(T) = \mathfrak{D}(T_1) + S,$$

where S is the finite dimensional space generated by the vectors ϕ_i ($i = 1, 2, \cdots, k$). Hence, if $\mathfrak{R}(T)$ denotes the range of T, we have

$$\mathfrak{R}(T) = \mathfrak{R}(T_1) + \hat{S},$$

where $\hat{S}$ is a finite dimensional space. Hence, we have only to show that $\mathfrak{R}(T_1)$ is closed. Now, suppose that $\mathfrak{R}(T_1)$ is not closed. Then there exists an element g and a sequence $f_n \in \mathfrak{D}(T_1)$ such that $T_1 f_n \longrightarrow g$, but $g \notin \mathfrak{R}(T_1)$. It then follows from the closure of the operator T_1 that f_n does not converge. Hence there exists $\epsilon > 0$ and a sequence m_i, n_i of indices approaching infinity such that

$$|f_{m_i} - f_{n_i}| > \epsilon.$$

Putting

$$f_{m_i} - f_{n_i} = g_i,$$

we have $|g_i| > \epsilon$, $T_1 g_i \longrightarrow 0$. If we then put

$$\hat{g}_i = \hat{g}_i / |g_i|,$$

we have $|\hat{g}_i| = 1$, $T_1 g_i \longrightarrow 0$. A subsequence of $\hat{g}_i$ converges weakly: we can suppose without loss of generality that this subsequence is the sequence $\hat{g}_i$ itself. It then follows by Lemma 6 that $\hat{g}_i$ converges weakly in the topology of A^n, and hence in the topology of C^0. Therefore $\hat{g}_i(x)$ is a uniformly bounded sequence which converges at each x $(0 \leq x \leq 1)$; this implies that $\hat{g}_i$ converges in the topology of $L_2[0,1]$. From the closure of T_1 we find, putting

$$\hat{g} = \lim_{i \to \infty} \hat{g}_i,$$

that $|\hat{g}| = 1$, $T_1 \hat{g} = 0$. But then $\hat{g}$ is a nonzero function in C^n which satisfies the equation $\tau g = 0$ and the boundary conditions

$$\hat{g}(0) = \hat{g}'(0) = \cdots = \hat{g}^{(n-1)}(0) = 0;$$

this contradiction proves Lemma 7.

If we examine the part of the foregoing proof which concerns the operator T_1, we see that we have actually shown:

COROLLARY. *Let T be a differential operator with an inverse T^{-1}. Then T^{-1} is a continuous mapping from the range $\Re(T)$ of T to L_2.*

We strengthen this conclusion in:

LEMMA 8. *Let T be a differential operator with an inverse T^{-1}. Then T^{-1} is a continuous mapping from the range $\Re(T)$ of T into A^n, and a compact mapping from $\Re(T)$ into $L_2[0,1]$.*

Proof. We know that if Tf_n converges, f_n converges. It follows by Lemma 6 that f_n converges in the topology of A^n, proving the first part of the lemma. Now suppose that Tf_n converges weakly: since T^{-1} is continuous, f_n converges weakly. It follows by Lemma 6 that f_n converges weakly in the topology of A^n, and hence in the topology of C^0; so $f_n(x)$ is a uniformly bounded sequence of functions converging at each $x \in [0,1]$. Then it follows that f_n converges in the topology of L_2. Since T^{-1} thus transforms weakly convergent sequences into strongly convergent sequences, T^{-1} is compact.

LEMMA 9. *Let T be the differential operator defined by the formal operator τ of order n and by the boundary conditions* (5). *Then T^* is the differential* *operator T_1 defined by the formal operator τ^* and by a set of boundary conditions*

$$B_i(f) = \sum_{j=0}^{n-1} B_{ij} f^{(j)}(0) + \sum_{j=0}^{n-1} \hat{B}_{ij} f^{(j)}(1) = 0 \qquad (i = 1, 2, \cdots, k')$$

obtained from the conditions (5) *as follows:*

Let $S_i\,[\sigma_i^0 \cdots \sigma_i^{2n-1}]$ $(i = 1 \cdots k')$ be a basis for the set of solutions of the equations

$$A_i(\sigma) = \sum_{j=0}^{n-1} A_{ij}\sigma^j + \sum_{j=0}^{n-1} \hat{A}_{ij}\sigma^{n+j} \qquad (i = 1 \cdots k)$$

derived from equations (5), *and let*

$$F_1(f, g) - F_0(f, g) = \sum_{i,j=0}^{n-1} \{\alpha_{ij} f^{(i)}(1)\overline{g^{(j)}(1)} - \beta_{ij} f^{(i)}(0)\overline{g^{(j)}(0)}\}$$

be the bilinear functional arising in Green's formula (4). *Then*:

$$B_{ij} = \sum_{l=0}^{n-1} \overline{\alpha}_{lj}\, \overline{\sigma}_i^{\,l} \text{ and } \hat{B}_{ij} = -\sum_{l=0}^{n-1} \overline{\beta}_{lj}\overline{\sigma}_i^{\,n+l}.$$

Proof. It follows immediately from Green's formula that $T_1 \subseteq T^*$. To prove the converse, let ϕ_i be a C^n function such that

$$\phi_i^{(j)}(0) = \sigma_i^j, \quad \phi_i^{(j)}(1) = \sigma_i^{j+n-1}.$$

Then $A_m(\phi_i) = 0$ $(m = 1 \cdots k)$, so that $\phi_i \in \mathfrak{D}(T)$. If $f \in \mathfrak{D}(T^*)$, it follows that

$$0 = (T\phi_i, f) - (\phi_i, T^*f) = F_1(\phi_i, f) - F_0(\phi_i, f)$$

$$= \sum_{j=0}^{n-1} B_{ij} f^{(j)}(0) + \sum_{j=0}^{n-1} \hat{B}_{ij} f^{(j)}(1),$$

so that $f \in \mathfrak{D}(T_1)$. From this it follows immediately that $T_1 = T^*$.

LEMMA 10. *Let T be a differential operator, and suppose that for some* *complex* λ *both* $T-\lambda$ *and* $T^*-\bar{\lambda}$ *have an inverse. Then* $T(T^*)$ *is a regular operator,* T *and* T^* *have spectra related by* $\sigma(T)=\bar{\sigma}(T^*)$, *and determine spectral measures* E_1 *and* E_2 *related by* $E_1(\lambda)=E_2^*(\bar{\lambda})$.

In this case, we call T a regular differential operator.

Proof. By Lemma 7 and its corollary, the range of $T-\lambda$ is closed and $(T-\lambda)^{-1}$ is continuous. To show that $(T-\lambda)^{-1}$ is everywhere defined, that is, that

$$\mathfrak{R}(T-\lambda)=H,$$

we have then only to show that no nonzero $z\in H$ is orthogonal to $(T-\lambda)\mathfrak{D}(T)$. However, any such z would satisfy $(T^*-\bar{\lambda})z=0$, and we have ruled out this possibility in our hypothesis. This, together with Lemma 8, proves the first part of our lemma. The remaining parts follow, via the remark after Lemma 1, from the corresponding results for bounded operators, all of which are well known (Cf. [7, Lemma V.4].)

For application to the spectral theory of differential operators we shall need the criterion contained in:

LEMMA 11. *Let* T *be a regular operator in a Banach space* $\mathfrak{X}$ *and let* $\lambda_0\in\sigma(T)$. *Let* $f_1^*, f_2^*, \cdots, f_n^*$ *be a basis for the solutions of* $(T^*-\lambda_0)f=0$,[6] *and let* Σ *be the space of solutions of* $(T-\lambda_0)\sigma=0$. *Then* λ_0 *is a multiple pole of the resolvent* $(T-\lambda)^{-1}$ *if and only if some nonzero* $\sigma\in\Sigma$ *satisfies* $f_i^*(\sigma)=0$ $(i=1,2,\cdots,n)$.

Proof. We can readily see, by Lemma 1 (c), that λ_0 is a multiple pole of the resolvent if and only if there exists a solution g of the equation $(T-\lambda_0)^2g=0$ which is not a solution of $(T-\lambda_0)g=0$; that is, if and only if some nonzero $\sigma\in\Sigma$ is in the range of $(T-\lambda_0)$. Now, if $\sigma=(T-\lambda_0)g$, then

$$f_i^*(\sigma)=f_i^*((T-\lambda_0)g)=(T^*-\lambda_0)f_i^*(g)=0.$$

Conversely, if $f_i^*(\sigma)=0$, then it follows that σ is in the closure of the range of $T-\lambda_0$, and our lemma will be proved once we show that $T-\lambda_0$ has a closed

[6] The general theory of adjoint unbounded operators in a Banach space is discussed more fully in Lemmas 18 and 19 below. It is well to remark, however, that we are faced with the usual confusion as to adjoints in Hilbert space, where, contrary to our practice in other Banach spaces, we make use of the *Hermitian*, rather than the pure Banach-space, adjoint. This has the effect of introducing complex conjugates in many of the Hilbert-space formulas where the corresponding Banach-space formulas do not have complex conjugates. This should not cause any essential difficulty to the reader.

range. This, however, is easy to show since

$$(T-\lambda_0)\mathfrak{D}(T) = (T-\lambda_0)E(\lambda_0)\mathfrak{D}(T) + (T-\lambda_0)(I-E(\lambda_0))\mathfrak{D}(T)$$

$$= (T-\lambda_0)E(\lambda_0)\mathfrak{D}(T) + (I-E(\lambda_0))\mathfrak{X}.$$

The first space on the right is finite dimensional and the second is closed, so that $(T-\lambda_0)\mathfrak{D}(T)$ is closed.

LEMMA 12. *Let E be a projection of a B-space $\mathfrak{X}$ onto a finite dimensional range, and let $E^*: \mathfrak{X}^* \longrightarrow \mathfrak{X}^*$ be its adjoint. Then, if $\phi_1, \phi_2, \cdots, \phi_n$ is a basis for $E\mathfrak{X}$ we can find a unique basis $\psi_1^*, \psi_2^*, \cdots, \psi_n^*$ of $E\mathfrak{X}^*$ such that $\psi_i^*(\phi_j) = \delta_{ij}$; and then*

$$Ef = \sum_{i=1}^{n} \phi_i\psi_i^*(f) \quad \text{for any} \quad f \in \mathfrak{X}.$$

Proof. Any element Ef can be written uniquely as

$$Ef = \sum_{i=1}^{n} \phi_i\alpha_i(f),$$

where the $\alpha_i(f)$ are linear functionals. If $f_m \longrightarrow f$ and $\alpha_i(f_m) \longrightarrow \alpha_i$, it is clear that $\alpha_i = \alpha_i(f)$. Hence, by the closed graph theorem of Banach spaces [7, Theorem 11.8] the uniquely determined linear functionals α_i are continuous. Hence $\alpha_i(f) = \psi_i^*(f)$ for some $\psi_i^* \in \mathfrak{X}^*$.

From

$$Ef = \sum_{i=1}^{n} \phi_i\psi_i^*(f)$$

it follows readily that

$$E^*\psi^* = \sum_{i=1}^{n} \psi_i^*\psi^*(\phi_i),$$

so that $\psi_1^*, \psi_2^*, \cdots, \psi_n^*$ span $E^*\mathfrak{X}^*$. To see that $\psi_1^*, \psi_2^*, \cdots, \psi_n^*$ are linearly independent, let $\sum_{i=1}^{n} \alpha_i\psi_i^* = 0$; then

$$\alpha_j = \left(\sum_{i=1}^{n} \alpha_i\psi_i^*\right)\phi_j = 0,$$

so that Lemma 12 is completely proved.

As the final lemma of this section, we state a useful elementary principle in the theory of spectral differential operators.

LEMMA 13. *Let T be a spectral differential operator, and let λ_i be an enumeration of the points in $\sigma(T)$. Then, if $f \in \mathfrak{D}(T)$, the "expansion"*

$$\sum_{i=1}^{\infty} E(\lambda_i) f$$

converges unconditionally in the topology of A^n.

Proof. The series $\sum_{i=1}^{\infty} (\lambda_i) f$ certainly converges unconditionally in the topology of L_2. On the other hand, so does the series

$$T\left(\sum_{i=1}^{\infty} E(\lambda_i) f\right) = \sum_{i=1}^{\infty} E(\lambda_i)(Tf).$$

Hence, by Lemma 6, the original series converges unconditionally in the topology of A^n.

6. Application. The second order differential operator. In this section we wish to apply the theory developed up to now to various second order differential operators arising out of the formal differential operator

$$\tau = -\left(\frac{d}{dx}\right)^2 + q(x).$$

Our peturbation theorem, Theorem 1, reduces the study of this operator to the much simpler operator $-(d/dx)^2$. What we need about the latter is summarized, however, in:

LEMMA 14. *The unbounded operator T defined by the formal differential operator $\tau = -(d/dx)^2$ and the boundary conditions*

(6) $\quad f(0) - k_0 f'(0) = 0, \quad f(1) - k_1 f'(1) = 0, \qquad k_0, k_1$ *arbitrary,*

is a spectral operator satisfying all the hypotheses of case (b) *of Theorem* 1.

REMARK. We can also admit the boundary conditions determined by $k_0 = \infty$ and/or $k_1 = \infty$; that is, the conditions $f'(0) = 0$ and $f'(1) = 0$, respectively.

Proof. Since it is easy to treat all special cases in which k_0 or k_1 is zero or infinity by a separate argument much like the argument given below, we shall assume for simplicity that we have none of these special cases to deal with. If we put $\lambda = s^2$, the general solution of the equation

$$-f''(x) - \lambda f(x) = 0$$

is $\sin s(x + \alpha)$, where α is an arbitrary constant. This satisfies the boundary condition at zero if

$$\tan s\alpha = k_0 s,$$

and satisfies the boundary conditions at one if

$$\tan s(1 + \alpha) = k_1 s.$$

Thus, $T - \lambda$ can only fail to have an inverse if $\lambda = s^2$, where s is a root of the equation

$$(7) \qquad \tan s = \frac{(k_1 - k_0)s}{1 + k_0 k_1 s^2} = \frac{cs}{1 + ds^2}, \qquad d \neq 0.$$

It is readily seen by making use of Lemma 9 that T^* is the differential operator defined by τ^* and by the adjoint boundary conditions

$$f(0) - \overline{k}_0 f'(0) = 0; \; f(1) - \overline{k}_1 f'(1) = 0.$$

Thus the adjoint operator $T^* - \overline{\lambda}$ can only fail to have an inverse if $T - \lambda$ fails to have an inverse; that is, if and only if s satisfies (7). Since not every s satisfies (7), it follows immediately from Lemma 10 that T is regular.

Our next task is to locate the zeros of (7) more exactly. Since $\tan s$ is periodic of period π and has only the zero $s = 0$ in its period-strip, it follows readily that (7) has a countable sequence $z_k, z_{k+1}, \cdots$ of zeros which can be numbered in such a way that

$$z_n = n\pi + O(1).$$

From this preliminary estimate we readily obtain the estimate

$$\tan z_n \sim \frac{cn\pi}{1 + d(n\pi)^2} \sim c(dn\pi)^{-1}$$

Hence it follows that

$$z_n = n\pi + c(dn\pi)^{-1} + O(n^{-2}).$$

We thus obtain an enumeration λ_n $(n = k, k+1, \cdots)$[7] of the eigenvalues of T such that

$$\lambda_n = (n\pi)^2 + 2cd^{-1} + O(n^{-1}).$$

Hence, if d_n is the distance from λ_n to the remainder of the spectrum,

$$d_n \sim \pi^2(2n+1),$$

so that

$$\sum_{n=k}^{\infty} d_n^{-2} < \infty .$$

It is evident from the form of the boundary conditions defining our operator that each λ_n can correspond to at most one function ϕ_n (up to a scalar multiple) which satisfies

$$(T - \lambda_n)\phi_n = 0.$$

Thus, if $E(\lambda_n)$ is to be anything but a projection onto a one-dimensional range, λ_n must be a multiple pole of the resolvent. By Lemma 11, the condition for this is $(\phi_n, \psi_n) = 0$, where ψ_n is the (unique) solution of

$$(T^* - \overline{\lambda}_n)\psi_n = 0.$$

Since, however, T^* is defined by the complex-conjugate boundary conditions of those that define T, it is clear that

$$\psi_n(x) = \overline{\phi_n(x)}.$$

Hence, λ_n can only be a multiple pole of the resolvent of T if

$$\int_0^1 (\phi_n(x))^2 dx = 0.$$

Now, we have

$$\phi_n(x) = \sin z_n(x + \alpha_n) = \sin(z_n x + \beta_n),$$

where β_n must be determined so as to satisfy

$$k_0^{-1} z_n^{-1} \sin \beta_n = \cos \beta_n .$$

It follows readily that

[7] Note: k need not be equal to one.

$$\beta_n = \pi/2 - (n\pi k_0)^{-1} + O(n^{-2}),$$

so that

$$\phi_n(x) = \cos(z_n x + \delta_n), \ \delta_n = (n\pi k_0)^{-1} + O(n^{-2}).$$

It follows that

$$\int_0^1 (\phi_n(x))^2\,dx \sim \int_0^1 \cos^2 n\pi x\,dx = \frac{1}{2},$$

so that only a finite set of λ_n can be multiple poles of the resolvent of T. For those λ_n which are simple poles of the resolvent of T, the projection $E(\lambda_n)$ is, by Lemma 12, the operator determined by the integral kernel

$$\hat{\phi}_n(x)\hat{\phi}_n(y) = E_n(x, y),$$

where $\hat{\phi}_n$ is a scalar multiple of ϕ_n, the scalar being chosen so as to make

$$\int_0^1 (\hat{\phi}_n(x))^2\,dx = 1.$$

We have $\hat{\phi}_n = c_n\phi_n$, and a simple computation reveals that

$$c_n = 2^{-1/2} + O(n^{-2});$$

hence it follows that

$$E_n(x, y) = \frac{1}{2}\cos n\pi x \cos n\pi y - \frac{(cd^{-1}x + k_0)}{\sqrt{2}\,n\pi}\sin n\pi x \cos n\pi y$$
$$- \frac{(cd^{-1}y + k_0)}{\sqrt{2}\,n\pi}\sin n\pi y \cos n\pi x + O(n^{-2}),$$

which gives a decomposition of E_n into four terms

$$(8) \qquad E_n = \hat{E}_n + A_n + B_n + \Delta_n.$$

It is now trivial to find a uniform bound for

$$\Big|\sum_{n \in J} E_n\Big|,$$

J an arbitrary finite set of integers, by making use of the decomposition (8).

We have

$$|\sum_{n\in J} \hat{E}_n| \leq 1,$$

since the $\hat{E}_n$ are a family of orthogonal projections. We have

$$|\sum_{n\in J} \Delta_n| \leq M,$$

since

$$|\Delta_n| = O(n^{-2}) \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-2} < \infty.$$

The operators A_n and B_n have the form

$$A_n = \hat{E}_n\hat{A}_n \quad \text{and} \quad B_n = \hat{B}_n\hat{E}_n,$$

where

$$|\hat{A}_n| = O(n^{-1}) \quad \text{and} \quad |\hat{B}_n| = O(n^{-1}),$$

a situation studied above in the proof of part (b) of Theorem 1, where the argument given proves not only the uniform boundedness of $\sum_{n\in J} A_n$, but also, with suitable slight modifications, the law

$$\lim_{n\to\infty} \left|\sum_{m=n}^{\infty} A_m\right| = 0.$$

All that remains to complete the proof of our lemma is a proof that

$$\sum_{i=k}^{\infty} E(\lambda_i) = I.$$

By Lemma 15 below,

$$E_\infty = I - \sum_{i=k}^{\infty} E(\lambda_i)$$

either projects onto an infinite dimensional space or is zero. But,

$$\lim_{m\to\infty} \left|(I - \sum_{n=m}^{\infty} E(\lambda_n)) - (I - \sum_{n=m}^{\infty} \hat{E}_n)\right| = 0.$$

Hence, by Lemma 4,

$$I - \sum_{n=m}^{\infty} E(\lambda_n)$$

has a finite dimensional range for all sufficiently large m, and hence, a fortiori, E_∞ has a finite dimensional range.

THEOREM 2. *Let T be the unbounded differential operator defined by the formal differential operator $\tau = -(d/dx)^2$ and the boundary conditions*

$$(9) \qquad f(0) - k_0 f'(0) = 0 \quad f(1) - k_1 f'(1) = 0,$$

where k_0 and k_1 are arbitrary, possibly infinite, complex numbers. Then if B is an arbitrary bounded operator, $T + B$ is a spectral operator.

Proof. This follows from Lemma 14 and Theorem 1.

COROLLARY 1. *Let T be the unbounded differential operator defined by the formal differential operator*

$$\tau = -\left(\frac{d}{dx}\right)^2 + q(x)$$

and by the boundary conditions (9), where $q(x) \in C^\infty$.[8] *Then T is a spectral operator.*

This corollary is the "convergence in mean" form of the theorem of Birkhoff-Hilb. As far as pointwise convergence is concerned, we can state:

COROLLARY 2. *Let T be as in Corollary 1, and let $f \in \mathfrak{D}(T)$. Then if λ_i is an enumeration of $\sigma(T)$, the series*

$$\sum_{i=1}^{\infty} E(\lambda_i) f$$

converges unconditionally in the topology of A^2.[9]

Proof. This follows immediately from Corollary 1 and Lemma 13.

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[8] This much is what we have proved explicitly. But, with a little more "analytic care," we would see that it is sufficient that $q(x)$ be measurable and bounded.

[9] We shall see (Corollary 2 of Theorem 3) that this series converges to f.

It may be noted, moreover, that Theorem 1 and Lemma 14 yield a much wider class of spectral operators than the analytic method of Birkhoff-Hilb. For instance, the differential-difference operator

$$\tau f(x) = \left(\frac{d}{dx}\right)^2 f(x) + q(x) f(x+\alpha)$$

(in which $x + \alpha$ is understood to be taken modulo 1, and $q(x)$ is bounded and measurable), with appropriate boundary conditions, is immediately seen to be spectral, as is the integro-differential operator

$$\tau f(x) = \left(\frac{d}{dx}\right)^2 f(x) + \int_0^1 K(x,y) f(y)\, dy,$$

provided only that the integral kernel K defines a bounded operator.

7. Theorems on the spectral measure of infinity. Suppose that T is an unbounded regular spectral operator in a Banach space $\mathfrak{X}$, and that $\{\lambda_i\}$ is its spectrum. Let $E(\lambda_i)$ be the associated spectral measure. Then we put

$$E(\infty) = I - \sum_{i=1}^{\infty} E(\lambda_i).$$

It is clear that $E(\infty) f = f$ if and only if

$$E(\lambda_i) f = 0, \qquad \text{for } 1 \leq i < \infty.$$

This leads us to the following more general:

DEFINITION 3. If T is an unbounded regular operator in the Banach space $\mathfrak{X}$, with spectrum $\{\lambda_i\}$ and spectral measure $E(\lambda_i)$, we put

$$S_\infty(T) = \{f \mid E(\lambda_i) f = 0, \quad 1 \leq i < \infty\}.$$

LEMMA 15. *The space $S_\infty(T)$ either is infinite dimensional or consists only of zero.*

Proof. We can suppose without loss of generality that $0 \notin \sigma(T)$, and put $U = T^{-1}$. It then follows by the remark following Lemma 1 that

$$\sigma(U) = \{\lambda_i^{-1}\} \cup \{0\},$$

and that the spectral measure $\hat{E}$ of U is defined by

$$\hat{E}(\lambda_i^{-1}) = E(\lambda_i).$$

Hence, if $f \in S_\infty = S_\infty(T)$, we have

$$\hat{E}(\lambda_i^{-1})Uf = U\hat{E}(\lambda_i^{-1})f = 0,$$

so that $US_\infty \subseteq S_\infty$. Moreover, by [**15**, Theorem 8.2c], $(U-\lambda)^{-1}f$ is regular at every point λ_i^{-1} if $f \in S_\infty$; thus if $f \in S_\infty$, $(U-\lambda)^{-1}$ has no singularity other than the origin. Hence U, regarded as an operator in S_∞, is quasi-nilpotent. If S_∞ were finite dimensional, it would follow that for some finite k, $U^k S_\infty = 0$. Since U has the inverse T, this would imply that S_∞ contained no nonzero vector.

LEMMA 16. *The space $S_\infty(T)$ is the set of all $f \in \mathfrak{X}$ for which $(T-\lambda)^{-1}f$ is an entire function of λ.*

Proof. If $(T-\lambda)^{-1}f$ is entire, then if we let C be a small circle around λ_i we find that

$$0 = \frac{1}{2\pi i}\int_C (T-\lambda)^{-1} f d\lambda = -E(\lambda_i)f$$

Conversely, if $E(\lambda_i)f = 0$, it follows from [**15**, Theorem 8.2c] that $(T-\lambda)^{-1}f$ is regular at λ_i. Since this holds for every $\lambda_i \in \sigma(T)$, it follows that $(T-\lambda)^{-1}f$ is entire.

LEMMA 17. *Let T be a regular spectral operator in a Banach space $\mathfrak{X}$. Suppose that all but a finite number of the poles μ_i of the resolvent function $(T-\lambda)^{-1}$ are simple, and that $S_\infty(T) = 0$. Let*

$$d_i = \operatorname{dist}(\mu_i, \sigma(T)),$$

and let B be bounded.

(a) *If $d_i \longrightarrow \infty$, $T + B$ is regular.*

(b) *If $\underline{\lim}_{i\to\infty} d_i > 0$, there exists an $\epsilon > 0$ such that $T + B$ is regular whenever $|B| < \epsilon$.*

(c) *If $\underline{\lim}_{i\to\infty} d_i > 0$ and B is compact, $T + B$ is regular.*

Proof. This lemma is needed to make the statement of Theorem 3 below plausible and possible. The proof results incidentally from the proof of Theorem

3, so that it is not necessary to give the details here.

THEOREM 3. *Let T be a regular spectral operator in the Banach space $\mathfrak{X}$. Suppose that all but a finite number of the points in $\sigma(T)$ are simple poles of the resolvent function $(T-\lambda)^{-1}$ and that $S_\infty(T)=0$. Let U_i be a sequence of bounded domains with $\bigcup_{i=1}^{\infty} U_i$ the entire plane, and put V_i = boundary (U_i); let*

$$V_i \cap \sigma(T) = \phi \text{ and } d_i = \operatorname{dist}(V_i, \sigma(T));$$

and let B be a bounded operator.

(a) *If $d_i \longrightarrow \infty$, $S_\infty(T+B)=0$.*

(b) *If $\varliminf_{i\to\infty} d_i > 0$, there exists an $\epsilon > 0$ such that $S_\infty(T+B)=0$ whenever $|B| \leq \epsilon$.*

(c) *If $\varliminf_{i\to\infty} d_i > 0$, and B is compact, $S_\infty(T+B)=0$.*[10]

Proof. We first show that if $\mu_1, \mu_2, \cdots, \mu_N$ is a finite set of points in the plane, we can find a domain U containing all of them such that V = boundary (U) has a minimum distance from $\sigma(T)$ greater than $d = 1/2 \varliminf d_i$ (or, in case (a), greater than an arbitrarily prescribed d) and such that the minimum distance from V to μ_i is greater than a constant D which may be as large as we please. This is done as follows: we take j_0 so large that

$$d_i > \frac{1}{2} \varliminf_{k\to\infty} d_k \qquad \text{if } i \geq j_0,$$

and let K be a prescribed very large closed circular domain. Put

$$K' = K \cup \bigcup_{i=1}^{j_0} \overline{U_i},$$

and let $U_1, U_2, \cdots, U_M$ be a covering of K'. Then we have only to take

$$U = \bigcup_{i=1}^{M} U_i.$$

[10] It would be interesting to know that in case (b) of Theorem 3 we can dispense with the restriction $|B| < \epsilon$, but I do not know whether or not this is possible.

Now, let $f \in S_\infty(T+B)$, and let

$$f(\lambda) = (T + B - \lambda)^{-1} f.$$

We shall show that the entire function $f(\lambda)$ is uniformly bounded, so that $f(\lambda)$ is constant, $f(\lambda) = g$, and hence $f = (T + B - \lambda) g$ for all λ. From this it is evident that $g = 0$, so that $f = 0$. To demonstrate the uniform boundedness we proceed as follows: Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the set of all multiple poles of the resolvent, and let Λ be an arbitrary point in the complex plane. Take, in the first part of this proof,

$$\mu_1, \mu_2, \cdots, \mu_N = \Lambda, \lambda_1, \cdots, \lambda_n.$$

Then, by Lemma 3, there exists an absolute constant c such that $|R_\lambda| < cd^{-1}$ for $\lambda \in V$, where $R_\lambda = (T - \lambda)^{-1}$. If we put

$$\overline{R}_\lambda = (T + B - \lambda)^{-1},$$

we have (cf. formula (1) in the proof of Theorem 1)

$$\overline{R}_\lambda = (I + R_\lambda B)^{-1} R_\lambda.$$

Hence, if

$$|B| \leq c^{-1} d(1 - \delta)$$

with $\delta > 0$, $\overline{R}_\lambda$ exists for $\lambda \in V$, and

$$|\overline{R}_\lambda| < \delta^{-1} cd^{-1}.$$

But then

$$|\overline{R}_\lambda f| \leq \delta^{-1} cd^{-1} |f|$$

for $\lambda \in V$, so that, by the maximum modulus principle,

$$|\overline{R}_\lambda f| \leq \delta^{-1} cd^{-1} |f|$$

everywhere in U. Hence we have

$$|f(\Lambda)| = |\overline{R}_\Lambda f| \leq \delta^{-1} cd^{-1} |f|;$$

that is, $f(\lambda)$ is uniformly bounded. This proves Theorem 3 in cases (a) and (b).

To handle case (c), we observe that since $\sum_{i=1}^{N} E(\lambda_i)$ converges strongly to I, $\sum_{i=1}^{N} E(\lambda_i)f$ converges to f uniformly as f ranges over any compact subset of $\mathfrak{X}$. Since we now assume that B is compact, it follows that $\sum_{i=1}^{N} E(\lambda_i)B$ converges to B in the uniform topology of operators. We choose N_0 so large that

$$\left| B - \sum_{i=1}^{N_0} E(\lambda_i)B \right| \leq c^{-1} d(1 - \delta).$$

Then, if we put

$$C = B - \sum_{i=1}^{N_0} E(\lambda_i)B,$$

we have

$$\overline{R}_\lambda = \left(I + R_\lambda C + \sum_{i=1}^{N_0} R_\lambda E(\lambda_i)B \right)^{-1} R_\lambda .$$

However, if d_1 is the minimum distance from λ to any of the points λ_i, it follows by the discussion of the functional calculus of T preceding Lemma 3 that there exists an absolute constant c_1 such that

$$|R_\lambda E(\lambda_i)| \leq c_1 d_1^{-1}$$

for $1 \leq i \leq N_0$ and for d_1 sufficiently large. We now determine the domain U of the first paragraph of this proof by putting

$$\mu_1, \mu_2, \cdots, \mu_N = \Lambda, \lambda_1, \cdots, \lambda_{N_1},$$

where $N_1 \geq N_0$ is so large that the set $\lambda_1, \lambda_2, \cdots, \lambda_{N_1}$ includes all the multiple poles of the resolvent, and where

$$D = 2|B| N_0 c_1 \delta^{-1}.$$

It then follows, as in the proof of parts (a) and (b) of Theorem 3, that $\overline{R}_\lambda$ exists for $\lambda \in \bar{V}$, and that

$$|\overline{R}_\lambda| < z\delta^{-1} cd^{-1};$$

from this point on we can argue just as in cases (a) and (b).

Thus all cases of Theorem 3 are proved.

COROLLARY 1. *Under the hypotheses of Theorem* 1, $T + B$ *is a spectral operator such that* $S_\infty(T + B) = 0$.

Proof. We choose the domains U_i of Theorem 3 as follows: If i is even, $i = 2n$, we take U_i to be the interior of a circle of radius d_i about the point λ_i, where d_i is the distance from λ_i to the rest of $\sigma(T)$. If i is odd, $i = 2n + 1$, we take U_i to be the set of all points z with $|z| < n$ but $|z - \lambda_i| > d_i/4$ for all i.

COROLLARY 2. *If* T *is the differential operator of Theorem* 2, *and* B *is bounded, then every function* $f \in L_2[0, 1]$ *can be expanded in a series of eigenfunctions* (*including, possibly, a finite number of solutions of equations of the type*

$$(T + B - \lambda)^k f = 0)$$

of $T + B$ *which converges unconditionally in the topology of* L_2. *Any function of class* A^2 *which satisfies the appropriate boundary conditions can be expanded in a series of eigenfunctions converging unconditionally in the topology of* A^2.

Theorem 3 also applies to a class of operators which are not necessarily spectral. To discuss this class of operators, we shall first extend the elementary theory of the adjoint from closed operators in Hilbert space to closed operators in an arbitrary reflexive Banach space. If $\mathfrak{X}$ is a reflexive Banach space, so is the direct sum $\mathfrak{X} \oplus \mathfrak{X}$ (in any suitable norm), and we have evidently

$$(\mathfrak{X} \oplus \mathfrak{X})^* = \mathfrak{X}^* \oplus \mathfrak{X}^*.$$

The space $\mathfrak{X} \oplus \mathfrak{X}$ admits the evident automorphisms

$$A_1 : (x, y) \longrightarrow (y, x),$$

$$A_2 : (x, y) \longrightarrow (-y, x).$$

We have

$$A_1^2 = -A_2^2 = I, \quad A_1 A_2 = -A_2 A_1.$$

If M is a closed manifold in a Banach space Y, its annihilator $M^\perp$ is the closed subspace of Y^* defined by

$$M^\perp = \{y^* \in Y^* \mid y^*(M) = 0\}.$$

If Y is reflexive, we have evidently $M^{\perp\perp} = M$. If T is a linear transformation in

$\mathfrak{X}$ (Note: we continue to suppose that $\mathfrak{D}(T)$ is dense in $\mathfrak{X}$.), its graph $\Gamma(T)$ is the subset of $\mathfrak{X} \oplus \mathfrak{X}$ defined by

$$\Gamma(T) = \{(x, Tx) \mid x \in D(T)\}.$$

Clearly, $\Gamma(T)$ is closed if and only if T is closed. We have evidently

$$\Gamma(T^{-1}) = A_1\Gamma(T),$$

whenever T^{-1} is defined (or, equivalently, whenever $A_1\Gamma(T)$ is the graph of a single-valued operator). We define the closed linear operator T^* in $\mathfrak{X}^*$ by putting

$$\Gamma(T^*) = [A_2\,\Gamma(T)]^{\perp}.$$

The operator T^* is single valued, since $(0, y^*) \in \Gamma(T^*)$ is equivalent to $y^*(x) = 0$ for all $x \in D(T)$; and since $D(T)$ is dense in $\mathfrak{X}$, this gives $y = 0$. It may also be remarked that if T is bounded, this definition of T^* agrees with the usual one.

LEMMA 18. (a) *$D(T^*)$ is dense.*

(b) $T^{**} = T$.

(c) *T and T^* have both bounded inverses if either does, and* $(T^{-1})^* = (T^*)^{-1}$.

(d) *If B is a bounded operator,* $(T + B)^* = T^* + B^*$.

Proof. The proofs are exactly like those in the Hilbert-space case. If $D(T^*)$ is not dense, we can find an $x \in \mathfrak{X}$ such that

$$xD(T^*) = 0,$$

while $x \neq 0$. Then

$$A_2(0, x) = (-x, 0) \in \Gamma(T^*) = A_2\Gamma(T),$$

so that

$$(0, x) \in A_2^2\,\Gamma(T) = \Gamma(T),$$

and hence $x = T(0) = 0$, a contradiction. This proves (a).

To prove (b), we observe that

$$\Gamma(T^{**}) = (A_2\Gamma(T^*))^{\perp} = A_2(\Gamma(T^*))^{\perp} = A_2(A_2\,\Gamma(T)) = -\,\Gamma(T).$$

To prove (c), we observe that

$$\Gamma((T^*)^{-1}) = A_1\,\Gamma(T^*) = A_1\,(A_2\Gamma(T))^{\perp} = (A_2(A_1\Gamma(T)))^{\perp} = \Gamma((T^{-1})^*).$$

Thus $(T^*)^{-1} = (T^{-1})^*$ even if either or both of the transformations are unbounded, multi-valued, or not everywhere defined, so that (c) follows as a special case.

To prove (d) we note that it is evident that

$$\Gamma(T^* + B^*) \subseteq \Gamma((T + B)^*).$$

On the other hand, if $x^* \in D((T + B)^*)$, so that

$$x^*((T + B)y) = (T + B)^* x^*(y)$$

for every $y \in D(T)$, we have clearly

$$x^*(Ty) = \{(T + B)^* x^* - B^* x^*\}(y)$$

for $y \in D(T)$; thus $x^* \in D(T^*)$, and

$$T^* x^* + B^* x^* = (T + B)^* x^*.$$

LEMMA 19. (a) *If one of T and T^* is regular, both are.*

(b) *We have $\sigma(T) = \sigma(T^*)$.*

(c) *If T and T^* are regular, their spectral measures E and $\hat{E}$ are related by $\hat{E}(\lambda) = E^*(\lambda)$.*

(d) *If T and T^* are regular and one is spectral, so is the other.*

Proof. By Lemma 18 (c) and (d), we have

$$((T - \lambda)^{-1})^* = ((T - \lambda)^*)^{-1} = (T^* - \lambda)^{-1}$$

with both sides of this equation existing as bounded operators for exactly the same λ. This proves (b) and (a), since for bounded operators U and U^* are either both compact or both not compact.

To prove (c), we note that $E(\lambda)$ may be characterized as

$$E(\lambda) = -\frac{1}{2\pi i}\int_C (T - \lambda)^{-1}\,d\lambda,$$

where C is a sufficiently small circle about λ. But then

$$E^*(\lambda) = -\frac{1}{2\pi i}\int_C (T^* - \lambda)^{-1} d\lambda = \hat{E}(\lambda)$$

is evident. However, since (d) follows immediately from (c), Lemma 19 is entirely proved.

Suppose that T is a regular operator in $\mathfrak{X}$. Then by sp(T), the *spectral span* of T, we denote the smallest closed manifold containing all the manifolds $E(\lambda)\mathfrak{X}$. Thus, $x \in \text{sp}(T)$ if and only if x can be approximated by linear combinations of solutions f of equations

$$(T - \lambda)^k f = 0,$$

that is, by generalized eigenvectors of T. Thus, if T is known to be a regular spectral operator,

$$\text{sp}(T) = \left(\sum_{i=1}^{\infty} E(\lambda_i)\right)\mathfrak{X}.$$

For nonspectral regular operators in a reflexive space, however, we may state:

LEMMA 20. *If T is a regular operator in the (reflexive) Banach space $\mathfrak{X}$, then $sp(T) = S_\infty(T^*)^\perp$.*

REMARK. For spectral operators, the conditions

$$\text{sp}(T) = \mathfrak{X} \text{ and } S_\infty(T) = 0$$

are clearly equivalent; but for nonspectral operators the condition for sp$(T) = \mathfrak{X}$ given by the lemma is $S_\infty(T^*) = 0$ and not $S_\infty(T) = 0$. Indeed, H. Hamburger [**10**, pp. 74-79] has constructed an example of a compact operator U in Hilbert space $\mathfrak{X}$ whose generalized eigenvectors span $\mathfrak{X}$, and which is such that an infinite dimensional closed subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ exists such that $U\mathfrak{X}_0 \subseteq \mathfrak{X}_0$, and U is quasi-nilpotent in $\mathfrak{X}_0$. If we put $T = (U^*)^{-1}$, we have sp$(T) \neq \mathfrak{X}$, while $S_\infty(T) = 0$.

Proof of Lemma 20. It is clear that if $\lambda \in \sigma(T)$ and we have

$$E(\lambda)f = f, \text{ while } E(\mu)^* g^* = 0 \text{ for every } \mu \in \sigma(T) = \sigma(T^*),$$

then

$$g^*(f) = g^*(E(\lambda)f) = E(\lambda)^* g^*(f) = 0.$$

Thus, it is clear that $\mathrm{sp}(T) \subseteq S_\infty(T^*)^\perp$. Conversely, if $f \notin \mathrm{sp}(T)$, there exists a functional $g^* \in \mathfrak{X}^*$ such that

$$g^*(f) = 1, \quad g^*(\mathrm{sp}(T)) = 0.$$

Since $g^*(E(\lambda) f') = 0$ for any $f' \in \mathfrak{X}$ and any $\lambda \in \sigma(T)$, it follows that

$$E(\lambda)^* g^* = 0 \text{ for every } \lambda \in \sigma(T) = \sigma(T^*).$$

Thus $g^* \in S_\infty(T^*)$; and since $g^*(f) = 1$, it follows that $f \notin S_\infty(T^*)^\perp$.

Lemma 20 and Theorem 3 together give us a fairly general insight into the range of situations in which a "spectral density" property $\mathrm{sp}(T) = \mathfrak{X}$ is to be expected of an operator T. However, in applying these results it is convenient to be able to deal, wherever possible, with solutions of the equation $(T - \lambda) f = 0$, rather than with solutions of the equation $(T - \lambda)^k f = 0$. The next lemma describes a simple case in which this is possible.

LEMMA 21. *Let T be a regular spectral operator in the Banach space $\mathfrak{X}$. Suppose that all but a finite number of the countable set $\{\lambda_n\}$ of points in $\sigma(T)$ are simple poles of the resolvent function and correspond to one-dimensional eigenspaces. Let d_n be the minimum distance from λ_n to the other points in $\sigma(T)$. Then all but a finite number of points in $\sigma(T + B)$ are simple poles corresponding to one-dimensional eigenspaces if*

(a) *d_i approaches infinity, and B is bounded; or*

(b) *$\underline{\lim}_{i \to \infty} d_i > 0$, and $|B|$ is less than some positive constant $\epsilon(T)$; or*

(c) *$\underline{\lim}_{i \to \infty} d_i > 0$, and B is compact.*

Proof. The proof in each of these three cases is very much like the proof in the corresponding case of Theorem 3. We shall show that there exists an N such that $\mu \in \sigma(T + B)$ and $|\mu| \geq N$ imply that μ is a simple pole of the resolvent $\overline{R}_\lambda$ of $T + B$ and corresponds to a one-dimensional eigenspace. Indeed, if $\lambda \in \sigma(T + B)$, there exists an $f \in \mathfrak{X}$ such that $|f| = 1$ and such that

$$(T + B - \lambda) f = 0, \text{ so that } (T - \lambda) f = -Bf.$$

From this last equation it is evident that if $(T - \lambda)^{-1} = R_\lambda$ exists, it must have a norm which is at least $|B|^{-1}$. By Lemma 3, there exists an absolute constant $c = c(T)$ such that

$$|(T - \lambda)^{-1}| \leq c d^{-1}$$

if λ is not within a distance d of any point in $\sigma(T)$, and if N is so great that every multiple pole λ_0 of R_λ satisfies $|\lambda_0| < N$. It follows that every point μ of $\sigma(T+B)$ with $|\mu| \leq N$ is within a distance $c^{-1}|B|$ of a point $\lambda_n \in \sigma(T)$. Moreover, if we suppose that $c^{-1}|B| < d_n/2$ (which covers cases a and b), then we see as in the proof of Theorems 1 and 3 that the resolvent

$$\overline{R}_\lambda = (T + B - \lambda)^{-1}$$

exists everywhere on the circle C_n with center λ_n and radius $d_n/2$, at least if N is chosen to be sufficiently large (or, in case b, for $|B|$ sufficiently small). We have, as usual,

$$\overline{R}_\lambda = (I + R_\lambda B)^{-1} R_\lambda;$$

and, for N sufficiently large (or $|B|$ sufficiently small), this leads, as in the proof of Theorems 1 and 3, to an estimate

$$|E(\lambda_n) - E_n| < \frac{1}{2} |E(\lambda_n)|^{-1}.$$

In this formula, E_n is the sum of all the projections $\overline{E}(\mu)$ for μ interior to C_n, where $\overline{E}$ is the spectral measure corresponding to $T+B$. If follows by Lemma 4 that E_n is a projection onto a one-dimensional subspace, so that there is exactly one point $\mu_n \in \sigma(T)$ interior to C_n, and $\overline{E}(\mu_n)$ is a one-dimensional projection. Since we have already shown that every $\mu \in \sigma(T)$ with $|\mu| \leq N$ must belong to the interior of some C_n, our Lemma is proved in cases (a) and (b).

It is not hard to see that the same argument will work in case (c) as soon as we are able to show that $|R_{\lambda_n'} B| \longrightarrow 0$ if λ_n' is a sequence with $|\lambda_n'| \longrightarrow \infty$, and with

$$\operatorname{dist}(\lambda_n', \sigma(T)) > \epsilon > 0.$$

However, since it is evident from the functional calculus that $R_{\lambda_n'}$ converges strongly to zero as $n \longrightarrow \infty$, and since B is compact, it follows that $|R_{\lambda_n'} B| \longrightarrow 0$ as $n \longrightarrow \infty$. In this way we are able to dispose successfully of case (c), so that Lemma 21 is proved in entirety.

REMARK. It is not hard to see that a proof like that of Lemma 21 will establish the existence of certain cases in which the hypothesis that the resolvent R_λ of T has only simple poles corresponding to one-dimensional eigenspaces will yield the corresponding property for $T+B$, so that we can be sure that not even one pole of the resolvent $\overline{R}_\lambda$ of $T+B$ is multiple. In general, the situation is this: Multiple poles of $\overline{R}_\lambda$ can only arise out of multiple poles of R_λ, or out of simple poles of R_λ which are multiple eigenvalues, or, finally, out of the "fusion" of several poles of T under the influence of the perturbation B. If we rule out the first two causes, and demand that B be too small to move any pole of R_λ far enough to cause two poles of R_λ to meet, we can be sure that $\overline{R}_\lambda$ has only simple poles. On the other hand, it is clear that if R_λ has multiple poles or multiple eigenvalues, no demand that B be small can be strong enough to ensure that $\overline{R}_\lambda$ has no multiple poles.[11]

8. Applications to differential equations. Theorem 1 is usually inapplicable in the theory of partial and singular ordinary differential operators because the very simple behavior of the eigenvalues required in the hypotheses of Theorem 1 ordinarily fails. However, even in these cases, Theorem 3 can often be applied to yield interesting results. Let us begin by considering the ordinary singular differential operator

$$\tau = -\left(\frac{d}{dx}\right)^2 + q(x)$$

on the half-open interval $I = [0, \infty)$, and make the assumption that $q'(x) > 0$, $q(x) \longrightarrow \infty$. Then, as is well known (cf. [**16**, p. 19]), any boundary condition

$$f(0) + kf'(0) = 0 \qquad (0 \le k \le \infty)$$

determines a self-adjoint operator T as follows:

(a) $\mathfrak{D}(T)$ is the set of all functions f which belong to $A^2[0, N]$ for every $N > 0$, such that $\tau f \in L_2$, and such that $f(0) + kf'(0) = 0$.

(b) $Tf = \tau f$ for $f \in \mathfrak{D}(T)$.

Moreover (cf. [**16**, p. 113 and p. 134]), the operator T is without continuous spectrum, and has only a finite number $N(\lambda)$ of eigenvalues (counted with appropriate multiplicities) below any fixed λ. This number is given asymptotically as $\lambda \longrightarrow +\infty$ by the formula

$$N(\lambda) = \int_0^{\mu(\lambda)} (\lambda - q(x))^{1/2},$$

where $\mu(\lambda)$ is the uniquely determined solution of $q(\mu(\lambda)) = \lambda$. This formula makes it easy for us to evaluate

[11] For a detailed discussion of this type of question, c.f. [**18**].

$$c = c(\tau) = \varliminf_{\lambda\to\infty} \lambda^{-1}N(\lambda),$$

and by use of Theorem 3 we are able to state:

THEOREM 4. (a) *If the singular differential operator* τ *is such that* $c(\tau) = +\infty$, *and* T *is the self-adjoint operator in Hilbert space* $\mathfrak{X}$ *associated above with* τ, *then* $\mathrm{sp}(T + B) = \mathfrak{X}$ *for every bounded operator* B.

(b) *If instead of* $c(\tau) = +\infty$ *we have* $c(\tau) > 0$, *then* $\mathrm{sp}(T + B) = \mathfrak{X}$ *for all bounded operators* B *with* $|B| < \epsilon = \epsilon(\tau)$, *and for every compact operator* B.

REMARK. It is easy to see that $\epsilon(\tau) = 1/2c(\tau)$ is an acceptable determination.

Proof. The proof results immediately from Lemma 20 and Theorem 3, the only point in question being the method by which we are to choose the domains U_i of Theorem 3. However, it is clearly possible to choose arbitrarily large real λ_i such that the distance from λ_i to $\sigma(T)$ is not less than $c(\tau)/2$. If we put

$$U_i = \{x + iy \mid x < \lambda_i\},$$

we complete our proof.

The same argument evidently applies to any self-adjoint operator T which is is without continuous spectrum, and for which we have

$$c(T) = \varliminf_{\lambda\to\infty} \lambda^{-1}N(\lambda) > 0,$$

where $N(\lambda)$ is the number of eigenvalues μ (counted with multiplicities and supposed finite) such that $-\lambda \le \mu \le \lambda$. This observation applies to an extensive class of elliptic partial differential operators. Thus, for instance, Hilbert-Courant [**12**, Chap. 6, Theorem 17] gives the value

$$c(T) = (4\pi)^{-1} \iint_G p^{-1}(x, y)\, dxdy$$

for the partial differential operator T defined in terms of the formal operator

$$\tau = -\frac{\partial}{\partial x} p(x,y) \frac{\partial}{\partial x} - \frac{\partial}{\partial y} p(x,y) \frac{\partial}{\partial y} + q(x,y)$$

and in terms of any one of a wide family of boundary conditions. Here, G is a bounded domain whose boundary is of measure zero, and T is an unbounded self-adjoint operator in the Hilbert space $\mathfrak{X} = L_2(G)$. The functions $p(x,y)$ and $q(x,y)$ are required to be real and infinitely differentiable in a neighborhood of the closure of G, while we assume that $p(x,y) > 0$ everywhere on the closure of G. This means, however, that the corresponding partial differential operator $T + B$, defined in terms of the formal operator[12]

$$\tau' = -\frac{\partial}{\partial x}\, p(x,y)\, \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\, p(x,y)\, \frac{\partial}{\partial y} + q(x,y) + iq'(x,y),$$

has the property $\operatorname{sp}(T+B) = \mathfrak{X}$, provided only that $|q'(x,y)| < \epsilon$ for some sufficiently small $\epsilon > 0$.

Many other instances are known in which a self-adjoint formal elliptic operator τ has nonzero constant $c(\tau)$. For instance, Gårding [8] shows that if the domain $G \subseteq E^n$ is bounded, and τ is real, formally self-adjoint, or order m, and has constant coefficients, then we have an asymptotic expression of the form

$$N(\lambda)\lambda^{-n/m} \sim d(\tau).$$

This allows us to apply Theorem 3, case (a) whenever $n < m$, and cases (b) and (c) of Theorem 3 whenever $n = m$.

To apply Theorem 3 when $n > m$, we must proceed in a slightly different way. Let us suppose that T is an unbounded self-adjoint operator without continuous spectrum such that $N(\lambda)$ is finite and

$$\varliminf_{\lambda \to \infty} N(\lambda)\lambda^{-\epsilon} > 0,$$

where $\epsilon > 0$. Then the operator T^k satisfies

$$\lim_{\lambda \to \infty} N_1(\lambda)\lambda^{-1} = \infty$$

for some sufficiently large k ($N_1(\lambda)$ is the number of eigenvalues μ_1 of T^k such

[12] To define exactly the functional domains and boundary conditions involved in the theory of partial differential equations would involve us in a very extensive analytic discussion, which has, after all, nothing to do with our problem, since we can take the same domain for $T + B$ as was required to make T self-adjoint (or, more generally, spectral). This difficulty leads to a slight vagueness in the formulations of the rest of this section, but not to any real lack of rigor in the results.

that $-\lambda \leq \mu_1 \leq \lambda$; that is, the number of eigenvalues μ of T such that $-\lambda^{1/k} \leq \mu \leq \lambda^{1/k}$). Now, if B is a bounded operator such that every product

$$T^{i_1} B^{j_1} T^{i_2} \ldots B^{j_n},$$

with at least one j_i nonzero, is a bounded operator, it follows readily that $(T + B)^k$ satisfies all the hypotheses of Theorem 3. It then follows that

$$\mathrm{sp}((T + B)^k) = \mathfrak{X}.$$

However, from [**15**, Theorem 9.4] it follows readily that

$$\mathrm{sp}(S) = \mathfrak{X} \quad \text{and} \quad \mathrm{sp}(S^k) = \mathfrak{X}$$

are equivalent restrictions on a regular operator S. That is, we can conclude that $\mathrm{sp}(T + B) = \mathfrak{X}$.

To give a concrete example of a case in which this argument applies, we have only to use the result of Gårding, and consider the formal operator $\tau + K$, in which τ is self-adjoint elliptic partial differential operator with constant coefficients in a bounded domain G, and K is an integral operator

$$Kf(x) = \int_G K(x, y) f(y)\, dy$$

in which the kernel is a C^∞ function of both its arguments, defined when each argument is in a neighborhood of the closure of G. We are able to conclude that the appropriate unbounded operators T defined in terms of such formal operators also have the "spectral spanning" property $\mathrm{sp}(T) = \mathfrak{X}$.

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ON A THEOREM OF BEURLING AND KAPLANSKY

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1. Introduction. The object of this paper is to remark that a natural and simple proof of the theorem of Beurling and Kaplansky (Theorem 1 below) can be obtained by adapting to general groups a classical proof already given in the books of Wiener [8] and Zygmund [9]. In fact, Theorem 1 is an immediate consequence of a lemma (Lemma 1 below) which was proved by these authors in the case when the group is the integers or the real numbers. An easy generalization of Lemma 1 (Lemma 2 below) yields immediately the generalization of the Beurling and Kaplansky theorem stated as Theorem 2 below. For the history of the development of this theorem, see [3, p. 149] and [5]; the book [3] did not appear until the present paper had been submitted, but it seemed wise to add the reference.

2. Statement of results. Let $A = \{a, b, \cdots\}$ be a locally compact abelian group and $X = \{x, y, \cdots\}$ the dual group (the group operations will be written multiplicatively). Let

$$L^1(A) = \{f, g, h, p, \cdots\}$$

denote the set of all integrable functions with respect to the Haar measure of A,

$$||f|| = ||f||_1$$

the L^1-norm of f, $\hat{f}(x)$ the Fourier transform of $f(a)$,

$$f_1 * f_2$$

the product of convolution (that is, the product in the group algebra),

$$f_1 f_2 = f_1(a) f_2(a)$$

the ordinary product of functions, and

$$(x, a) = x(a) = a(x)$$

Received January 12, 1953. The author is a fellow of the John Simon Guggenheim Memorial Foundation.

Pacific J. Math. 4 (1954), 459-465

the value of the character $x \in X$ at the point $a \in A$. Subsets of A will be denoted by $C, D, \cdots$, subsets of X by $P, Q, S, \cdots$, and subsets of $L^1(A)$ by $I, J, \cdots$.

The spectrum $S(f)$ of a function $f \in L^1(A)$ is the set of the points $x \in X$ such that $\hat{f}(x) = 0$, and the spectrum $S(I)$ of a set $I \subset L^1(A)$ is the set of the points $x \in X$ such that $\hat{f}(x) = 0$ for all $f \in I$.

We suppose known the following Tauberian theorem of Segal and Godement (see [**1**] or [**4**]).

THEOREM A. *If I is a closed ideal of $L^1(A)$, and $f \in L^1(A)$ is such that $S(I)$ is interior to $S(f)$, then $f \in I$.*

Theorem A is a consequence of the regularity (in the sense of Silov) of the algebra $L^1(A)$, and the following Lemma A (see [**7**], [**1**], or [**4**]).

LEMMA A. *Given $f \in L^1(A)$ and $\epsilon > 0$, there is a function $g \in L^1(A)$ with the following properties:*

(i) *$\hat{f}(x) = 0$ implies $\hat{g}(x) = 0$; that is, $S(f) \subset S(g)$.*

(ii) *If $h = f - g$, then $\hat{h}(x)$ vanishes in a neighborhood of the point ∞ (that is outside of a compact set $P \subset X$).*

(iii) *$\|g\| \leq \epsilon$.*

It is known [**6**] that Theorem A is not true if $S(f)$ is merely contained in but not interior to $S(f)$; however, if $S(I)$ consists of a single point, the following theorem is true:

THEOREM 1 (Beurling and Kaplansky). *If I is a closed ideal such that $S(I)$ consists of a single point x_0, then $S(f) \supset S(I)$ implies $f \in I$.*

This is a special case of the following:

THEOREM 2. *Let I be a closed ideal such that the boundary P of $S(I)$ is a reducible set (or that the intersection of P with the boundary of $S(f)$ is a reducible set). Then $S(f) \supset S(I)$ implies $f \in I$.*

A set is said to be reducible if it contains no nonvoid perfect subsets.

Theorem 1 was proved by Beurling in the case when A consists of the real numbers, using complex-variable methods. Kaplansky proved the theorem in the general case using the structure theory of groups. A direct and simple proof of Theorem 1 is given in a recent paper of Helson [**2**], and in the same paper is given a complete proof of Theorem 2.

We want to show that a still more natural and simple proof of Theorems 1 and 2 can be obtained as follows.

2. Proofs. We first reduce Theorem 1 to the following Lemma 1 (observe that Lemma A is obtained from Lemma 1 by replacing the point x_0 by ∞).

LEMMA 1. *Given a point* $x_0 \in S(f)$, $f \in L^1(A)$, *and* $\epsilon > 0$, *there is a function* $g \in L^1(A)$ *with the following properties*:

(i) $S(f) \subset S(g)$:

(ii) *if* $h = f - g$, *then* $\hat{h}(x)$ *vanishes in a neighborhood* $U(x_0)$ *of the point* x_0;

(iii) $||g|| \le \epsilon$.

It is easy to see that Theorem 1 is an immediate consequence of Lemma 1 and Theorem A. In fact, if $S(I)$ consists of a single point $x_0 \in S(f)$, then by Lemma 1 there is a function h such that $||f-h|| < \epsilon$, and x_0 is interior to $S(h)$; hence, by Theorem A, $h \in I$. Since ϵ is arbitrary and $||f-h|| \le \epsilon$, it follows that $f \in I$, and this proves Theorem 1.

Similarly it is easy to see that Theorem 2 is an immediate consequence of Theorem A, Lemma A, and the following Lemma 2.

LEMMA 2. *Given a compact reducible set* $Q \subset S(f)$, $f \in L^1(A)$, *and* $\epsilon > 0$, *there is a function* $g \in L^1(A)$ *with the following properties*:

(i) $S(f) \subset S(g)$;

(ii) *if* $h = f - g$, *then* $\hat{h}(x)$ *vanishes in a neighborhood* $U(Q)$ of the set Q;

(iii) $||g|| \le \epsilon$.

Hence Theorems 1 and 2 will be proved if we prove Lemmas 1 and 2.

3. Proof of Lemma 1. Without loss of generality we may suppose $x_0 = 1 =$ unit of X. Then by hypothesis

$$\hat{f}(x_0) = \int_A f(a)\, da = 0.$$

Given $\epsilon > 0$, there is a compact set $C \subset A$ such that

$$\int_{A-C} |f(a)|\, da < \epsilon/4, \tag{1}$$

hence also

$$(2) \qquad \left| \int_C f(a)\, da \right| = \left| \int_{A-C} f(a)\, da \right| < \epsilon/4.$$

If $p(a)$ is any function from $L^1(A)$, and $g = p * f$, we have

$$g(a) = \int_A f(b)\, p(ab^{-1})\, db = \int_C + \int_{A-C} f(b)\, p(ab^{-1})\, db,$$

$$(3) \qquad \|g\| \leq \int_A \left| \int_C f(b)\, p(ab^{-1})\, db \right| da$$

$$+ \int_A \left| \int_{A-C} f(b)\, p(ab^{-1})\, db \right| da = M + N.$$

Using (1) and (2), and denoting the characteristic function of the set $C' = A - C$ by $\phi_{C'}$, we have

$$(3a) \qquad N = \int_A \left| \int_A f(b)\, \phi_{C'}(b)\, p(ab^{-1})\, db \right| da$$

$$= \|(f\phi_{C'}) * p\| \leq \|f\phi_{C'}\| \cdot \|p\|$$

$$= \|p\| \cdot \int_{C'} |f(a)|\, da \leq \epsilon/4 \cdot \|p\|,$$

$$(3b) \qquad M \leq \int_A \left| \int_C f(b)\, [p(ab^{-1}) - p(a)]\, db \right| da$$

$$+ \int_A \left| \int_C f(b)\, db \right| |p(a)|\, da$$

$$\leqq \{ \sup_{b \in C} \int_A |p(ab^{-1}) - p(a)|\, da \} \|f\| + \epsilon/4 \|p\|.$$

Let us denote $p(ab^{-1})$ by $p^b(a)$; then

$$(4) \qquad \|g\| \leq \epsilon/2 \|p\| + \|f\| \sup_{b \in C} \|p^b - p\|.$$

Since

$$\hat{g}(x) = \hat{f}(x)\,\hat{p}(x),$$

$\hat{f}(x) = 0$ implies $\hat{g}(x) = 0$, and inequality (4) shows that Lemma 1 will be proved if we prove the following proposition.

PROPOSITION A. *Given* $\epsilon > 0$ *and a compact set* $C \subset A$, *there is a function* $p(a)$ *such that*:

a) $p \in L^1(A)$ *and* $\|p\| \leq 2$;

b) *there is a neighborhood* $U(1)$ *of the point* $1 \in X$ *such that* $\hat{p}(x) = 1$ *for* $x \in U(1)$;

c) $\|p^b - p\| < \epsilon$ *for* b *in the compact set* C.

Proof of Proposition A. Take two compact neighborhoods V and V' of the $1 \in X$, of measures η and η', and such that

$$\overline{V} \subset V'; \; \eta' \leq 4\eta, \tag{5}$$

and define

$$\hat{p}(x) = 1/\eta\,\{\hat{\phi}_V * \hat{\phi}_{V'}\} = 1/\eta\,\{\hat{\phi} * \hat{\phi}'\}, \tag{6}$$

where $\hat{\phi} = \hat{\phi}_V$ $(\hat{\phi}' = \hat{\phi}_{V'})$ is the characteristic function of the set $V\,(V')$. Since $\hat{\phi}$, $\hat{\phi}' \in L^2(X)$, by Plancherel's theorem $\hat{p}(x)$ is the Fourier transform of a function $p(a) \in L^1(A)$. Since $\overline{V} \subset V'$, there is a neighborhood $U = U(1)$ such that $V \cdot U \subset V'$, and from (6) it is clear that $\hat{p}(x) = 1$ for $x \in U$. Using the Plancherel theorem it is easy to see that $p(a)$ satisfies also the conditions a) and c), provided V' is taken small enough (cfr. [5]). For instance, let us prove condition c). Since the Fourier transform of $\phi^b - \phi$ is $\hat{\phi}(x)\,[(x, b) - 1]$, and since $\hat{\phi}(x) = 0$ outside of $V' \cdot V'$, it follows that if $b \in C$, and V' is small enough, then

$$\|\phi^b - \phi\|_2 = \|[(x, b) - 1]\,\hat{\phi}\|_2 \leq \epsilon_1\,\|\hat{\phi}\|_2 = \epsilon_1 \eta^{1/2},$$

for every $b \in C$, where $\epsilon_1 > 0$ is arbitrarily small. Since

$$p(a) = \phi(a)\,\phi'(a)/\eta,$$

by Plancherel's theorem,

$$\|p^b - p\|_1 = 1/\eta\,\|\phi\phi' - \phi^b\phi'^b\| \leq 1/\eta\,[\|\phi'(\phi - \phi^b)\| + \|\phi^b(\phi' - \phi'^b)\|]$$

$$\leq 1/\eta[\|\phi'\|_2\,\epsilon_1\,\|\phi\|_2 + \|\phi\|_2\,\epsilon_1\,\|\phi'\|_2] \leq 2\epsilon_1(\eta\eta')^{1/2}/\eta \leq 4\epsilon_1,$$

and this proves condition c).

REMARK. As we already mentioned, the foregoing proof of Lemma 1 is an adaptation of a proof given in Zygmund's book. Zygmund considers the particular case when A consists of the integers and X is the unit circle, so that the functions $\hat{f}(x)$ are periodic functions with absolutely convergent Fourier series, and he takes for $\hat{p}(x)$ the function

$$\hat{p}(x) = 1 \text{ if } |x| \leq \eta,$$

$$\hat{p}(x) = 0 \text{ if } |x| \geq 2\eta,$$

$$\hat{p}(x) \text{ linear if } \eta \leq |x| \leq 2\eta.$$

Then he proves that the total variation of the derivative of the function is bounded by a fixed number, and from this he deduces properties a), b), c) of the function $p(a)$. This is the only point in Zygmund's proof which does not apply to general groups; however, it is easy to see that the function $\hat{p}$ used by Zygmund is exactly what formula (6) reduces to when V is taken to be an interval, and thus the proof can be adapted to the general case.

4. Proof of Lemma 2. Let $Q \subset S(f)$ be a compact reducible set, and let $Q^{(1)} = Q'$ be the set of the points x such that any neighborhood of x contains an infinite subset of Q. Define

$$Q^{(2)} = (Q^{(1)})',$$

and form in the usual way the sequence of derivative sets:

$$Q \supset Q^{(1)} \supset Q^{(2)} \supset \cdots \supset Q^{(\alpha)} \supset \cdots.$$

Let w be such that

$$Q^{(w)} = Q^{(w+1)};$$

then $Q^{(w)}$ is a perfect set; and since Q is reducible, $Q^{(w)} = 0$. If $w = 1$, then Q is a finite set and n successive applications of Lemma 1 yields Lemma 2 in this case. We will now prove Lemma 2 by induction on w.

Suppose that Lemma 2 is true if $Q^{(w)} = 0$ for $w < w_0$; we shall prove that it is also true if $Q^{(w)} = 0$ for $w = w_0$. Consider first the case when $w_0 = w' + 1$. Then $Q^{(w')}$ is a finite set, and hence there is a function $h \in L^1(A)$ such that

$$||f - h|| \leq \epsilon/2, \; S(f) \subset S(h),$$

and $\hat{h}(x)$ vanishes on an open set $U \supset Q^{(w')}$. Since $Q - U$ has the property

$$(Q - U)^{(w')} = 0,$$

and $w' < w_0$, by the inductive assumption there is a function h' such that

$$S(f) \subset S(h) \subset S(h'), \; ||h - h'|| \leq \epsilon/2,$$

and $\hat{h}'(x)$ vanishes on an open set $U' \supset Q - U$. Hence $\hat{h}'(x)$ vanishes on $U \cup U' \supset Q$, and

$$||f - h'|| \leq ||f - h|| + ||h - h'|| \leq 2\epsilon/2 = \epsilon .$$

If w_0 is not of the form $w' + 1$, then by definition

$$Q^{(w_0)} = \bigcap_{w < w_0} Q^{(w)} ;$$

hence for some $w' < w_0$ we must have $Q^{(w')} = 0$, and by the inductive assumption Lemma 2 is true in this case.

This proves Lemma 2.

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ASYMPTOTIC LOWER BOUNDS FOR THE FREQUENCIES OF CERTAIN POLYGONAL MEMBRANES

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1. Background. Let the bounded, simply connected, open region R of the (x, y) plane have the boundary curve C. If a uniform elastic membrane of unit density is uniformly stretched upon C with unit tension across each unit length, the square $\lambda = \lambda(R)$ of the fundamental frequency satisfies the conditions (subscripts denote differentiation)

$$\text{(1a)} \qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u \quad \text{in } R, \\ \lambda = \text{minimum}, \end{cases}$$

with the boundary condition

$$\text{(1b)} \qquad u(x, y) = 0 \quad \text{on } C.$$

The solution u of problem (1) is unique up to a constant factor. It is known [**13**, p. 24] that λ is the minimum over all piecewise smooth functions u satisfying (1b) of the Rayleigh quotient

$$\text{(2)} \qquad \rho(u) = \iint_R |\nabla u|^2 \, dxdy \Big/ \iint_R u^2 \, dxdy,$$

where $|\nabla u|^2 = u_x^2 + u_y^2$. In many practical methods for approximating λ one essentially determines $\rho(u)$ for functions u satisfying (1b) which are close to a solution of the boundary value problem (1). See [**9**, p. 112; **6**, p. 276; **11**, and **12**]. By (2) these approximations are known to be *upper bounds* for λ; they can be made arbitrarily good with sufficient labor. It is obviously of equal importance to obtain close lower bounds for λ; cf. [**14**].

The lower bounds for λ given by Pólya and Szegö [**13**] are ordinarily far

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Presented to the American Mathematical Society May 2, 1953, under a slightly different title; received by the editors May 15, 1953. The work on this report was sponsored in part by the Office of Naval Research, USN.

Pacific J. Math. 4 (1954), 467-480

from close. Those obtainable from $\rho(u)$, $\iint_R u^2 dxdy$, and $\iint_R |\Delta u|^2 dxdy$ by methods due to Temple [**15**], D. H. Weinstein [**17**], Wielandt [**18**], and Kato [**8**] (for expositions see [**3**] and [**16**]) are arbitrarily good, but presuppose knowledge of a lower bound for the second eigenvalue λ_2 of the problem (1). The same is true of Davis's proposals in [**4**]. It is possible, following Aronszajn and Zeichner [**1**], to get close lower bounds for λ by minimizing $\rho(u)$ over a class of functions u permitted some discontinuity in R (method of A. Weinstein); the author has no knowledge of the practicability of the method.

A common method of approximating λ is to replace the boundary value problem (1) by a similar problem in finite differences. Divide the plane into *squares* of side h by the network of lines $x = \mu h$, $y = \nu h$ $(\mu, \nu = 0, \pm 1, \pm 2, \cdots)$. The points $(\mu h, \nu h)$ are the *nodes* of the net. A *half-square* is an isosceles right triangle whose vertices are three nodes of one square of the net. Assume that

(3) R is the union of a finite number of squares and half-squares.

Then every interior node of R has four neighboring nodes in $R \cup C$.

Define Δ_h, a finite-difference approximation to Δ, by the relation

$$h^2 \Delta_h v(x, y) = v(x + h, y) + v(x - h, y) + v(x, y + h) + v(x, y - h) - 4v(x, y).$$

Let λ_h be the least number satisfying the following difference equation for a net function v defined on the nodes (x, y) of the net:

(4a) $$\Delta_h v = -\lambda_h v \quad \text{at the nodes in } R,$$

with the boundary condition

(4b) $$v = 0 \quad \text{at the nodes on } C.$$

One can interpret λ_h as the square of the fundamental frequency of a network of massless strings with uniform tension h, fastened to C, and supporting a particle of mass h^2 at each node. That is, a certain lumping of the distributed masses and tensions of problem (1) yields problem (4).

It is easily verified for a rectangular region of commensurable sides π/p, π/q, and for h such that (3) holds, that one has $u = v = \sin px \sin qy$, and that

(5) $$\frac{\lambda_h}{\lambda} = \frac{\sin^2(ph/2) + \sin^2(qh/2)}{(ph/2)^2 + (qh/2)^2} = 1 - \frac{p^4 + q^4}{p^2 + q^2}\frac{h^2}{12} + o(h^2) \quad (h \longrightarrow 0).$$

Hence $\lambda_h < \lambda$ for all h, and one can use λ_h as a lower bound for λ. However, since λ is known exactly for rectangular regions, relation (5) contributes nothing to its computation. For general regions R, it was stated [**3**, p. 405] in 1949 that nothing could be said about the relation of λ_h to λ.

2. A new result. An asymptotic relation resembling (5) will now be established for any *convex* polygonal region R satisfying (3). Such regions are polygons of at most eight sides, having interior vertex angles of 45°, 90°, or 135°. The following theorem[1] will be proved in § 3 by use of the lemmas of § 4:

THEOREM. *Let R be a convex region which is a finite union of squares and half-squares for all h under consideration. Let u solve problem* (1) *for R, and let*

$$a = a(R) = \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R (u_x^2 + u_y^2)\,dxdy}.$$

Then, as $h \longrightarrow 0$, one has

$$(6) \qquad \frac{\lambda_h}{\lambda} \leq 1 - \frac{a}{12} h^2 + o(h^2) \qquad (h \longrightarrow 0).$$

It is a consequence of the theorem that, for all sufficiently small h, say for $h \leq h_0$, λ_h is a *lower bound* for λ. The ordinary finite-difference method thus complements any method based on Rayleigh quotients; and, since $\lambda_h \longrightarrow \lambda$ as $h \longrightarrow 0$, together two such methods can confine λ to an arbitrarily short interval. In particular, Pólya [**11** and **12**] devises modified finite-difference approximations to problem (1) which furnish upper bounds to λ for all h. Hence arbitrarily good two-sided bounds to λ can be found by finite-difference methods alone.

The constant a of the theorem is the best possible for a rectangle R of sides π/p, π/q. For this region, we have $a = (p^4 + q^4)\cdot(p^2 + q^2)^{-1}$, and (6) is seen by (5) to be actually an equality up to terms $o(h^2)$.

Using heuristic reasoning, Milne [**9**, p. 238, (97.5)] finds an approximate formula which, specialized to the fundamental eigenvalue and set in our notation, says

$$(7) \qquad \frac{\lambda_h}{\lambda} \doteq 1 - \frac{\lambda h^2}{24} + o(h^2) \qquad (h \longrightarrow 0).$$

[1] The author gratefully acknowledges many helpful conversations with his colleague Dr. Wolfgang Wasow on the subject of this paper.

For a rectangle of sides π/p, π/q, the coefficient of $-h^2/12$ in (7) is $(p^2+q^2)/2$. Since

$$\frac{p^2+q^2}{2} + \frac{(p^2-q^2)^2}{p^2+q^2} = \frac{p^4+q^4}{p^2+q^2},$$

the coefficient of h^2 in (7) is low for all rectangles with $p \neq q$, and exact for squares. Hence (7) cannot ordinarily be expected to be exact in its h^2 term.

The use of the theorem to bound λ is limited by our lack of knowledge of h_0. However, it is the author's conjecture that, for the regions R of the theorem, $\lambda_h < \lambda$ for all h.

The convexity of R is vital to the statement and proof of the theorem; in fact, by the remark after Lemma 4, $a = \infty$ for nonconvex polygons. A heuristic argument, supported by the numerical example of § 5, has in fact convinced the author that, for nonconvex polygons, $\lambda_h > \lambda$ for all sufficiently small h.

The restriction of R and h to satisfy (3) is less essential, but is used in two ways: (i) to be sure that no interior node has a neighboring node outside R; (ii) to prove that $\Gamma = 0$ in Lemma 7. With an appropriate alteration of Δ_h near C, and with a modification of Lemma 7, one can extend the present method to obtain formulas of type (6) without assuming (3)—and even for convex regions R bounded by piecewise analytic curves C. See [5]. Analogous results can be expected in n dimensions.

3. Proof of the theorem. Let K be the class of functions u which vanish on C, such that $(uu_x)_x$ and $(uu_y)_y$ are continuous in $R \cup C$. Applying Gauss's divergence formula (27) with $p = uu_x$, $q = uu_y$, one finds that, for all u in K, Green's formula is valid in the form

$$\iint_R |\nabla u|^2 dxdy = -\iint_R u\Delta u dxdy.$$

Hence, for all $u \in K$, $\rho(u)$ in (2) can be rewritten with $-\iint_R u\Delta u\, dxdy$ in the numerator.

Since, by Lemma 1, the function u which minimizes (2) and solves (1) belongs to K, and since any function in K is piecewise smooth, one may alternatively define λ as the minimum, over all functions in K, of the quotient

$$\rho(u) = -\iint_R u\Delta u dxdy \Big/ \iint_R u^2 dxdy.$$

Analogously, without having to worry about function classes, one can show that λ_h is the minimum, over all net functions v satisfying (4b), of the quotient

$$(8) \qquad \rho_h(v) = -h^2 \sum_{N_h}\sum v\Delta_h v \Big/ h^2 \sum_{N_h}\sum v^2,$$

where the sums are extended over all nodes N_h of the net inside R.

The key to proving the theorem is to set the solution u of problem (1) into the Rayleigh quotient (8) of problem (4). It will be shown that

$$(9) \qquad \frac{\rho_h(u)}{\lambda} = 1 - \frac{1}{12} ah^2 + o(h^2) \qquad (h \longrightarrow 0).$$

Since $\lambda_h \le \rho_h(u)$, the theorem follows from (9). Henceforth u will always denote a solution of problem (1).

The denominator of $\rho_h(u)$ is a Riemann sum for $\iint_R u^2\,dxdy$. Since u^2 is continuous and hence Riemann integrable over R,

$$(10) \qquad h^2 \sum_{N_h}\sum u^2 = \iint_R u^2\,dxdy + o(1) \qquad (h \longrightarrow 0).$$

(It can be shown that one can replace $o(1)$ by $o(h^2)$ in (10), but we shall not need to do this.)

The nodes N_h inside R are divided into two classes:

N_h' : those at a distance h from some 135° vertex of C;

N_h'': the other nodes of N_h.

Split the numerator of $\rho_h(u)$ accordingly:

$$(11) \qquad -h^2 \sum_{N_h}\sum u\Delta_h u = -h^2 \sum_{N_h'}\sum u\Delta_h u - h^2 \sum_{N_h''}\sum u\Delta_h u = S_h'(u) + S_h''(u).$$

To estimate $S_h'(u)$ note that, since there are at most eight 135° vertices, the number of nodes in N_h' is at most 8, for any h. At any node in N_h',

$$h^2 |u\Delta_h u| \le h^2\left(\frac{u-0}{h}\right) \sum_{i=1}^{4} \left|\frac{u-u_i}{h}\right| \le 4h^2 \max |\nabla u|^2,$$

where the maximum of $|\nabla u|^2$ is taken for all points (x, y) within a distance $2h$ of some 135° vertex. Hence, by Lemma 2, as $h \longrightarrow 0$ through values such that (3) holds,

$$|S_h'(u)| \leq 32h^2 \max |\nabla u|^2 = o(h^2) \qquad (h \longrightarrow 0). \tag{12}$$

Now, using the notation and assertion of Lemma 5, one obtains

$$S_h''(u) = -h^2 \sum_{N_h''}\sum u\Delta u - \frac{h^4}{12} \sum_{N_h''}\sum u(u_{xxxx}' + u_{yyyy}''). \tag{13}$$

Since u satisfies (1a),

$$-h^2 \sum_{N_h''}\sum u\Delta u = \lambda h^2 \sum_{N_h''}\sum u^2 = \lambda h^2 \sum_{N_h}\sum u^2 + o(h^2) \quad (h \longrightarrow 0); \tag{14}$$

the last step is correct because $u(x, y) \longrightarrow 0$ as $(x, y) \longrightarrow C$.

Combining (13) and (14), one finds that, as $h \longrightarrow 0$,

$$\begin{aligned} S_h''(u) &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^4}{12} \sum_{N_h''}\sum u(u_{xxxx}' + u_{yyyy}'') + o(h^2) \\ &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^2}{12} \iint_R u(u_{xxxx} + u_{yyyy})\,dxdy + o(h^2), \end{aligned} \tag{15}$$

by Lemma 6. The integrals used in this proof exist, by Lemma 3. Using (11), (12), (15), and Lemma 7, one finds that

$$\begin{aligned} &-h^2 \sum_{N_h}\sum u\Delta_h u \\ &= \lambda h^2 \sum_{N_h}\sum u^2 - \frac{h^2}{12} \iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy + o(h^2) \qquad (h \longrightarrow 0). \end{aligned} \tag{16}$$

Dividing (16) by the denominator of $\rho_h(u)$, one gets

$$\rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{h^2 \sum\sum_{N_h} u^2} + o(h^2).$$

Hence, by (10),

$$(17) \qquad \rho_h(u) = \lambda - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R u^2\,dxdy} + o(h^2) \qquad (h \longrightarrow 0).$$

If one divides (17) by λ, and notes from (2) that $\lambda \iint_R u^2 dxdy = \iint_R |\nabla u|^2 dxdy$, it is seen that

$$\frac{\rho_h(u)}{\lambda} = 1 - \frac{h^2}{12} \frac{\iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy}{\iint_R |\nabla u|^2\,dxdy} + o(h^2) \qquad (h \longrightarrow 0).$$

By the definition of a we have proved (9) and hence the theorem.

4. Some lemmas. Lemma 1, suggested to the author by Professor Max Shiffman, is used to establish Lemmas 2 to 7, which were applied to prove the theorem. In all the lemmas R is the convex union of squares and half-squares of the network, while $u = u(x, y)$ is a function solving problem (1) in R.

LEMMA 1. *The function u is an analytic function of x and y in $R \cup C$, except at the* 135° *vertices of C. Let r, θ be local polar coordinates centered at a* 135° *vertex P_k, with $0 < \theta < 3\pi/4$ in R. Then*

$$(18) \qquad u = \gamma_k r^{4/3} \sin(4\theta/3) + r^{7/3} E_k(r, \theta),$$

where γ_k is a constant, and where $E_k(r, \theta)$, together with all its derivatives, is bounded in a neighborhood of P_k.

Proof. By reflection one can continue u antisymmetrically across each straight segment of C, and (1a) is satisfied by the extended u at all points of $R \cup C$ except the 135° vertices. The first sentence of the lemma then follows from [2, p. 179].

For $(\xi, \eta) \in R$, write $t = \xi + i\eta$. For each t, let $w = f(z, t)$ be an analytic function of the complex variable $z = x + iy$ which maps R into the unit circle $|w| < 1$, with $f(t, t) = 0$. To study f near a vertex z_k of C, one may assume that $f(z_k, t) = 1$. Let the interior vertex angle of C at z_k be π/α_k ($\alpha_k = 4, 2$, or $4/3$). It is a property of the Schwarz-Christoffel transformation [**10**, p. 189] that

$$(19) \qquad f(z,t) = 1 + (z - z_k)^{\alpha_k} g_k(z,t),$$

where g_k is an analytic function of z regular at z_k.

Let $G(z,t) = G(x,y;\xi,\eta)$ be Green's function for Δu in R. Now $G(z,t) = -(2\pi)^{-1} \log |f(z,t)|$; see [**10**, p. 181]. It then follows from (19) that, in the notation of the lemma, when $\alpha_k = 4/3$,

$$(20) \qquad G(z,t) = \gamma_k(t) r^{4/3} \sin (4\theta/3) + r^{7/3} E_k(r,\theta,t).$$

Moreover, $\gamma_k(t)$ and $E_k(r,\theta,t)$ are integrable over R, since the only discontinuity of $G(z,t)$ is a logarithmic one at $t = z$.

The function u is representable by the integral [**2**, pp. 182-3]

$$(21) \qquad u(x,y) = \lambda \iint_R G(x,y;\xi,\eta) u(\xi,\eta)\, d\xi d\eta.$$

Substituting (20) into (21) proves (18) and the lemma.

LEMMA 2. *$|\nabla u(x,y)| \longrightarrow 0$ as $(x,y) \longrightarrow$ any 135° vertex of C.*

Proof. By (18), $|\nabla u| = O(r^{1/3})$, as $(x,y) \longrightarrow$ any 135° vertex of C.

LEMMA 3. *The functions u_{xx}^2, $u_x u_{xxx}$, $u u_{xxxx}$, u_{yy}^2, $u_y u_{yyy}$, and $u u_{yyyy}$ are Lebesgue-integrable in R.*

Proof. By Lemma 1 these functions are continuous in $R \cup C$, except at the 135° vertices P_k. At these vertices (18) implies that they are $O(r^{-4/3})$ and are hence integrable.

LEMMA 4. *The Lebesgue integrals $\int_C u_y u_{yy} dx$ and $\int_C u_x u_{xx} dy$ exist.*

Proof. Analogous to that of Lemma 3.

REMARK. Lemmas 2, 3, and 4 are false for polygonal regions R which are not convex, since in general the exponent in (18) is α_k, where π/α_k is the interior angle at the vertex P_k.

LEMMA 5. *At each node (x,y) in R of the network of section 1, one has*

$$\Delta_h u = \Delta u + \frac{1}{12} h^2 (u'_{xxxx} + u''_{yyyy}), \tag{22}$$

where

$$\begin{cases} u'_{xxxx} = u_{xxxx}(x + \theta' h, y) \ , \ -1 < \theta' < 1; \\ u''_{yyyy} = u_{yyyy}(x, y + \theta'' h), \ -1 < \theta'' < 1. \end{cases} \tag{23}$$

Proof. By Lemma 1, u_{xxxx} is continuous in the open line segment from $(x-h, y)$ to $(x+h, y)$ (though infinite at any 135° vertex). Since u is continuous in $R \cup C$, it follows from Taylor's formula [7, p. 357] that, if we fix y and set $\phi(x) = u(x, y)$,

$$\phi(x+h) + \phi(x-h) - 2\phi(x)$$

$$= h^2 \phi''(x) + \frac{1}{24} h^4 [\phi''''(x + \theta_1 h) + \theta''''(x - \theta_2 h)],$$

where $0 < \theta_i < 1$ $(i = 1, 2)$. By the continuity of ϕ'''', the last bracket equals $2\phi''''(x + \theta' h)$, where $-1 < \theta' < 1$.

A similar formula for $\psi(y) = u(x, y)$, when added to the above and divided by h^2, yields (22) and (23).

LEMMA 6. *Define N''_h as in* § 3. *For each node (x, y) in N''_h, use the notation of* (23). *Then, as $h \to 0$ over values such that* (3) *holds, one has*

$$h^2 \sum_{N''_h}\sum u(u'_{xxxx} + u''_{yyyy}) = \iint_R u(u_{xxxx} + u_{yyyy})\,dxdy + o(1) \quad (h \to 0). \tag{24}$$

Proof. For all (x, y) in the entire plane E_2 define

$$f(x, y) = \begin{cases} u(u_{xxxx} + u_{yyyy}), & \text{if } (x, y) \in R; \\ 0, & \text{elsewhere}. \end{cases}$$

By the proof of Lemma 3 one sees that $f(x, y)$ is $O(r^{-4/3})$ in the neighborhood of each 135° vertex P_k of C, and continuous elsewhere. Divide the nodes $(x, y) = (\mu h, \nu h)$ of $N''_h \subset R$ into four classes $K^{(i)}$ $(i = 1, 2, 3, 4)$ according to the parity of (μ, ν). Fix any class $K^{(i)}$. For each vertex (x, y) in $K^{(i)}$ let $S(x, y)$ be the union of the four closed network squares of E_2 which contain (x, y). The area of each $S(x,y)$ is $4h^2$; ordinarily certain of the $S(x,y)$ contain points not in R. Define

$$f_h^{(i)}(\xi,\eta) = \begin{cases} u(x,y)(u'_{xxxx} + u''_{yyyy}), & \text{for } (\xi,\eta) \in S(x,y); \\ 0, & \text{for } (\xi,\eta) \notin \bigcup S(x,y). \end{cases}$$

Then $f_h^{(i)}(\xi,\eta) \longrightarrow f(\xi,\eta)$, as $h \longrightarrow 0$, for almost all (ξ,η) in the plane. Using the fact that no node of N_h'' is adjacent to a 135° vertex of C, one can show that for all i, uniformly in h, $|f_h^{(i)}(\xi,\eta)| \le F(\xi,\eta)$, where F is an integrable function in E_2.

Each term of the sum (24) for which $(x,y) \in K^{(i)}$ is equal to

$$\frac{1}{4} \iint_{S(x,y)} f_h^{(i)}(\xi,\eta)\, d\xi d\eta .$$

Hence, applying Lebesgue's convergence theorem, one sees that, as $h \longrightarrow 0$, for each i,

$$\text{(25)} \qquad \begin{aligned} \sum\sum_{N_h'' \cap K^{(i)}} u(u'_{xxxx} + u''_{yyyy}) &= \frac{1}{4} \iint_{E_2} f_h^{(i)}(\xi,\eta)\, d\xi d\eta \\ &\longrightarrow \frac{1}{4} \iint_{E_2} f(\xi,\eta)\, d\xi d\eta \qquad (h \longrightarrow 0). \end{aligned}$$

Summing (25) over $i = 1, 2, 3, 4$ proves (24) and the lemma.

LEMMA 7. *One has*

$$\text{(26)} \qquad \iint_R u(u_{xxxx} + u_{yyyy})\, dxdy = \iint_R (u_{xx}^2 + u_{yy}^2)\, dxdy .$$

Proof. The following applications of Gauss's divergence theorem in the form

$$\text{(27)} \qquad \iint_R (p_x + q_y)\, dxdy = \int_C (pdy - qdx)$$

can be justified by integrating over the region R^* interior to a smooth convex curve C^* inside R, and then letting $C^* \longrightarrow C$ appropriately. The continuity of the integrals in the limit follows from Lemmas 1, 3, and 4.

In the divergence theorem for $p = uu_{xxx}$, $q = uu_{yyy}$, the line integral vanishes, and one finds

$$(28) \qquad \iint_R u(u_{xxxx} + u_{yyyy})\,dxdy = -\iint_R (u_x u_{xxx} + u_y u_{yyy})\,dxdy.$$

A second application of the divergence theorem with $p = u_x u_{xx}$, $q = u_y u_{yy}$, combined with (28), shows that

$$(29) \qquad \iint_R u(u_{xxxx} + u_{yyyy})\,dxdy = \iint_R (u_{xx}^2 + u_{yy}^2)\,dxdy + \Gamma,$$

where $\Gamma = \int_C (u_y u_{yy}dx - u_x u_{xx}dy)$.

By (1a), $u_{xx} = -u_{yy}$ on C, whence $\Gamma = \int_C u_{yy}(u_y dx + u_x dy)$. On the segments of C parallel to the axes, $u_{xx} = u_{yy} = 0$, so that there the contribution to Γ is zero.

Now the vector $\nabla u = (u_x, u_y)$ is perpendicular to C. On the segments of C making a 45° or 135° angle with the x-axis, (u_y, u_x) is parallel to (u_x, u_y), whence (u_y, u_x) is perpendicular to C. Thus $u_y dx + u_x dy \equiv 0$ when (dx, dy) is tangent to C, so that the contribution to Γ from these 45° and 135° segments of C is also zero.

Hence $\Gamma = 0$, and the lemma follows from (29).

5. Numerical example. Let R_1 be the six-sided, nonconvex, L-shaped region whose closure is the union of the three unit squares

$$\begin{cases} -1 \le x \le 0, & 0 \le y \le 1; \\ 0 \le x \le 1, & 0 \le y \le 1; \\ 0 \le x \le 1, & -1 \le y \le 0. \end{cases}$$

The fundamental frequencies $\lambda_h = \lambda_h(R_1)$ and corresponding net functions v were computed by B. F. Handy on the SWAC (National Bureau of Standards Western Automatic Computer) for $1/h = 3, 4, \cdots, 8$. The computation used a *power method*; for some initial net function v_0, $(h^2\Delta_h + 5I)^m v_0$ was determined for large positive integers m, where I is the identity operator. On the basis of Collatz's inclusion theorem [**3**, p. 289], the values in the accompanying table are believed to have errors less than 5×10^{-6}. Observe that $\lambda_h(R_1)$ is less for $h = 1/8$ than for $h = 1/7$.

TABLE

h	$\lambda_h(R_1)$	$\lambda_h(R_2)$
1/2	9.07180	12.00000
1/3	9.52514	13.73700
1/4	9.64143	14.37340
1/5	9.67860	14.67081
1/6	9.69083	14.83259
1/7	9.69384	14.93003
1/8	9.69316	14.99315

Since R_1 is not convex, the theorem of § 2 does not apply, but a heuristic argument suggests that $\lambda_h(R_1) - \lambda(R_1) = O(h^{4/3})$. A least-squares fit to the values of $\lambda_h(R_1)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_1) \doteq \alpha_1 + \beta_1 h^{4/3} + \gamma_1 h^2 = \phi_1(h)$$

yielded the values

$$\alpha_1 = 9.63632, \qquad \beta_1 = 2.40286, \qquad \gamma_1 = -5.97212. \tag{30}$$

The maximum of $|\lambda_h(R_1) - \phi_1(h)|$ for the five values of h is .00013. Hence α_1 is a working estimate of $\lambda(R_1)$.

The fact that $\beta_1 > 0$ in (30) supports the author's conjecture that, for nonconvex polygonal domains satisfying (3), $\lambda_h > \lambda$ for all sufficiently small h.

The table also gives Handy's values for the second eigenvalues of R_1, which are the fundamental eigenvalues $\lambda_h(R_2)$ of the trapezoidal halfdomain R_2 of R_1 for which $x > y$. Since the theorem does apply to R_2, a least-squares fit to the values of $\lambda_h(R_2)$ for $1/8 \leq h \leq 1/4$ of a function of type

$$\lambda_h(R_2) \doteq \alpha_2 + \beta_2 h^2 = \phi_2(h)$$

seemed appropriate, and yielded the values

$$\alpha_2 = 15.19980, \qquad \beta_2 = -13.22219.$$

The maximum of $|\lambda_h(R_2) - \phi_2(h)|$ for the five values of h was .00010. Hence α_2 is a working estimate of $\lambda(R_2)$.

The value of β_2 is negative, in agreement with (6), but the quantity

$-12\beta_2/\alpha_2 = 10.4387$ is something like one-fifth larger than an estimate of the corresponding quantity $a(R_2)$ of the theorem. One therefore suspects that a is not the best possible constant in (6) for the region R_2.

In the table, note the relative closeness of the values of $\lambda_h(R_2)$ to the working estimate, α_2, of $\lambda(R_2)$, even for a coarse net. Thus the value 12 for $\lambda_{1/2}(R_2)$, which is obtained by pencil and paper from a simple quadratic equation, is comparable to the lower bounds 12.1 and $5\pi^2/4$ obtained respectively by comparison with λ for the circular membrane of equal area [**13**, p. 8] and with λ for the rectangular region $0 < x < 1$; $-1 < y < 1$. The value $\lambda_{1/3}(R_2) = 13.737$ requires getting the least eigenvalue of a 7th-order matrix, a relatively easy procedure with a desk machine.

The monotonicity of $\lambda_h(R_2)$ supports the author's conjecture[2] that, for the R of the theorem, $\lambda_h < \lambda$ for all h.

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[2]See page 470.

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(Now with the University of California, Los Angeles)

Pacific Journal of Mathematics

Vol. 4, No. 3 July, 1954