

# Pacific Journal of Mathematics

**ORTHONORMAL CYCLIC GROUPS**

PAUL CIVIN

## ORTHONORMAL CYCLIC GROUPS

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In an earlier paper [1] a characterization was given of the Walsh functions in terms of their group structure and orthogonality. The object of the present note is to present a similar result concerning the complex exponentials.

**THEOREM.** *Let  $\{A_n(x)\}$  ( $n = 0, \pm 1, \dots; 0 \leq x \leq 1$ ) be a set of complex-valued measurable functions which is a multiplicative cyclic group. A necessary and sufficient condition that  $\{A_n(x)\}$  be an orthonormal system over  $0 \leq x \leq 1$  is that the generator of the group admit a representation  $\exp(2\pi i c(x))$  almost everywhere, with  $c(x)$  equimeasurable with  $x$ .*

As the sufficiency is immediate, we present only the proof of the necessity. Let the notation be chosen so that the generator of the group is  $A_1(x)$ , and

$$A_n(x) = (A_1(x))^n \quad (n = 0, \pm 1, \dots).$$

The normality implies  $|A_1(x)| = 1$  almost everywhere. Hence there is a measurable  $a(x)$ ,  $0 \leq a(x) < 1$ , such that

$$A_1(x) = \exp(2\pi i a(x))$$

almost everywhere. Let  $b(x)$  be a function [2, p.207] monotonically increasing and equimeasurable with  $a(x)$ . Also let

$$c(x) = m\{u : 0 \leq u \leq 1, b(u) \leq x\} \quad (-\infty < x < \infty).$$

The orthonormal condition becomes

$$\delta_{0,n} = \int_0^1 \exp(2\pi ni b(x)) dx = \int_{-\infty}^{\infty} \exp(2\pi niy) dc(y),$$

where the latter integral is a Lebesgue-Stieltjes integral. Thus for any  $\epsilon > 0$ ,

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$$\begin{aligned}\delta_{0,n} &= \int_{b(0)-\epsilon}^{b(1)} \exp(2\pi niy) dc(y) \\ &= \int_{b(0)}^{b(1)} \exp(2\pi niy) dc(y) + \exp(2\pi ni b(0)) m\{x : b(x) = b(0)\},\end{aligned}$$

and the latter integral is interpretable as a Riemann-Stieltjes integral.

Integration by parts yields

$$(1) \quad \delta_{0,n} = \exp(2\pi ni b(1)) - 2\pi ni \int_{b(0)}^{b(1)} c(y) \exp(2\pi niy) dy.$$

If  $f(y) = y$ ,  $0 < y \leq 1$ , and  $f(y+1) = f(y)$ , a direct calculation shows that

$$(2) \quad \delta_{0,n} = \exp(2\pi ni b(1)) - 2\pi ni \int_0^1 f(y - b(1)) \exp(2\pi niy) dy.$$

Formulas (1) and (2), and the completeness of the complex exponentials, imply the existence of a constant  $k$  such that for almost all  $y$ ,  $0 < y \leq 1$ ,

$$f(y - b(1)) + k = \begin{cases} 0, & 0 < y \leq b(0) \\ c(y), & b(0) < y \leq b(1) \\ 0, & b(1) < y \leq 1. \end{cases}$$

Since the supremum of  $c(y)$  is one, and  $f(y)$  has no interval of constancy, one infers that  $k = 0$ ,  $b(0) = 0$ , and  $b(1) = 1$ . Thus  $c(y) = y$ ,  $0 < y \leq 1$ , which is equivalent to the proposition that was asserted.

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