ORTHONORMAL CYCLIC GROUPS

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In an earlier paper [1] a characterization was given of the Walsh functions in terms of their group structure and orthogonality. The object of the present note is to present a similar result concerning the complex exponentials.

THEOREM. Let \( \{ A_n(x) \} \) \( (n = 0, \pm 1, \cdots; 0 \leq x \leq 1) \) be a set of complex-valued measurable functions which is a multiplicative cyclic group. A necessary and sufficient condition that \( \{ A_n(x) \} \) be an orthonormal system over \( 0 \leq x \leq 1 \) is that the generator of the group admit a representation \( \exp(2\pi i c(x)) \) almost everywhere, with \( c(x) \) equimeasurable with \( x \).

As the sufficiency is immediate, we present only the proof of the necessity. Let the notation be chosen so that the generator of the group is \( A_1(x) \), and

\[
A_n(x) = (A_1(x))^n \quad (n = 0, \pm 1, \cdots).
\]

The normality implies \( |A_1(x)| = 1 \) almost everywhere. Hence there is a measurable \( a(x), 0 \leq a(x) < 1 \), such that

\[
A_1(x) = \exp(2\pi i a(x))
\]

almost everywhere. Let \( b(x) \) be a function [2, p. 207] monotonically increasing and equimeasurable with \( a(x) \). Also let

\[
c(x) = \inf \{ u : 0 \leq u \leq 1, b(u) \leq x \} \quad (-\infty < x < \infty).
\]

The orthonormal condition becomes

\[
\delta_{0,n} = \int_0^1 \exp(2\pi ni b(x)) \, dx = \int_{-\infty}^{\infty} \exp(2\pi niy) \, dc(y),
\]

where the latter integral is a Lebesgue-Stieltjes integral. Thus for any \( \varepsilon > 0 \),

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481
\[ \delta_{0,n} = \int_{b(0)}^{b(1)} \exp(2\pi niy) \, dc(y) \]

\[ = \int_{b(0)}^{b(1)} \exp(2\pi niy) \, dc(y) + \exp(2\pi ni \, b(0)) \, m \{ x : b(x) = b(0) \}, \]

and the latter integral is interpretable as a Riemann-Stieltjes integral.

Integration by parts yields

\[ \delta_{0,n} = \exp(2\pi ni \, b(1)) - 2\pi ni \int_{b(0)}^{b(1)} c(y) \exp(2\pi niy) \, dy. \]

If \( f(y) = y, 0 < y \leq 1, \) and \( f(y + 1) = f(y) \), a direct calculation shows that

\[ \delta_{0,n} = \exp(2\pi ni \, b(1)) - 2\pi ni \int_{0}^{1} f(y - b(1)) \exp(2\pi niy) \, dy. \]

Formulas (1) and (2), and the completeness of the complex exponentials, imply the existence of a constant \( k \) such that for almost all \( y, 0 < y \leq 1, \)

\[ f(y - b(1)) + k = \begin{cases} 0, & 0 < y \leq b(0) \\ c(y), & b(0) < y \leq b(1) \\ 0, & b(1) < y \leq 1. \end{cases} \]

Since the supremum of \( c(y) \) is one, and \( f(y) \) has no interval of constancy, one infers that \( k = 0, b(0) = 0, \) and \( b(1) = 1. \) Thus \( c(y) = y, 0 < y \leq 1, \) which is equivalent to the proposition that was asserted.

**References**

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Paul Civin, *Orthonormal cyclic groups* .............................................. 481
Kenneth Lloyd Cooke, *The rate of increase of real continuous solutions of algebraic differential-difference equations of the first order* .............. 483
Philip J. Davis, *Linear functional equations and interpolation series* .... 503
F. Herzog and G. Piranian, *Sets of radial continuity of analytic functions* . 533
P. C. Rosenbloom, *Comments on the preceding paper by Herzog and Piranian* .......................................................... 539
Donald G. Higman, *Remarks on splitting extensions* .......................... 545
Margaret Jackson, *Transformations of series of the type* $\Psi_3$ .......... 557
Herman Rubin and Patrick Colonel Suppes, *Transformations of systems of relativistic particle mechanics* .......................................................... 563
A. Seidenberg, *On the dimension theory of rings. II* ............................ 603
Bertram Yood, *Difference algebras of linear transformations on a Banach space* .......................................................... 615