SETS OF RADIAL CONTINUITY OF ANALYTIC FUNCTIONS

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**FRITZ HERZOG AND GEORGE PIRANIAN**

1. **Introduction.** A point set $E$ on the unit circle $C$ ($|z|=1$) will be called a *set of radial continuity* provided there exists a function $f(z)$, regular in the interior of $C$, with the property that $\lim_{r \to 1} f(re^{i\theta})$ exists if and only if $e^{i\theta}$ is a point of $E$. From Cauchy's criterion it follows that the set $E$ of radial continuity of a function $f(z)$ is given by the formula

$$E = \prod_{k=1}^{\infty} \sum_{n=1}^{\infty} \prod_{\epsilon \bar{\epsilon}} E \left\{ |f(r_1 e^{i\theta}) - f(r_2 e^{i\theta})| \leq \frac{1}{k} \right\},$$

where the inner intersection on the right is taken over all pairs of real values $r_1, r_2$ with $1 - 1/n \leq r_1 < r_2 < 1$. From the continuity of analytic functions it thus follows that every set of radial continuity is a set of type $F_{\sigma\delta}$. The main purpose of the present note is to prove the following result.

**Theorem 1.** If $E$ is a set of type $F_{\sigma}$ on $C$, it is a set of radial continuity.

The theorem will be proved by means of a refinement of a construction which was used by the authors in an earlier paper [2] to show that every set of type $F_{\sigma}$ on $C$ is the set of convergence of some Taylor series.

2. **A special function.** That the set consisting of all points of $C$ is a set of radial continuity is trivial. In proving Theorem 1, it may therefore be assumed that the complement of $E$ is not empty. In order to surmount difficulties one at a time, we begin with a new proof of the well-known fact that the empty set is a set of radial continuity (see [1, vol. 2, pp. 152-155]).

Let

$$f(z) = \sum_{n=N}^{\infty} C_n(z),$$

where

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533
\[ C_n(z) = \frac{z^{k_n}}{n^2} \left\{ 1 + \frac{z}{\omega_{n_1}} + \left(\frac{z}{\omega_{n_1}}\right)^2 + \cdots + \left(\frac{z}{\omega_{n_1}}\right)^{n^2-1} \right\} \]

\[ + z^{n^2} \left[ 1 + \frac{z}{\omega_{n_2}} + \left(\frac{z}{\omega_{n_2}}\right)^2 + \cdots + \left(\frac{z}{\omega_{n_2}}\right)^{n^2-1} \right] \]

\[ + \cdots \]

\[ + z^{(n+1)n^2} \left[ 1 + \frac{z}{\omega_{n_1}} + \left(\frac{z}{\omega_{n_1}}\right)^2 + \cdots + \left(\frac{z}{\omega_{n_1}}\right)^{n^2-1} \right] \right\}; \]

where \( \omega_{n_j} = e^{\pi i j/n} \)

and \( \{ k_n \} \) is a sequence of nonnegative integers which increases rapidly enough so that no two of the polynomials \( C_n(z) \) contain terms of like powers of \( z \), and so that a certain other requirement is met; the positive integer \( N \), which is the lower limit of the foregoing series, will be determined later.

If \( z \) is one of the points \( \omega_{n_j} \), then \( |C_n(z)| = 1 \). On the other hand, let \( z \) lie on the unit circle, and let \( \Gamma_n(z) \) be any sum of consecutive terms from (1). If \( z \) is different from each of the roots of unity \( \omega_{n_j} \) that enter into \( \Gamma_n(z) \), and \( \delta \) denotes the (positive) angular distance between \( z \) and the nearest of these \( \omega_{n_j} \), then

\[ |\Gamma_n(z)| < \frac{A_1}{\delta n^2}, \]

where \( A_1 \) is a universal constant (see [2, Lemma A]). Now, if

\[ z = e^{i\theta} \omega_{n_j}, \quad |\theta| < \frac{\pi}{n^2}, \]

and \( R_{n_j}(z) \) denotes the sum of the terms in the \( j \)th row of (1) (including the factor \( z^{k_n/n^2} \)), then

\[ |R_{n_j}(z)| = \frac{\sin (n^2\theta/2)}{n^2 \sin (\theta/2)} > A_2, \]

where \( A_2 \) is again a positive universal constant. But if the angular distance
between $z$ and $\omega_{nj}$ is less than $\pi/n^2$, the angular distances between $z$ and the remaining $n$th roots of unity are all greater than $1/n$, and therefore (3) implies that, for sufficiently large $n$, by (2) and (4),

$$|C_n(z)| > A_2 - 2A_1/n > 5A_3,$$

where $A_3 = A_2/6$. We now choose $N$ so large that the second of these inequalities holds whenever $n \geq N$.

Let $k_N = 0$; let $r_N$ be a number $(0 < r_N < 1)$ such that

$$|C_N(re^{i\theta}) - C_N(e^{i\theta})| < \frac{A_3}{N!}$$

for $r_N \leq r \leq 1$ and all $\theta$. Next, let $k_{N+1}$ be large enough so that

$$|C_{N+1}(rNe^{i\theta}) - C_{N+1}(e^{i\theta})| < \frac{A_3}{(N+1)!}$$

for all $\theta$; and let $r_{N+1} > r_N$, and near enough to 1 so that

$$|C_{N+1}(re^{i\theta}) - C_{N+1}(e^{i\theta})| < \frac{A_3}{(N+1)!}$$

for $r_{N+1} \leq r \leq 1$ and all $\theta$. Let this construction be continued indefinitely.

Now let $L$ be a line segment joining the origin to a point $e^{i\theta}$, and let $n$ be an integer such that $n > N$ and

(5) $$|C_n(e^{i\theta})| > 5A_3.$$

We then write

$$f(re^{i\theta}) - f(r_{n+1}e^{i\theta}) = C_n(e^{i\theta}) + [C_n(r_ne^{i\theta}) - C_n(e^{i\theta})] - C_n(r_{n+1}e^{i\theta})$$

$$+ \sum_{j=N}^{n-1} \{ [C_j(r_ne^{i\theta}) - C_j(e^{i\theta})] - [C_j(r_{n+1}e^{i\theta}) - C_j(e^{i\theta})] \}$$

$$+ \sum_{j=n+1}^{\infty} \{ C_j(r_ne^{i\theta}) - C_j(r_{n+1}e^{i\theta}) \}$$

and obtain from the inequalities above.
It follows that, if there exist infinitely many integers \( n \) for which (5) is satisfied \( f(z) \) does not approach a finite limit as \( z \) approaches \( e^i \) along the line \( L \). But for each real \( \theta \) there exist infinitely many integers \( n \) with the property that, for some integer \( j_n \),

\[
\frac{\theta}{2\pi} - \frac{j_n}{n} < \frac{1}{2n^2}
\]

(see [3, p. 48, Theorem 14]), so that each \( z \) on \( C \) admits infinitely many representations (3). It follows that \( \lim_{r \to 1} f(re^{i\theta}) \) does not exist for any value \( \theta \).

3. Closed sets of radial continuity. Let \( E \) be a closed set on \( C \), and let \( G \) denote its (nonempty) complement. Again, let \( f(z) \) be the function defined in 3.2, except for the following modification. In the polynomial \( C_n(z) \), let \( \omega_{n1}, \omega_{n2}, \ldots, \omega_{np_n} \) denote those \( n \)th roots of unity which lie in \( G \) and have the additional property that the angular distance of each one of them from \( E \) is greater than \( n^{-\frac{3}{2}} \). The exponent of \( z \) in the factor outside of the brackets in the last row of the right member of (1) becomes \( (p_n - 1)n^2 \). And the \( p_n \) \( n \)th roots of unity \( \omega_{nj} \) that occur in \( C_n(z) \) must be so labelled that their arguments increase as the index \( j \) increases, with \( \arg \omega_{nj} > 0 \) and \( \arg \omega_{np_n} \leq 2\pi \). Then every partial sum \( \Gamma_n(z) \) of consecutive terms of \( C_n(z) \) satisfies the inequality \( | \Gamma_n(z) | < A_1n^{-3/2} \) for all \( z \) belonging to \( E \), and therefore the Taylor series of \( f(z) \) converges on \( E \). On the other hand, let the exponents \( k_n \) in (1) be chosen in a manner similar to that of 3.2, and let \( L \) be a line segment joining the origin to a point \( e^{i\theta} \) in the (open) set \( G \). Then there exist infinitely many integers \( n \) for which (5) is satisfied by our newly constructed polynomials \( C_n(z) \), and therefore \( \lim_{r \to 1} f(re^{i\theta}) \) does not exist.

4. The general case. Suppose finally that \( E \) is a set of type \( F_\sigma \) on \( C \). Then the complement \( G \) of \( E \) is of type \( G_\delta \); that is, it can be represented as the intersection of open sets \( G_1, G_2, \ldots \), with \( G_k \supset G_{k+1} \) for all \( k \). In turn, we can represent \( G_1 \) as the union of closed intervals \( I_{1h} \) in such a way that no two distinct intervals \( I_{1h} \) and \( I_{1h'} \) contain common interior points, and in such a way that no point of \( G_1 \) is a limit point of end points of intervals \( I_{1h} \). Similarly,
each set \( G^*_k \) can be represented as the union of closed intervals \( I_{kh} \) satisfying similar restrictions.

Let \( n_0 \) be any positive integer. Since the denumerable set of all open arcs

\[
z = e^{i\theta}, \quad |\theta - 2\pi j/n| < \pi/n^2 \quad (j = 1, 2, \ldots, n, \ n > n_0)
\]
covers the entire unit circle, there exists a set of finitely many such arcs covering the unit circle. It follows that we can choose a finite number of terms \( C_n(z) \) (see (1)), modified as in \( \S \ 3 \), such that their sum \( f(z) \) has the following properties:

i) for each \( \theta \) in \( I_{11} \), there exist two values \( \rho' \) and \( \rho'' \), \( 0 < \rho' < \rho'' < 1 \), such that \( |f_1(\rho'e^{i\theta}) - f_1(\rho''e^{i\theta})| > A_3 \);

ii) for each point \( e^{i\theta} \) outside of \( I_{11} \) and outside of the two neighboring intervals \( I_{1h} \) and \( I_{1h'} \), and for each \( n \) for which \( C_n(z) \) occurs in \( f(z) \), the modulus of any sum of consecutive terms of \( C_n(e^{i\theta}) \) is less than \( A_1 n^{-3/2} \).

Next we accord a similar treatment to \( I_{12} \), then to \( I_{21}, I_{13}, I_{22}, I_{31}, I_{14} \), and so forth. The sum \( f(z) \) of the polynomials \( f_1(z), f_2(z), \ldots \) thus constructed has the following properties: if \( e^{i\theta} \) lies in \( E \), that is, lies in only finitely many of the intervals \( I_{kh} \), the Taylor series of \( f(z) \) converges at \( z = e^{i\theta} \); if \( e^{i\theta} \) lies in \( G \), there exist pairs of values \( \rho' \) and \( \rho'' \) arbitrarily near to 1 and such that

\[
|f(\rho'e^{i\theta}) - f(\rho''e^{i\theta})| > A_3.
\]

It follows that \( E \) is the set of radial continuity of \( f(z) \), and the proof of Theorem 1 is complete.

5. Sets of uniform radial continuity. The following theorem is analogous to Theorem 2 of [2].

Theorem 2. If \( E \) is a closed set on \( C \), then there exists a function \( f(z) \), regular in \( |z| < 1 \), such that \( \lim_{r \to 1} f(re^{i\theta}) \) exists uniformly with respect to all \( e^{i\theta} \) in \( E \) and does not exist for any \( e^{i\theta} \) not in \( E \).

For the proof of Theorem 2, we refer to the function \( f(z) \), constructed in \( \S \ 3 \). Note that \( |\Gamma_n(z)| < A_1 n^{-3/2} \) for all \( z \) in \( E \). Hence the Taylor series of \( f(z) \) converges uniformly in \( E \). It then follows easily, by the use of Abel's summation, that the convergence

\[
\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})
\]
is also uniform in $E$.

6. **An unsolved problem.** The converse of Theorem 1 is false, since a set of radial continuity can be the complement of a denumerable set which is dense on $C$. We do not know whether there exist sets of type $F_{\sigma\delta}$ that are not sets of radial continuity.

**References**

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Paul Civin, *Orthonormal cyclic groups* ........................................ 481
Kenneth Lloyd Cooke, *The rate of increase of real continuous solutions of algebraic differential-difference equations of the first order* ........ 483
Philip J. Davis, *Linear functional equations and interpolation series* .... 503
F. Herzog and G. Piranian, *Sets of radial continuity of analytic functions* ... 533
P. C. Rosenbloom, *Comments on the preceding paper by Herzog and Piranian* ................................................................. 539
Donald G. Higman, *Remarks on splitting extensions* ....................... 545
Margaret Jackson, *Transformations of series of the type $\sum_{\psi}$* ........ 557
Herman Rubin and Patrick Colonel Suppes, *Transformations of systems of relativistic particle mechanics* ........................................... 563
A. Seidenberg, *On the dimension theory of rings. II* ......................... 603
Bertram Yood, *Difference algebras of linear transformations on a Banach space* . ................................................................. 615