TRANSFORMATIONS OF SERIES OF THE TYPE $3\Psi_3$

MARGARET JACKSON
TRANSFORMATIONS OF SERIES OF THE TYPE $\Psi_3$

MARGARET JACKSON

1. Sears [3] has given relations between series of the type $\Psi_2$. Generalizations of some of these results are included in, or may be obtained from, the following two formulae established by Slater [4]:

$$
\prod_{r=0}^{\infty} \frac{(1 - x q^r)(1 - q^{r+1}/x q^r)(1 - b_1 q^r) \cdots (1 - b_M q^r)}{(1 - a_1 q^r) \cdots (1 - a_M q^r)}
\times \frac{(1 - q^{r+1}/a_{M+2}) \cdots (1 - q^{r+1}/a_{M+1})}{(1 - q^{r+1}/a_1) \cdots (1 - q^{r+1}/a_M)}
\left[ \frac{a_{M+2}, \ldots, a_{M+1} ; x}{b_1, \ldots, b_M} \right]

= \frac{q/a_1}{\prod_{r=0}^{\infty}} \left[ \frac{(1 - a_1 x q^{r-1})(1 - q^{r+2}/a_1 x q^r)(1 - b_1 q^{r+1}/a_1) \cdots}{(1 - a_1 q^r)(1 - q^{r+1}/a_1)(1 - a_1 q^{r+2}/a_2) \cdots} \right]
\times \frac{(1 - b_M q^{r+1}/a_1) \cdots (1 - a_1 q^{r'/a_M+2}) \cdots (1 - a_1 q'/a_{M+1})}{(1 - a_1 q'/a_M)(1 - a_2 q'/a_1) \cdots (1 - a_M q'/a_1)}
\times \frac{q a_{M+2}/a_1, \ldots, q a_{M+1}/a_1 ; x}{q b_1/a_1, \ldots, q b_M/a_1}

+ (M - 1) \text{ similar terms obtained by interchanging } a_1 \text{ with } a_2, a_3, \ldots, a_M,

= \frac{q/a_1}{\prod_{r=0}^{\infty}} \left[ \frac{(1 - a_1 x q^{r-1})(1 - q^{r+2}/a_1 x q^r)(1 - b_1 q^{r+1}/a_1) \cdots}{(1 - a_1 q^r)(1 - q^{r+1}/a_1)(1 - a_1 q'/a_2) \cdots} \right]

Received February 12, 1953. This work was included in a doctoral thesis approved by the University of Nottingham.

Pacific J. Math. 4 (1954), 557-562

557
\( (1 - b_M q^{r+1}/a_1)(1 - a_1 q^r/a_{M+2}) \cdots (1 - a_1 q^r/a_{2M+1}) \)
\[
\frac{(1 - a_1 q^r/a_M)(1 - a_2 q^{r+1}/a_1) \cdots (1 - a_M q^{r+1}/a_1)}{(1 - a_1 q^r/a_M)(1 - a_2 q^{r+1}/a_1) \cdots (1 - a_M q^{r+1}/a_1)}
\]
(1.2)
\[
\times \Psi_{M}^{\Psi_{M+1}, M+2} \frac{a_1/b_1 \cdots, a_1/b_M}{b_1 \cdots b_M, a_1/a_{M+2}, \cdots, a_1/a_{2M+1}}
\]
\[+ (M - 1) \text{ similar terms obtained as in (1.1)},\]

where

\[ M \geq 1, \quad \xi = \frac{a_{M+2} \cdots a_{2M+1}}{a_1 \cdots a_M}, \quad |x| < 1, \quad |q| < 1.\]

In particular we see that (1.2), with \( M = 3 \), is a generalization of the basic analogue of the fundamental three-term relation [3, §10, result IVa] for \( {}_3 F_2 \) to which it reduces if we take \( a_1 = aq, \ a_2 = bq, \ a_3 = cq, \ a_5 = a, \ a_6 = b, \ a_7 = c, \ b_1 = q, \ b_2 = e, \ b_3 = f, \) and \( x = ef/abc \). Similarly, (1.1) and (1.2) may be used to obtain many more of the relations given by Sears. It will be noted, however, that the parameters occurring in the \( \Psi \) series in (1.1) and (1.2) are related in a very symmetrical way, and consequently these formulae can only be expected to provide generalizations of the two-, three-, and four-term relations between \( {}_3 \Phi_2 \) which are of a symmetrical nature; in particular, they do not provide a generalization of the basic analogue of the fundamental two-term relation [3, §10, 1]. In this paper, one such generalization is obtained which, when used in conjunction with (1.1), will yield generalizations of all Sears’ formulae and provide basic analogues of known transformations [2] of \( {}_3 H_3 \).

2. To obtain the required generalization, we establish the basic analogue of the formula [2, §2.1] which was used to obtain the generalization of the fundamental two-term relation between \( {}_3 F_2 \). The method by which this result can be obtained has been indicated by Bailey [1], who obtained a particular case of the following formula (2.1). We use the fact that a basic bilateral series \( {}_8 \Psi_8 \) which terminates below can be expressed in terms of an \( {}_8 \Phi_7 \), which can in turn be transformed into two series \( {}_4 \Phi_3 \), one of which can be replaced by a \( {}_4 \Psi_4 \) which terminates below. Then, proceeding to the limit, we obtain a transformation which can be restated in the form (2.1). The analysis is straightforward, though rather lengthy, so we just state the result:
We obtain a generalization of the basic analogue of the fundamental two-term relation by interchanging both $b$ and $d$ and $c$ and $e$ in (2.1), then replacing $a$ by $def/qa^2$, $d$ by $ef/aq$, $e$ by $df/aq$, $f$ by $de/aq$, leaving $b$ and $c$ unaltered, and replacing $def/abcq$ by $\sigma$, we obtain:

\[
\prod_{r=0}^{\infty} \frac{1 - \sigma q^r}{(1 - aq^{r+1}) (1 - aq^r/aq) (1 - aq^r/bc) (1 - dq^r/a)(1 - eq^r/a)(1 - fq^r/a)}
\]

\[
\times \prod_{r=0}^{\infty} \frac{1 - a^2 q^{r+2}/bdef) (1 - a^2 q^{r+2}/cdef) (1 - q^{r+1})}{(1 - a^2 q^{r+2}/def) (1 - defq^{r-1}/a^2)}
\]

\[
\times 3\Phi_2 \left[ \begin{array}{c} aq/ef, aq/df, aq/de ; q \\ a^2 q^2/bdef, a^2 q^2/cdef \end{array} \right].
\]
\[
\times \left[ \prod_{r=0}^{\infty} \frac{(1-aq^{r+1}/d)(1-aq^{r+1}/e)(1-aq^{r+1}/f)}{(1-aq^r)} \right] \Psi_3^{3} \left[ \begin{array}{c} a, b, c; \\ d, e, f \end{array} \right] \frac{def}{abcq}
\]

\[
+ \prod_{r=0}^{\infty} \frac{(1-q^{r+1}/d)(1-q^{r+1}/e)(1-q^{r+1}/f)}{(1-q^{r+1}/b)(1-q^{r+1}/c)}
\]

\[
\times \frac{(1-aq^r/b)(1-aq^r/c)(1-q^{r+1})}{(1-aq^{r+1})(1-q^r/a)} \Phi_2^{3} \left[ \begin{array}{c} aq/d, aq/e, aq/f; q \\ aq/b, aq/c \end{array} \right]
\]

(2.2)

\[
= \prod_{r=0}^{\infty} \frac{(1-aq^{r+1}/f)}{(1-q^{r+1}/b)(1-q^{r+1}/c)(1-dq^r)(1-eq^r)}
\]

\[
\times \left[ \prod_{r=0}^{\infty} \frac{(1-\sigma q^r)(1-fq^r/b)(1-fq^r/c)}{(1-fq^{r+1})} \right] \Psi_3^{3} \left[ \begin{array}{c} ef/aq, df/aq, f/q; aq \\ b, c, f \end{array} \right]
\]

\[
+ \prod_{r=0}^{\infty} \frac{(1-q^{r+1}/c\sigma)(1-q^{r+1}/b\sigma)(1-q^{r+1})}{(1-aq^{r+2}/ef)(1-aq^{r+2}/df)}
\]

\[
\times \frac{(1-q^{r+1}/f)(1-dfq^r/bc)(1-efq^r/bc)}{(1-\sigmafq^r)(1-q^{r+1}/f\sigma)} \Phi_2^{3} \left[ \begin{array}{c} f/c, f/b, \sigma; q \\ df/bc, ef/bc \end{array} \right].
\]

The two \( \Phi_2^{3} \) which occur in this formula are not connected by a two-term relation, and it would appear therefore that (2.2) is probably the simplest generalization of the fundamental two-term relation for \( \Phi_2^{3} \) to which it reduces when \( f = q \). This is the only relation between \( \Phi_2^{3} \) which can be obtained from (2.2).

There are some relations involving \( \Psi_3^{3} \), which generalize more than one \( \Phi_2^{3} \) transformation. Such a formula can be obtained from (2.1) by interchanging the parameters \( b \) and \( d \), then replacing \( a \) by \( def/aq^2 \), \( d \) by \( ef/aq \), \( e \) by \( df/aq \), \( f \) by \( de/aq \), but leaving \( b \) and \( c \) unaltered:
TRANSFORMATIONS OF SERIES OF THE TYPE $\psi_3$

\[(2.3)\]

\[
\prod_{r=0}^{\infty} \frac{(1 - aq^r)(1 - aqu^{r+2}/ef)(1 - aqu^{r+1}/b)}{(1 - q^{r+1}/b)(1 - dq^r)(1 - aqu^r)}
\]

\[
\times \frac{(1 - dq^r/c)(1 - eq^r/b)(1 - fq^r/b)}{(1 - aqu^{r+1}/e)(1 - eq^{r+1}/f)(1 - efq^{r+1}/b)} 3\psi_3[c, ef/aq, ef/bq; d \atop \sigma c, e, f \atop \sigma c, e, f] \]

\[
+ \sum_{r=0}^{\infty} \frac{(1 - q^{r+1}/e)(1 - q^{r+1}/f)(1 - aqu^{r+1}/b)}{(1 - aqu^{r+1}/d)(1 - q^{r+1}/b)(1 - q^{r+1}/c)}
\]

\[
\times \frac{(1 - q^{r+1})(1 - aqu^r)(1 - aqu^{r+1})}{(1 - aqu^{r+1}/e)(1 - aqu^{r+1}/f)(1 - aqu^{r+1})} 3\Psi_3 \]

\[
\times \left\{ \prod_{r=0}^{\infty} \frac{(1 - q^{r+1}/c)(1 - efq^r/bc)(1 - dq^r/c)}{(1 - efq^r/b)(1 - bq^{r+1}/ef)(1 - dq^r)} 3\Phi_2[aq/d, ef/b, e/b; q \atop \sigma a, e, f/c \atop \sigma a, e, f/c] \right\}
\]

\[
- \sum_{r=0}^{\infty} \frac{(1 - aqu^r)(1 - q^{r+1}/d)(1 - aqu^{r+1}/c)}{(1 - aqu^{r+1}/c)(1 - aqu^{r+1}/a)(1 - q^r/a)} 3\Phi_2[aq/d, aqu/e, aqu/f; q \atop \sigma a, aq/c \atop \sigma a, aq/c].
\]

If $e$ (or $f$) = $q$, (2.3) reduces to a two-term relation; but it reduces to a four-term relation between $3\Phi_2$ when $c = 1$. This particular result is not stated explicitly by Sears but can be deduced from his results.

It will be seen that the $3\Psi_3$ transformations are more complicated than the analogous $3H_3$ transformations. For this reason, no more such results are given, but they can all be obtained from (1.1) and (2.2).

3. Corrigenda. In (2.3) and (2.4) of [2], the terms $\Gamma'(1 + b - \sigma), \Gamma'(1 + c - \sigma)$ should be $\Gamma'(1 - b - \sigma), \Gamma'(1 + c - \sigma)$, in (5.1) the factor $\Gamma'(d - c)$ on the left should be in the denominator of the first term on the right, and there should be a factor $\Gamma'(d)$ in the denominator on the left.

REFERENCES

1. W. N. Bailey, Series of hypergeometric type which are infinite in both directions, Quart. J. Math., Oxford Ser., 7 (1936), 105-115.

2. M. Jackson, Transformations of series of the types $3H_3$ with unit arguments, J. London Math. Soc. 27 (1952), 116-123.


THE UNIVERSITY, NOTTINGHAM
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, E.G. Straus, at the University of California Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50; back numbers (Volumes 1, 2, 3) are available at $2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

* To be succeeded in 1955, by H.L. Royden, Stanford University, Stanford, California.

** To be succeeded in 1955, by E.G. Straus, University of California, Los Angeles 24, Calif.
Paul Civin, *Orthonormal cyclic groups* ........................................ 481
Kenneth Lloyd Cooke, *The rate of increase of real continuous solutions of algebraic differential-difference equations of the first order* .......... 483
Philip J. Davis, *Linear functional equations and interpolation series* ...... 503
F. Herzog and G. Piranian, *Sets of radial continuity of analytic functions* ... 533
P. C. Rosenbloom, *Comments on the preceding paper by Herzog and Piranian* .......................................................... 539
Donald G. Higman, *Remarks on splitting extensions* ......................... 545
Margaret Jackson, *Transformations of series of the type \( \psi_3 \) ............ 557
Herman Rubin and Patrick Colonel Suppes, *Transformations of systems of relativistic particle mechanics* ...................................... 563
A. Seidenberg, *On the dimension theory of rings. II* .......................... 603
Bertram Yood, *Difference algebras of linear transformations on a Banach space* ................................................................. 615