A CLASS OF GENERALIZED WALSH FUNCTIONS

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1. Introduction. Let α denote a fixed integer, α > 2, and put \( \omega = \exp \left( \frac{2\pi i}{\alpha} \right) \).

**Definition 1.** The Rademacher functions of order \( \alpha \) are defined by

\[
\phi_k(x) = \omega^k \text{ if } k/\alpha \leq x < (k+1)/\alpha, \quad k = 0, \ldots, \alpha - 1;
\]

and for \( n \geq 0 \)

\[
\phi_n(x + 1) = \phi_n(x) = \phi_0(\alpha^n x).
\]

**Definition 2.** The Walsh functions of order \( \alpha \) are defined by

\[
\psi_0(x) = 1,
\]

and if \( n = a_1 \alpha^{n_1} + \cdots + a_m \alpha^{n_m} \) where \( 0 < a_j < \alpha \) and \( n_1 > n_2 > \cdots > n_m \), then

\[
\psi_n(x) = \phi_{a_1}^{n_1}(x) \cdots \phi_{a_m}^{n_m}(x).
\]

For convenience we let \( \Psi_\alpha \) denote the set of Walsh functions of order \( \alpha \). We may observe that \( \Psi_2 \) is the orthonormal system of functions defined by Walsh [4]. R.E.A.C. Paley's proof that \( \Psi_2 \) is orthonormal and complete in \( L(0,1) \) may be modified by the reader to establish the same properties for \( \Psi_\alpha, \alpha = 3, 4, \ldots \) [3; pp. 242-244].

It is the purpose of this paper to study Fourier expansions in the sets \( \Psi_\alpha \). The results obtained here will include known results for ordinary Walsh Fourier series, most of which are contained in a paper of N. J. Fine [1]. In fact, most
of the properties of Fourier expansions in \( \Psi_2 \) are shared by expansions in \( \Psi_\alpha \).

The system \( \Psi_\alpha \) is in fact the character group of \( G_\alpha \), the countable product of cyclic groups of order \( \alpha \), transferred to the unit interval. The operation +, introduced in §2, is precisely the image of the group operation. Some of our results and many of our methods readily admit interpretations in \( G_\alpha \), although little mention of these will be made in the text. For example, in Lemma 1 we prove that the Haar integral in the group corresponds to the Lebesgue integral on \((0,1)\).

Using an obvious abbreviation, we summarize our most important results:

(i) The \( \mathcal{W}_a FS \) of \( f(x) \) converges to \( f(x) \) a.e. if \( f(x) \) is of bounded variation, and the convergence tests of Dini and Dini-Lipschitz are valid. (ii) If \( f(x) \) has variation \( V \) and if \( c_k \) is the coefficient of \( \psi_k(x) \) in the \( \mathcal{W}_a FS \) of \( f(x) \), then \( |c_k| \leq V k^{-1} \csc \pi/\alpha \). (iii) The continuity of \( f(x) \) is a sufficient condition for the uniform \((C,1)\) summability of the \( \mathcal{W}_a FS \).

2. Notation and preliminary results. Define

\[
l_{n,k} = l_{n,k}(\alpha) = \{ x : k \alpha^{-n} \leq x < (k + 1) \alpha^{-n} \},
\]

\( k = 0, \ldots, \alpha^n - 1, n = 1, 2, \ldots \). Then if \( \phi_n(x) \) is the \( n \)th Rademacher function of order \( \alpha \), \( \phi_n(x) = \omega^k \) if \( x \in l_{n+1,k} \).

The term, \( \alpha \)-adic rational, will denote any number of the form \( k \alpha^{-n} \) where \( k \) and \( n \) are integers. Thus if \( x \) has the base \( \alpha \) expansion

\[
\sum_{j=1}^{\infty} x_j \alpha^{-j}, \quad 0 \leq x_j < \alpha,
\]

where the terminating expansion is taken in case \( x \) is an \( \alpha \)-adic rational, we see that \( \phi_n(x) = \omega^{x_n+1} \).

We introduce a binary operation, denoted by \( \dagger \), and defined as follows: If \( 0 \leq a < 1 \) and \( 0 \leq x < 1 \), and if \( a \) and \( x \) have base \( \alpha \) expansions

\[
\sum_{j=1}^{\infty} a_j \alpha^{-j} \quad \text{and} \quad \sum_{j=1}^{\infty} x_j \alpha^{-j}
\]

respectively, then \( a \dagger x \) will denote the number
where \( y_j = a_j + x_j \mod \alpha \), \( 0 \leq y_j < \alpha \). If we agree to take the terminating expansions for \( \alpha \)-adic rationals whenever possible, it follows that for any fixed \( a \) and all \( n \geq 0 \), \( \phi_n(a + x) = \phi_n(a) \phi_n(x) \), a.e. The exceptional values occur when \( a + x \) is the infinite expansion of an \( \alpha \)-adic rational. It is also true that \( \psi_n(a + x) = \psi_n(a) \psi_n(x) \), a.e.

**Lemma 1.** If \( f(x) \in L(0,1) \) then \( f(a + x) \in L(0,1) \) and

\[
\int_0^1 f(x) \, dx = \int_0^1 f(a + x) \, dx.
\]

The reader will have no difficulty in modeling a proof after the proof in the case \( \alpha = 2 \) [1, p. 379].

If \( f(x) \in L(0,1) \) and if

\[
c_n = \int_0^1 f(t) \, \bar{\psi}_n(t) \, dt
\]

we say that \( \sum_0^\infty c_n \psi_n(x) \) is the \( W_\alpha \) FS of \( f(x) \). Let \( s_k(x) \) denote the \( k \)th partial sum of this series, so that

\[
s_k(x) = \int_0^1 f(t) \sum_0^{k-1} \psi_j(x) \, \psi_j(t) \, dt = \int_0^1 f(t) \, D_k(x,t) \, dt
\]

where the kernel \( D_k(x,t) \) is defined accordingly. We will write \( D_k(t) = D_k(0,t) \). Observe that for all \( k \leq \alpha^n \), \( D_k(x,t) = D_k(x',t') \) provided only that \( x \) and \( x' \) are in the same \( l_{n,r} \) and that \( t \) and \( t' \) are in the same \( l_{n,r} \).

Let \( z = z(x,n) \) be that number satisfying

\[
(2.1) \quad x + z = 0
\]

except when this relation determines \( z \) as the nonterminating expansion of an \( \alpha \)-adic rational. In these cases let the first \( n \) digits in the expansion of \( z \) be determined by (2.1), and let the remaining digits be zeros. For all \( k \leq \alpha^n \) we have for almost all \( t \)
(2.2) \[ D_k(x, t) = \sum_{0}^{k-1} \psi_j(x) \bar{\psi}_j(t) = \sum \bar{\psi}_j(z) \bar{\psi}_j(t) = \sum \psi_j(z + t) = D_k(z + t). \]

If we use Lemma 1 we have the following useful result.

(2.3) \[ s_k(x) = \int_0^1 D_k(z + t) f(t) dt \]

\[ = \int_0^1 D_k(x + z + t) f(x + t) dt = \int_0^1 D_k(t) f(x + t) dt. \]

Unless otherwise stated all functions will be assumed to be periodic and integrable on \((0, 1)\).

3. Convergence.

**Lemma 2.**

\[ D_{\alpha n}(t) = \begin{cases} \alpha^n & \text{if } t \in I_{n, 0}, \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** We have from the definitions

(3.1) \[ D_{\alpha n}(t) = \sum_{r=0}^{\alpha n-1} \bar{\phi}_r(t) = \prod_{r=0}^{n-1} \left[ 1 + \bar{\phi}_r(t) + \cdots + \bar{\phi}_{\alpha-1}(t) \right]. \]

If \( t \in I_{n, 0} \) each \( \phi_r(t) = 1 \), while if \( t \notin I_{n, 0} \) at least one factor in the product vanishes. (The \( p \)th factor is zero if \( \phi_p(t) \neq 1 \).)

By translating under \( \dagger \) we see that Lemma 2 has the following equivalent form: If \( \rho = \rho(x, n) \) is such that \( x \in I_{n, \rho} \) then

\[ D_{\alpha n}(x, t) = \begin{cases} \alpha^n & \text{if } t \in I_{n, \rho}, \\ 0 & \text{otherwise}. \end{cases} \]

As an immediate consequence we have

**Theorem 1.** If \( f(x) \in L(0, 1) \) then \( \lim_{n \to \infty} s_{\alpha n}(x) = f(x) \) a.e. In particular, \( s_{\alpha n}(x) \to f(x) \) at a point of continuity of \( f(x) \) and the convergence is uniform in a closed interval of continuity. If \( x \) is an \( \alpha \)-adic rational then \( s_{\alpha n}(x) \to f(x) \) provided \( x \) is a point of right hand continuity of \( f(x) \).
Additional usefulness of Lemma 2 is seen from the identity

\[ D_n(x, t) = \sum_{j=1}^{m} \left\{ \phi_{n_1}^{a_1}(x) \overline{\phi}_{n_1}^{a_1}(t) \cdots \phi_{n_j}^{a_j-1}(x) \overline{\phi}_{n_j}^{a_j-1}(t) \right\}, \]

where the base \( \alpha \) expansion of \( n \) is given in Definition 2. To prove (3.2) notice that

\[ D_{\alpha n_1}(x, t)\left[ 1 + \phi_{n_1}(x) \overline{\phi}_{n_1}(t) + \cdots + \phi_{n_1}^{a_1-1}(x) \overline{\phi}_{n_1}^{a_1-1}(t) \right], \]

By using (3.3) recursively we obtain (3.2).

The usual method of establishing convergence of the full sequence of partial sums of the \( W_\alpha FS \) will be to reduce the convergence of \( s_n(x) \) to that of \( s_{\alpha n_1}(x) \) by showing that \( s_{\alpha n_1}(x) - s_n(x) \rightarrow 0 \) as \( n \rightarrow \infty \), where \( \alpha^{n_1} \leq n < \alpha^{n_1+1} \). In the following lemma we use the notation of Definition 2, with the additional convention of writing \( N \) for \( n_1 \).

**Lemma 3.** Let \( \nu \) be a fixed positive integer and let \( x \in l_{\nu, \rho} \). Then if \( \sigma \neq \rho \)

\[ \lim_{n \rightarrow \infty} \int_{l_{\nu, \sigma}} [D_n(x, t) - D_{\alpha N}(x, t)] f(t) dt = 0. \]

If also \( \gamma \in l_{\nu, \rho} \) and \( N \geq \nu \), then

\[ \left| \int_{\gamma} (\rho + 1) \gamma^{-\nu} [D_n(x, t) - D_{\alpha N}(x, t)] dt \right| < \alpha, \]

and in case \( \gamma = \rho \gamma^{-\nu} \), the integral (3.5) vanishes.

**Proof.** In proving (3.4) we may suppose \( N \geq \nu \). Let \( r \) be chosen so that \( n_r \geq \nu > n_{r+1} \); in case \( n_m \geq \nu \) take \( r = m \). By Lemma 2 all \( D_{\alpha k}(x, t) = 0 \) for \( t \in l_{\nu, \sigma} \) and \( k \geq \nu \). Thus \( D_n(x, t) = D_n(x, t) - D_{\alpha N}(x, t) \) and by (3.2) this is a sum of \( m - r \) terms, each of which is, for \( t \in l_{\nu, \sigma} \), a constant multiple of
say. A careful inspection of (3.2) shows that the sum of the moduli of the coefficients of \( \tilde{\psi}_{M(n)}(t) \) is bounded independent of \( n \). Also, \( M(n) \to \infty \) as \( n \to \infty \). We have now reduced (3.4) to a theorem of Mercer [2, p. 17].

The inequality (3.5) is proved by writing \( l_{\nu,\rho} \) as a sum of \( l_{N,s} \). On each \( l_{N,s} \) the integrand is a linear combination of \( \phi_N^b(t) \), \( 0 < b < \alpha \). On each complete \( l_{N,s} \) contained in \( (y, (\rho + 1)\alpha^{-\nu}) \) the integral vanishes. The remainder of the interval of integration has length less than \( \alpha^{-N} \), and from (3.3) we see that the integrand is numerically less than \( \alpha^{-N+1} \).

**Theorem 2.** If \( f(x) \) is of bounded variation and continuous from the right on \( [0, 1) \), then as \( n \to \infty \), \( s_n(x) \to f(x) \) at every point of continuity and at every \( \alpha \)-adic rational. If \( x \) is an \( \alpha \)-adic irrational which is a point of discontinuity, \( s_n(x) \) does not converge.

**Proof.** To prove convergence it is sufficient to show that for \( f(t) \) monotonic

\[
s_n(x) - s_n(x) = \int_0^1 [D_n(x, t) - D_{\alpha N}(x, t)] f(t) dt \to 0.
\]

Write this integral as

\[
\int_{l_{\nu,\rho}} + \int_{C l_{\nu,\rho}} [D_n(x, t) - D_{\alpha N}(x, t)] f(t) dt = J_1 + J_2,
\]

where \( C \) denotes the complement taken with respect to \( (0, 1) \). By the second theorem of the mean, there is \( \gamma \in l_{\nu,\rho} \) such that

\[
J_1 = f(\rho \alpha^{-\nu} + 0) \int_{\rho \alpha^{-\nu}}^\gamma [D_n - D_{\alpha N}] dt
\]

\[
+ f((\rho + 1)\alpha^{-\nu} - 0) \int_{\gamma}^{(\rho + 1)\alpha^{-\nu}} [D_n - D_{\alpha N}] dt.
\]

By (3.5)

\[
|J_1| \leq \alpha |f((\rho + 1)\alpha^{-\nu} - 0) - f(x)| + \alpha |f(x) - f(\rho \alpha^{-\nu} + 0)| < \varepsilon/2
\]

for \( \nu \) sufficiently large and for \( n \geq \alpha^{-\nu} \), since \( f(x + 0) = f(x) = f(x - 0) \). If
x is an α-adic rational, first choose ν large enough so that $p^\nu = x$, so that only right hand continuity is involved in (3.6). With ν fixed, $J_2 \to 0$ as $n \to \infty$ by (3.4).

Notice that for convergence at $x$, the hypothesis of bounded variation is needed only in a neighborhood of $x$.

The proof of the second part of Theorem 2 will be omitted, except to note that it is sufficient to consider the $W_a$ FS of $f(x)$, $f(x) = 0$ if $0 \leq x < a$, $f(x) = 1$ if $a < x \leq 1$, where $a$ is an α-adic irrational. The partial sums of the $W_a$ FS of $f(x)$ may be explicitly written in terms of the digits in the base α expansion of $a$, and the assertion follows directly.

Lemmas 2 and 3 provide a direct proof of the theorem of localization for $W_a$ FS.

**Theorem 3.** If $f(x) = g(x)$ a.e. for $a - \epsilon < x < a + \epsilon$, then the $W_a$ FS of $f(x)$ and $g(x)$ are equiconvergent at $a$. If $a$ is an α-adic rational it is sufficient that $f(x) = g(x)$ a.e. for $a < x < a + \epsilon$.

**Lemma 4.** The kernel $D_k(x, t)$ satisfies

\[ \int_0^1 D_k(x, t) dt = 1, \]

and for $0 < t < 1$

\[ |D_k(t)| < \alpha / t. \]

**Proof.** The first assertion is obvious.

For a proof of (3.8) the reader is referred to Fine's paper [1; pp. 391, 392].

**Theorem 4.** If for a fixed $x$,

\[ \frac{f(t) - c}{t - x} \in L(x - \delta, x + \delta) \text{ for some } \delta > 0, \]

then $s_n(x) \to c$.

**Proof.** Suppose the base α expansion of $x$ does not end in an infinite sequence of ones. Let $z$ be determined by (2.1). Then we have, using (2.2) and (3.7)
One may verify that
\[ |x - t| \leq \alpha (z + t). \]
Thus, with (3.8), we have
\[ |J_1| \leq \alpha^2 \int_{|t-x| < h} \frac{|f(t) - c|}{|t-x|} \, dt < \varepsilon \]
for \( h \) sufficiently small. With \( h \) fixed, \( J_2 \to 0 \) by Theorem 3 and the remark below equation (3.6).

In case \( x \) is of the form excluded in the argument above, the proof must be modified. We put \( z = z(x, n) \) where \( z(x, n) \) is defined in §2. Inequality (3.9) may not be satisfied on a set \( F_n \subset (x - \delta, x + \delta) \). One may show that \( F_n \) is a subset of an interval of length \( \alpha^{-n} \), so
\[
|J_1| \leq \alpha^2 \int_{|t-x| < h} \frac{|f(t) - c|}{|t-x|} \, dt + n \int_{F_n} |f(t) - c| \, dt = J'_1 + J''_1.
\]
\( J_1' < \varepsilon \) as before, and with \( h \) fixed,
\[
J_1'' \leq n \alpha^{-n} \int_{F_n} \frac{|f(t) - c|}{|t-x|} \, dt \to 0
\]
and \( J_2 \to 0 \) as \( n \to \infty \).

Lemma 1 and equation (2.2) provide a proof that
\[ \int_{0}^{1} |D_k(x, t)| \, dt = \int_{0}^{1} |D_k(t)| \, dt \quad \text{for all } x \in (0, 1). \]
We put \( L_k = \int_{0}^{1} |D_k(t)| \, dt \), the \( k \)th Lebesgue constant of the system \( \Psi_{\alpha} \).

**Lemma 5.** The Lebesgue constants satisfy \( L_k = O(\log k) \), where the \( O \) depends upon \( \alpha \).
Proof. By Lemma 4, \(|D_k(t)| \leq \min (\alpha/t, k)\). Thus

\[
L_k \leq \int_0^{a/k} k \, dt + \int_{a/k}^1 \alpha/t \, dt = O(\log k).
\]

In the statement of the next theorem, \(W(\delta; f)\) is the modulus of continuity of \(f(x)\);

\[
W(\delta; f) = \sup_{|h| \leq \delta, 0 \leq x < 1} |f(x + h) - f(x)|.
\]

**Theorem 5.** If \(f(x)\) satisfies \(W(\delta; f) = o((\log \delta^{-1})^{-1})\) as \(\delta \to 0\), then \(s_n(x) \to f(x)\) uniformly.

**Proof.** For this proof, write \(n = \alpha k + k'\) where \(0 < \alpha < \alpha, 0 \leq k' < \alpha^k\).

Since

\[
s_n - s_{\alpha k} = (s_n - s_{\alpha k}) + (s_{\alpha k} - s_{\alpha k}) = S_1 + S_2,
\]

it is sufficient to show that \(S_1 \to 0\) and \(S_2 \to 0\) uniformly. By using Lemma 2 and (3.3) we obtain

\[
S_2 = \int_{l_k, \rho} \left[ \phi_k(x) \Phi_k(t) + \cdots + \phi_{a-1}^a(x) \Phi_{a-1}^a(t) \right] \alpha^k f(t) \, dt,
\]

where \(\rho\) is chosen so that \(x \in l_k, \rho\). Since \(f(x)\) is uniformly continuous, \(S_2 \to 0\) as \(k \to \infty\). Again using (3.3),

\[
S_1 = \int_0^1 \phi_k^a(x) \Phi_k^a(t) D_k^* (x, t) f(t) \, dt.
\]

Replacing \(t\) by \(t + b\alpha^{-k-1}\), we have

\[
S_1 = \omega^{ab} \int_0^1 \phi_k^a(x) \Phi_k^a(t) D_k^* (x, t) f(t + b\alpha^{-k-1}) \, dt,
\]

so by subtraction

\[
S_1(1 - \omega^{ab}) = \phi_k^a(x) \int_0^1 D_k^* (x, t) \Phi_k^a(t) [f(t) - f(t + b\alpha^{-k-1})] \, dt.
\]

If \(b\) is chosen so that \(|1 - \omega^{ab}| \geq 3^{1/2}\), this becomes
where we have used Lemma 5.

4. Fourier coefficients.

**Theorem 6.** If

\[ f(x) \sim \sum_{0}^{\infty} c_n \psi_n(x), \]

then

\[ f(a + x) \sim \sum_{0}^{\infty} d_n \psi_n(x) \]

where \( d_n = c_n \psi_n(a) \).

**Proof.** This is a consequence of Lemma 1 and the relation \( \psi_n(a + x) = \psi_n(a) \psi_n(x), \) a.e.

By using Theorem 6 and the scheme from the proof of Theorem 5 we may establish the following.

**Theorem 7.** If

\[ f(x) \sim \sum_{0}^{\infty} c_j \psi_j(x), \]

then

\[ |c_n| \leq 3^{3/2} W((\alpha - 1)/n; f). \]

There is a similar result with \( W \) replaced by the integral modulus of continuity.

As a corollary to Theorem 7 there is the following.

**Theorem 8.** If \( f(x) \in \text{Lip}(\eta) \), then \( c_n = O(n^{-\eta}) \) where the \( O \) depends upon \( \alpha \).

For the next lemma we define
and we write \( n = a \alpha^k + k' \), where \( 0 < a < \alpha \), \( 0 < k' < \alpha^k \).

**Lemma 6.** For \( n \geq 0 \) and all \( x \),

\[
|I_n(x)| < n^{-1} \csc \pi/\alpha.
\]

**Proof.** If \( x \in I_{k,p} \) we have, from elementary properties of \( \psi_n(x) \),

\[
(4.1) \quad |I_n(x)| = \left| \int_{\rho \alpha^{-k}}^{x} \psi_n(t) dt \right| = \left| \psi_k', (\rho \alpha^{-k}) \int_{\rho \alpha^{-k}}^{x} \phi_k^a(t) dt \right|
\]

If \( \tau \) is defined by the relation \( x \in I_{k+1,\tau} \), we have by a direct calculation

\[
\left| \int_{\rho \alpha^{-k}}^{x} \phi_k^a(t) dt \right| \leq \max \left\{ \left| \int_{\rho \alpha^{-k}}^{\tau \alpha^{-k-1}} \phi_k^a(t) dt \right|, \left| \int_{\rho \alpha^{-k}}^{(\tau + 1)\alpha^{-k-1}} \phi_k^a(t) dt \right| \right\}
\]

\[
\leq \max \left\{ \alpha^{-k-1} \left| \frac{1 - \omega^{a\tau}}{1 - \omega} \right|, \alpha^{-k-1} \left| \frac{1 - \omega^{a(\tau + 1)}}{1 - \omega} \right| \right\}
\]

\[
\leq \alpha^{-k-1} \csc \pi/\alpha < n^{-1} \csc \pi/\alpha.
\]

**Theorem 9.** If \( f(x) \) has total variation \( V \) then

\[
|c_n| \leq V n^{-1} \csc \pi/\alpha.
\]

**Proof.** Since \( I_n(0) = I_n(1) = 0 \),

\[
(4.2) \quad c_n = -\int_{0}^{1} I_n(x) df(x),
\]

and the theorem is now seen to be a consequence of Lemma 6.

For \( \alpha = 2 \), Theorem 9 was proved by N. J. Fine [1, p. 383] and in this case \( \csc \pi/\alpha = 1 \). That this factor is necessary when \( \alpha > 2 \) is seen from the following example. For an arbitrary positive integer \( k \) define \( n = \alpha^{k+1} - 1 \). Let \( \beta \) denote the integral part of \( \alpha/2 \) and put \( \zeta = \beta \alpha^{-k-1} \) and \( \xi = \zeta + \beta/\alpha \). Let \( f(x) \) represent the characteristic function of the interval \( [\zeta, \xi] \). By using (4.1) and (4.2) we may calculate \( c_k \). It turns out that
\[ |c_k| = \left[ B(\alpha)/2 \right]^2 \alpha^{-n-1} \csc \pi/\alpha \ V, \]

where \( B(\alpha) = \max_{0 < b < \alpha} |1 - \omega^b| \) so that \( 3^{1/2} \leq B(\alpha) \leq 2. \)

5. \((C,1)\) summability. Let \( \sigma_k(x) \) represent the \( k \)th \((C,1)\) mean of \( \{s_n(x)\} \), and define the kernel,

\[ F_k(x, t) = k^{-1} \sum_{1}^{k} D_r(x, t). \]

We will write \( F_k(0, t) = F_k(t) \).

**Lemma 7.** For \( k \geq 1, \int_{0}^{1} F_k(x, t) \, dt = 1, \) and for \( 0 < t < 1, \) \( |F_k(t)| < \alpha/t. \)

**Proof.** These properties follow directly from the corresponding properties of \( D_k(x, t) \).

**Lemma 8.** There is a constant \( M \) such that for all \( k \geq 0 \)

\[ \int_{0}^{1} |F_{\alpha k}(x, t)| \, dt \leq M. \]

**Proof.** Write \( n \) in the form \( n = a \alpha^k + k' \) where \( 0 < a < \alpha \) and \( 0 \leq k' \leq \alpha^k \).

By a somewhat tedious calculation involving repeated use of (3.2) we obtain

\[ (5.1) \quad nF_n(t) = \left[ 1 + \cdots + \bar{\phi}_k^{a-1}(t) \right] \alpha^k F_{\alpha k}(t) + \bar{\phi}_k^{a}(t) k' F_{k'}(t) \]

\[ + \left[ 1 + [1 + \bar{\phi}_k(t)] + \cdots + [1 + \cdots + \bar{\phi}_k^{a-2}(t)] \right] \alpha^k D_{\alpha k}(t) \]

\[ + \left[ 1 + \cdots + \bar{\phi}_k^{a-1}(t) \right] k' D_{k'}(t). \]

If we take \( k' = \alpha^k \) and \( a = \alpha - 1 \) in (5.1) we obtain

\[ (5.2) \quad \alpha^{k+1} F_{\alpha k+1}(t) = R_k(t) \alpha^k F_{\alpha k}(t) + Q_k(t) \alpha^k D_{\alpha k}(t) \]

where

\[ (5.3) \quad R_k(t) = \begin{cases} \alpha & \text{if } \bar{\phi}_k(t) = 1, \\ 0 & \text{otherwise} \end{cases} \]

and
(5.4) \[ Q_k(t) = \begin{cases} \alpha(\alpha - 1)/2 & \text{if } \phi_k(t) = 1, \\ \alpha/(1 - \phi_k(t)) & \text{otherwise}. \end{cases} \]

By applying a simple induction argument to (5.2) we obtain

\begin{equation}
\alpha^{k+1} F_{\alpha k+1}(t) = Q_k(t) \alpha^k D_{\alpha k}(t) \\
+ \sum_{r=1}^{k} R_k(t) R_{k-1}(t) \cdots R_r(t) Q_{r-1}(t) \alpha^{r-1} D_{\alpha r-1}(t) \\
+ \prod_{r=0}^{k} R_r(t). \tag{5.5}
\end{equation}

Let

\[ S = \sum_{r=1}^{\alpha-1} |1 - \omega_r|^{-1}, \]

then equations (5.3)-(5.5) enable us to show that

\[ \alpha^{k+1} \int_0^1 |F_{\alpha k+1}(t)| \, dt \leq \alpha^k \left[ \left( \frac{\alpha - 1}{2} + S \right) + 1 + \left( \frac{\alpha - 1}{2} - S \right) \right] \sum_{r=1}^{k} \alpha^{r-1}, \]

from which the lemma follows.

Observe that by setting \( k = 0 \) in (5.2) we see that for \( \alpha > 2 \) the kernels \( F_{\alpha k}(t) \) are not positive. Fine showed that in case \( \alpha = 2 \), \( F_{\alpha k}(t) \geq 0 \) [1, p. 396].

**Lemma 9.** If \( t \) is not of the form \( t = d\alpha^{-m}, m \geq 1, 0 < d < \alpha \), then \( \lim_{k \to \infty} F_k(t) = 0 \).

**Proof.** Let \( t \) be given and choose \( n \) so that \( \alpha^{-n} < t < \alpha^{-n+1} \). Write \( k = p\alpha^n + q \) where \( 0 \leq q < \alpha^n \). Then

\[ k F_k(t) = \sum_{r=0}^{p-1} \sum_{s=1}^{\alpha^n} D_{r\alpha^n+s}(t) + \sum_{s=1}^{q} D_{p\alpha^n+s}(t). \]

One can show that \( D_{r\alpha^n+s}(t) = D_{\alpha^n}(t) D_r(\alpha^n t) + \psi_r(\alpha^n t) D_s(t) \). This gives

\[ D_{r\alpha^n+s}(t) = \psi_r(\alpha^n t) D_s(t), \]
so that
\[ kF_k(t) = \alpha^n F_{\alpha^n t}(t) D_p(\alpha^n t) + \psi_p(\alpha^n t) qF_q(t). \]

Put \( b \) equal to the integral part of \( \alpha^n t \). Since \( 0 < \alpha^n t - b < 1 \), we have by Lemma 4
\[ |D_p(\alpha^n t)| \leq \alpha(\alpha^n t - b)^{-1}. \]

Using Lemma 7 we obtain
\[ |kF_k(t)| \leq \alpha^{n-2} t^{-1}(\alpha^n t - b)^{-1} + q \alpha t^{-1}, \]
from which the conclusion follows.

**Theorem 10.** If \( f(x) \) is continuous then \( \sigma_{\alpha-k}(x) \to f(x) \) uniformly.

**Proof.** It follows from (2.3) and Lemma 7 that
\[ \sigma_n(x) - f(x) = \int_0^1 F_n(t) [f(x + t) - f(x)] dt. \]

By applying Lemmas 7-9 together with a standard argument we can show that
\[ \int_0^1 |F_{\alpha-k}(t)| |f(x + t) - f(x)| dt \to 0 \text{ uniformly.} \]

**Theorem 11.** If \( f(x) \) is continuous then \( \sigma_n(x) \to f(x) \) uniformly.

**Proof.** Let the base \( \alpha \) expansion of \( n \) be given in Definition 2. From (5.1) we obtain the estimate
\[ |nF_n(t)| \leq \sum_{r=1}^m \left| a_r \alpha^{nr} F_{\alpha^{nr} t}(t) \right| + \frac{1}{2} a_r (a_r + 1) \alpha^{nr} D_{\alpha^{nr} t}(t). \]

Let \( \epsilon_k = \epsilon_k(x) \) represent the larger of
\[ \int_0^1 |F_{\alpha-k}(t)| |f(x + t) - f(x)| dt \]
and
\[ \int_0^1 D_{\alpha k}(t) |f(x + t) - f(x)| \, dt, \]

so that by Theorems 1 and 10 \( \varepsilon_k \to 0 \) uniformly. Using (5.6) and (5.7)

\[ |s_n(x) - f(x)| \leq \alpha \sum_{r=1}^m a_r \alpha^{n_1} n^{-1} \varepsilon_{n_1} = \delta_n, \text{ say.} \]

One may readily verify that the transformation which sends \( \{ \varepsilon_k \} \) into \( \{ \delta_n \} \) is regular, so that \( \delta_n \to 0 \) uniformly, and the theorem is proved.

It is interesting to note that by virtue of a well known consequence of the Banach-Steinhaus theorem [5, p. 99], Theorem 11 implies that \( \int_0^1 |F_n(t)| \, dt \leq M. \)

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