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ON GROUPS OF ORTHONORMAL FUNCTIONS. II

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An orthonormal group on a measure-space Ω is defined as an orthonormal system of functions which is simultaneously a group with respect to multiplication. In an earlier note [1] we showed that, essentially, all such systems are derived from character groups of compact abelian groups.¹ It may be of interest to know, for a given Ω , what orthonormal groups it can support. The answer to this question for $\Omega = I$, the unit interval, is given by the following theorem.

THEOREM. *Let G be any countable abelian group. Then there exists on the unit interval I an orthonormal group G' isomorphic to G . If G is infinite, then G' is complete in $L^2(I)$.*

Proof. Assign the discrete topology to G , and let H be its character group. If G is of finite order n , then H and G are isomorphic. To each $h_k \in H$, $k = 1, 2, \dots, n$, we associate the interval $I_k = [(k-1)/n, k/n)$, and define the n functions f_j by

$$f_j(x) = h_k(g_j) \quad (x \in I_k),$$

where g_j are the elements of G . Then $\{f_j\}$ is the required orthonormal group.

If G is infinite, H is uncountable. The measure-algebra of H (with respect to normalized Haar measure) is non-atomic, separable, and normalized. Hence [2, p. 173] it is isomorphic to the measure-algebra of I . Now H is a complete separable metric space, and the outer Haar measure is a regular Caratheodory outer measure; the same is true of Lebesgue measure on I . Therefore we can apply a theorem of von Neumann [3, Th. 1] to obtain a measure-preserving transformation from H to I . The characters of H , transferred to I , then form the required orthonormal group, complete in $L^2(I)$.

¹Only the case of a countable orthonormal group was considered in [1], but the proofs carry over to the uncountable case with slight modification.

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We shall now examine a special case of this theorem, in which we can construct G' explicitly. Let G denote the additive group of rationals mod 1. G is the (weak) direct sum of the primary groups G_p consisting of all the rationals mod 1 with denominators powers of the prime p . The character group of G_p is H_p , the group of p -adic integers

$$(1) \quad \xi_p = c_{p0} + c_{p1}p + c_{p2}p^2 + \cdots \quad (c_{pm} = 0, 1, 2, \dots, p-1),$$

and H is the (strong) direct sum of all the H_p . The value of the character ξ_p at $1/p^m$, $m > 0$, is given by

$$(2) \quad \chi_{1/p^m}(\xi_p) = \exp 2\pi i A_m(\xi_p),$$

where

$$(3) \quad A_m(\xi_p) = \frac{1}{p^m} (c_{p0} + c_{p1}p + \cdots + c_{p,m-1}p^{m-1}).$$

For $n \leq m$, $p^n A_m(\xi_p) \equiv A_{m-n}(\xi_p) \pmod{1}$, so the definition

$$(4) \quad \chi_{k/p^m}(\xi_p) = \chi_{1/p^m}^k(\xi_p)$$

is unambiguous. For $h = (\xi_2, \xi_3, \xi_5, \dots) \in H$, and for $r \in G$, we decompose r into

$$(5) \quad r = \frac{k_1}{p_1^{m_1}} + \frac{k_2}{p_2^{m_2}} + \cdots + \frac{k_t}{p_t^{m_t}}$$

and have

$$(6) \quad \chi_r(h) = \prod_{s=1}^t \chi_{k_s/p_s^{m_s}}(\xi_{p_s}).$$

Next we construct a mapping λ of H onto I . For this purpose we remark that, given any sequence of integers $n_k > 1$, $k = 1, 2, \dots$, it is possible to get a representation of $x \in I$ by sequences $c_k(x)$ which generalizes the representation to the fixed base b . We divide I into n_1 equal intervals, each of these into n_2 equal intervals, and so on. If x falls in the j th interval ($j=0, 1, 2, \dots, n_k - 1$) at the k th stage, we define $c_k(x) = j$. We then have

$$(7) \quad x = \frac{c_1}{n_1} + \frac{c_2}{n_1 n_2} + \frac{c_3}{n_1 n_2 n_3} + \dots,$$

and the representation is unique except for a countable set. We can remove the ambiguity by choosing the finite expansion where two are possible. It is easy to see that the measure of the set of $x \in I$ for which c_1, c_2, \dots, c_k have given values is $(n_1 n_2 \dots n_k)^{-1}$, so that the functions $c_k(x)$ are statistically independent.

Returning to the mapping λ to be defined, let $h = (\xi_2, \xi_3, \xi_5, \dots) \in H$ be given. We run through the integers c_{pm} by the diagonal method, taking $n_k = p$ whenever we reach c_{pm} . Thus $n_1 = 2, n_2 = 3, n_3 = 2, n_4 = 5, n_5 = 3, n_6 = 2, n_7 = 7$, and so on. This defines a number $x \in I$ which we designate as $\lambda(h)$. Thus

$$(8) \quad \lambda(h) = \frac{c_{20}}{2} + \frac{c_{30}}{2 \cdot 3} + \frac{c_{21}}{2^2 \cdot 3} + \frac{c_{50}}{2^2 \cdot 3 \cdot 5} + \frac{c_{31}}{2^2 \cdot 3^2 \cdot 5} + \frac{c_{22}}{2^3 \cdot 3^2 \cdot 5} \\ + \frac{c_{70}}{2^3 \cdot 3^2 \cdot 5 \cdot 7} + \dots = \sum_{\substack{p \text{ prime} \\ m \geq 0}} \frac{c_{pm}}{D_{pm}};$$

D_{pm} is defined, for $p = k$ th prime p_k , by

$$(9) \quad D_{pm} = \prod_{j=1}^{k-1} p_j^{m+k-j} \prod_{j=k}^{m+k} p_j^{m+k-j+1},$$

empty products being interpreted as 1. With the convention given above, λ is one-to-one, and the inverse μ is given by

$$(10) \quad c_{pm}(x) \equiv [D_{pm}x] \pmod{p} \quad 0 \leq c_{pm}(x) < p.$$

It is easily verified that λ is continuous. More important, however, is the fact that it is measure-preserving, since the $c_{pm}(h)$ are statistically independent with respect to Haar measure on H , and have the same distribution as the $c_{pm}(x)$ on I ; the measures on both spaces are determined by the measures on the sets of constancy of finite collections of c_{pm} . It follows that the functions

$$(11) \quad f_r(x) = \chi_r(\mu(x)) \quad (r \in G; x \in I)$$

form a complete orthonormal group on I .

It may be of interest to note the following Fourier expansion of the function $((x)) = x - [x] - 1/2$:

$$(12) \quad ((x)) = - \sum_{p,m,n} \left\{ p^m D_{pm} \left(1 - e^{-2\pi i n/p^{m+1}} \right) \right\}^{-1} f_{n/p^{m+1}}(x),$$

where the summation is extended over all primes p , all $m \geq 0$, and $0 < n < p^{m+1}$, with $(n, p) = 1$. To derive (12), write

$$M = c_{p0}(x) + c_{p1}(x)p + \cdots + c_{pm}(x)p^m,$$

$$N = c_{p0}(x) + \cdots + c_{p,m-1}(x)p^{m-1},$$

so that

$$(13) \quad c_{pm}(x) = \frac{M - N}{p^m}.$$

On the other hand, from (2), (3), and (11),

$$f_{k/p^m}(x) = \exp 2\pi i \frac{kN}{p^m}.$$

Thus

$$(14) \quad \begin{aligned} N &= \sum_{r=0}^{p^m-1} r \delta(r, N) = \frac{1}{p^m} \sum_{r=0}^{p^m-1} r \sum_{k=0}^{p^m-1} e^{-2\pi i kr/p^m} f_{k/p^m}(x) \\ &= \sum_{k=0}^{p^m-1} f_{k/p^m}(x) \left\{ \frac{1}{p^m} \sum_{r=0}^{p^m-1} r e^{-2\pi i kr/p^m} \right\}. \end{aligned}$$

There is a similar expression for M . Combined with (13), they yield the Fourier expansion for $c_{pm}(x)$. Substituting this in (8) and rearranging, we obtain (12). The series converges uniformly and absolutely for all x . This, incidentally, furnishes an alternate proof of the completeness of $\{f_r\}$, since every power of x has an expansion of the same kind, and the polynomials are dense in $L^2(I)$.

If we wish to exhibit the rationals R as a complete orthonormal group, this time on the unit square I_2 , we need merely take

$$(15) \quad g_r(x, y) = e^{2\pi i r y} f_r(x) \quad (r \in R; (x, y) \in I_2).$$

To see this, let \hat{R} denote the character group of R . Every element $\sigma \in \hat{R}$ determines a number γ , $0 \leq \gamma < 1$, by $\sigma(1) = \exp 2\pi i \gamma$, and an element $h \in H$, defined by $\chi_r(h) = \sigma(r) \exp(-2\pi i r \gamma)$. Conversely, every pair (h, γ) determines a unique σ . If $h = (\xi_2, \xi_3, \dots)$, $h' = (\xi'_2, \xi'_3, \dots)$, then $\sigma(h, \gamma) \cdot \sigma(h', \gamma') = \sigma(h'', \gamma'')$, where $\gamma'' \equiv \gamma + \gamma' \pmod{1}$ and $h'' = (\xi''_2, \xi''_3, \dots)$ is given by $\xi''_p = \xi_p + \xi'_p + [y + y']$. Using these relations, we can show that the mapping $\sigma(h, \gamma) \rightarrow (\lambda(h), \gamma)$ is a continuous and measure-preserving transformation from \hat{R} to I_2 . Under this transformation R , realized as the character group of \hat{R} , goes over into the system (15). We could, of course, map I_2 onto I , preserving measure, and thus realize R as an orthonormal group on I .

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