A NOTE ON A PAPER BY L. C. YOUNG

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1. Introduction. Suppose that $f(x)$ is a real- or complex-valued function defined for all real $x$. For $0 \leq \alpha \leq 1$, we define the $\alpha$-variation of $f(x)$ over $a \leq x \leq b$ as the least upper bound of the sums

$$\left\{ \sum |\Delta f|^{1/\alpha} \right\}^\alpha$$

taken over all finite subdivisions of $a \leq x \leq b$. (When $\alpha = 0$, we denote by the above sum simply the maximum $|\Delta f|$.) We say that $f(x)$ is in $W_\alpha$ if it has finite $\alpha$-variation over the interval $0 \leq x \leq 1$. L. C. Young has proved the following result.

Theorem 1. (See [2, Theorem 4.2].) Suppose that $0 < \beta < 1$ and that $f(x)$, with period 1, satisfies the condition

$$\int_0^1 |f(\phi(t + h)) - f(\phi(t))| \, dt \leq h^\beta$$

for every monotone function $\phi(t)$ such that $\phi(t + 1) = \phi(t) + 1$

for all $t$. Then $f(x)$ is in $W_\alpha$ for each $\alpha < \beta$.

Young's argument does not suggest whether we can assert that $f(x)$ is in $W_\beta$. We present here an elementary proof for Theorem 1 and an example to show that this result is the best possible one in this direction.

2. Lemma. We require the following:

Lemma 2. Suppose that $a_1, a_2, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ are two sets of nonnegative numbers such that $a_1 \geq a_2 \geq \cdots \geq a_N$ and such that

$$\sum_{\nu=1}^{n} a_\nu \leq \sum_{\nu=1}^{n} b_\nu$$
for \( n = 1, \ldots, N \). Then for \( p > 1 \),
\[
\sum_{\nu=1}^{n} a_{\nu}^p \leq \sum_{\nu=1}^{n} b_{\nu}^p
\]
for \( n = 1, \ldots, N \).

Let
\[
S_n = \sum_{\nu=1}^{n} a_{\nu} \quad \text{and} \quad T_n = \sum_{\nu=1}^{n} b_{\nu}.
\]

With Abel's identity and Hölder's inequality, we have
\[
\sum_{\nu=1}^{n} a_{\nu}^p = \sum_{\nu=1}^{n} a_{\nu} a_{\nu}^{p-1}
\]
\[
= S_1(a_1^{p-1} - a_2^{p-1}) + \cdots + S_{n-1}(a_{n-1}^{p-1} - a_n^{p-1}) + S_n a_n^{p-1}
\]
\[
\leq T_1(a_1^{p-1} - a_2^{p-1}) + \cdots + T_{n-1}(a_{n-1}^{p-1} - a_n^{p-1}) + T_n a_n^{p-1}
\]
\[
= \sum_{\nu=1}^{n} b_{\nu} a_{\nu}^{p-1},
\]
\[
\leq \left( \sum_{\nu=1}^{n} b_{\nu} \right)^{1/p} \left( \sum_{\nu=1}^{n} a_{\nu} \right)^{(p-1)/p},
\]
from which the lemma follows.

3. **Proof of Theorem 1.** For a subdivision \( 0 = x_0 < x_1 < \cdots < x_N = 1 \), consider the numbers
\[
|f(x_1) - f(x_0)|, |f(x_2) - f(x_1)|, \ldots, |f(x_N) - f(x_{N-1})|,
\]
and label this set \( a_1, a_2, \ldots, a_N \) so that \( a_1 \geq a_2 \geq \cdots \geq a_N \). We say that the two points \( \xi' \) and \( \xi'' \) are associated with \( a_n \) if they are the two points of the subdivision for which
\[
a_n = |f(\xi'') - f(\xi')|;
\]
and, fixing \( n \), we consider the union of points associated with \( a_1, a_2, \ldots, a_n \). Labeling these \( \xi_1 < \xi_2 < \cdots < \xi_{m_n} \), we define
$\phi(t) = \xi_{\nu}$ for $\frac{\nu - 1}{mn} \leq t < \frac{\nu}{mn}$ \quad (\nu = 1, \ldots, mn),

and we extend this function so that

$$\phi(t + 1) = \phi(t) + 1.$$ 

Now $mn \leq 2n$ and, if $0 < h < 1/mn$,

$$h \sum_{\nu=1}^{n} a_{\nu} \leq h \sum_{\nu=2}^{m_n} |f(\xi_{\nu}) - f(\xi_{\nu - 1})|,$$

$$\leq \int_{0}^{1} |f(\phi(t) + h) - f(\phi(t))| \, dt \leq h^\beta.$$

Letting $h$ approach $1/m_n$, we have

$$\sum_{\nu=1}^{n} a_{\nu} \leq m_n^{1-\beta} \leq (2n)^{1-\beta}$$

for $n = 1, \ldots, N$. Finally selecting $b_1, b_2, \ldots, b_N$ so that

$$\sum_{\nu=1}^{n} b_{\nu} = (2n)^{1-\beta},$$

we have

$$b_1 = 2^{1-\beta} \quad \text{and} \quad b_n < 2^{1-\beta}(n - 1)^{-\beta}$$

for $n > 1$,

and applying Lemma 2 we conclude that

$$\left\{ \sum_{n=1}^{N} |\Delta_n f|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{n=1}^{N} b_n^{1/\alpha} \right\}^\alpha < 2 \left\{ \sum_{n=1}^{\infty} n^{-\beta/\alpha} \right\}^\alpha.$$

This completes the proof.

4. Further results. We now show that Theorem 1 is best possible.

Theorem 3. Suppose that $0 < \beta < \gamma \leq 1$. There exists a function $f(x)$, with period 1, which is not in $W_\beta$ and which satisfies the condition
\[ \left\{ \int_0^1 \left| f \{ \phi(t + h) \} - f \{ \phi(t) \} \right|^{1/\gamma} \right\}^\gamma \leq h^\beta \quad (h \geq 0) \]

for every monotone function \( \phi(t) \) such that
\[ \phi(t + 1) = \phi(t) + 1. \]

Consider two increasing sequences, \( \{x_n\} \) and \( \{y_n\} \), such that
\[ x_1 < y_1 < x_2 < \cdots < x_n < y_n < x_{n+1} < \cdots < x_1 + 1. \]

Define the function
\[ g(x) = \begin{cases} n^{-\beta} & \text{for } x_n < x < y_n, \\ 0 & \text{everywhere else in } x_1 \leq x < x_1 + 1, \end{cases} \]
and extend \( g(x) \) to have period 1.

**Lemma 4.** Suppose that \( 0 < \beta < \gamma \leq 1 \). The function \( g(x) \) defined above satisfies the condition
\[ \left\{ \int_0^1 \left| g(x + h) - g(x) \right|^{1/\gamma} dx \right\}^\gamma \leq \left( \frac{2\gamma}{\gamma - \beta} \right)^\gamma h^\beta \quad (h \geq 0). \]

Fix \( h \) in the range \( 0 < h \leq 1/2 \), and consider the finite sequence,
\[ \xi_0 < \xi_1 < \cdots < \xi_N = \xi_0 + 1, \]
defined as follows.

A. Let \( \xi_0 = x_1 - h \).

B. Suppose that \( \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_0 + 1 \) have been defined.

Let \( \xi_n = \text{Max} \{ \xi_{n-1} + 2h, y_n \} \) if this does not exceed \( \xi_0 + 1 \). Otherwise let \( \xi_n = \xi_0 + 1 \).

It is not difficult to show that
\[ \int_{\xi_{n-1}}^{\xi_n} \left| g(x + h) - g(x) \right|^{1/\gamma} dx \leq 2h n^{-\beta/\gamma} \]
for \( n = 1, \cdots, N \). Since \( \xi_n - \xi_{n-1} \geq 2h \) for \( n = 1, \cdots, N-1 \), we have \( Nh < 1 \) and
This completes the proof of Lemma 4.

Take any strictly increasing continuous function \( \phi(t) \) such that

\[
\phi(t + 1) = \phi(t) + 1.
\]

If \( \phi^{-1} \) is the inverse function, and

\[
u_n = \phi^{-1}(x_n) \quad \text{and} \quad v_n = \phi^{-1}(y_n),
\]

then \( u_1 < v_1 < u_2 < \cdots < u_n < v_n < u_{n+1} < \cdots < u_1 + 1 \) and

\[
g\{\phi(t)\} = \begin{cases} n^\beta & \text{for } u_n < t < v_n, \\ 0 & \text{everywhere else in } u_1 \leq t < u_1 + 1. \end{cases}
\]

Now \( g\{\phi(t)\} \) has period 1 in \( t \), and, by Lemma 4,

\[
\left( \int_0^1 |g\{\phi(t + h)\} - g\{\phi(t)\}|^{1/\gamma} dt \right)^\gamma \leq \left( \frac{2\gamma}{\gamma - \beta} \right)^\gamma h^\beta \quad (h \geq 0).
\]

The Lebesgue limit theorem allows us to conclude this holds for all nondecreasing \( \phi(t) \) such that

\[
\phi(t + 1) = \phi(t) + 1.
\]

To complete the proof of Theorem 3, observe that \( g(x) \) is not in \( W_\beta \) and let

\[
f(x) = \left( \frac{\gamma - \beta}{2\gamma} \right)^\gamma g(x).
\]

In the proof of Theorem 3, the fact that \( \beta < \gamma \) plays an important role. We have a different situation when \( \beta = \gamma \).

**Theorem 5.** Suppose that \( 0 \leq \beta \leq 1 \) and that \( f(x) \) is measurable and real-valued with period 1. The \( \beta \)-variation of \( f(x) \) over any interval of length 1 does not exceed 1 if and only if
\[ \left\{ \int_0^1 |f \{ \phi(t + h) \} - f \{ \phi(t) \}|^{1/\beta} dt \right\}^\beta \leq h^\beta \]

for each monotone function \( \phi(t) \) such that \( \phi(t + 1) = \phi(t) + 1 \).

For the sufficiency, let \( x_0 < \cdots < x_N = x_0 + 1 \) be a subdivision of some interval of length 1. Define the function

\[ \phi(t) = x_n, \quad \frac{n}{N} \leq t < \frac{n + 1}{N} \]  

\((n = 0, \ldots, N - 1)\),

and extend \( \phi(t) \) so that

\[ \phi(t + 1) = \phi(t) + 1; \]

for \( 0 < h < 1/N \) we get

\[ \left\{ \sum_{n=1}^N |\Delta_n f|^{1/\beta} \right\}^\beta \leq \left\{ \frac{1}{h} \int_0^1 |f \{ \phi(t + h) \} - f \{ \phi(t) \}|^{1/\beta} dt \right\}^\beta \leq 1. \]

For the necessity, we see that the \( \beta \)-variation for \( f \{ \phi(t) \} \) over any interval of length 1 does not exceed 1, and we can apply the following:

Theorem 6. (See [1, Theorem 1.3.3].) Suppose that \( 0 \leq \beta \leq 1 \), that \( f(x) \) is measurable and real-valued with period 1, and that the \( \beta \)-variation of \( f(x) \) over any interval of length 1 does not exceed 1. Then

\[ \left\{ \int_0^1 |f(x + h) - f(x)|^{1/\beta} dx \right\}^\beta \leq h^\beta \]  

\((h \geq 0)\).

References


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