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1. Introduction. Studies of nonlinear differential systems have become increasingly important with recent advances in all areas of applied mathematics. Linear systems, and methods based upon these, are inadequate to describe and investigate many of the phenomena associated with physical, chemical, and other systems. In many cases, linearizing processes applied to the equations impose properties of existence, uniqueness, oscillation character, or other nature, which effectively eliminate the phenomenon of prime concern from the investigation. New methods for studying nonlinear systems which avoid restrictions of the type just suggested are needed and are in process of development.

The present paper is concerned with a second-order nonlinear ordinary differential system in the real domain, with which are associated linear boundary conditions at two points. The methods used, and the results obtained, generalize and extend those given in an earlier paper [3]. In particular, the boundary conditions treated in the present paper fail to be self-adjoint when they are associated with linear differential equations. The paper establishes the existence of sets of characteristic numbers (eigenvalues) for the nonlinear systems, and gives oscillation theorems for the associated solutions.

2. Results. In the differential system,

$$(1)\qquad \begin{aligned} dy/dx &= K(x, y, z; \lambda)\, z, \\ dz/dx &= G(x, y, z; \lambda)\, y, \end{aligned}$$

let $K(x, y, z; \lambda)$, $-G(x, y, z; \lambda)$ be real positive functions that are continuous in $(y, z; \lambda)$ on

$$SL \begin{cases} S: -\infty < y, z < +\infty, \\ L: L_1 < \lambda < L_2, \end{cases}$$

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for each fixed x on X: $a \leq x \leq b$; measurable in x on X for each fixed $(y, z; \lambda)$ on SL; and bounded numerically on XSL by a function $M(x)$ that is summable (in the Lebesgue sense) on X. Let

$$\psi(x, \lambda) = \gamma(x, \lambda)\, z(x, \lambda) - \delta(x, \lambda)\, y(x, \lambda),$$

$$\phi(x, \lambda) = \alpha(x, \lambda)\, z(x, \lambda) - \beta(x, \lambda)\, y(x, \lambda),$$

where $\alpha(x, \lambda)$, $\beta(x, \lambda)$, $\gamma(x, \lambda)$, $\delta(x, \lambda)$ are continuous on XL, $\beta(x, \lambda)$ does not change sign on XL, $\delta(a, \lambda) \neq 0$ on L, and

$$\Delta(\lambda) = \alpha(a, \lambda)\, \delta(a, \lambda) - \beta(a, \lambda)\, \gamma(a, \lambda) \neq 0$$

on L.

The boundary conditions

$$(2) \qquad \begin{cases} \text{(a)} \quad \psi(a, \lambda) = 0, \\ \text{(b)} \quad \phi(a, \lambda) = \phi(b, \lambda) \end{cases}$$

are associated with system (1).

THEOREM 1. *There exists at least one solution*[1] $y(x, \lambda)$, $z(x, \lambda)$ *of* (1) *on* XL *such that*

$$y(a, \lambda) \equiv \gamma(a, \lambda), \quad z(a, \lambda) \equiv \delta(a, \lambda),$$

and hence $\psi(a, \lambda) \equiv 0$ *for all* λ *on* L.

Proof. Since $|K| < M(x)$, $|G| < M(x)$ on XSL, we have

$$K^2 z^2 + G^2 y^2 \leq M(x)^2 [y^2 + z^2].$$

Hence

$$[K^2 z^2 + G^2 y^2]^{1/2} \leq M(x)\,(y^2 + z^2)^{1/2} < M(x)\,[(y^2 + z^2)^{1/2} + 1]$$

$$\leq M(x)/g(y^2 + z^2), \qquad \text{where } g(t) = 1/(t^{1/2} + 1).$$

The functions $K(x, y, z; \lambda) z$, $G(x, y, z; \lambda) y$ thus satisfy the hypotheses of a

[1] A pair of functions $y(x, \lambda)$, $z(x, \lambda)$ each of which is continuous in (x, λ) on XL and absolutely continuous in x on X for each fixed λ on L, is a solution of system (1) if this pair satisfies (1) almost everywhere on X for each fixed λ on L.

fundamental existence theorem for differential systems.[2] An application of this theorem yields Theorem 1.

Let $y(x, \lambda)$, $z(x, \lambda)$ be a solution of system (1) as described in Theorem 1. This solution then satisfies boundary Condition (2) (a) for all values of λ on L. We now investigate conditions under which this solution will, for specific values of λ, also satisfy Condition (2) (b). Apply the transformation[3]

$$
\begin{aligned}
y(x, \lambda) &= u(x, \lambda) \sin v(x, \lambda), \\
z(x, \lambda) &= u(x, \lambda) \cos v(x, \lambda),
\end{aligned}
\tag{3}
$$

where

$$u(a, \lambda) = [\gamma(a, \lambda)^2 + \delta(a, \lambda)^2]^{1/2},$$

$$v(a, \lambda) = \tan^{-1}[\gamma(a, \lambda)/\delta(a, \lambda)] \qquad (-\pi/2 < v(a, \lambda) < \pi/2).$$

Substitution from (3) into (1) followed by simple reductions yields:

$$
\begin{aligned}
dv/dx &= K \cos^2 v - G \sin^2 v, \\
du/dx &= u[(K + G)/2] \sin 2v,
\end{aligned}
\tag{4}
$$

where

$$u(a, \lambda) = [\gamma(a, \lambda)^2 + \delta(a, \lambda)^2]^{1/2}, \quad v(a, \lambda) = \tan^{-1}[\gamma(a, \lambda)/\delta(a, \lambda)],$$

$$(-\pi/2 < v(a, \lambda) < \pi/2),$$

for all λ on L.

Existence of solutions $u(x, \lambda)$, $v(x, \lambda)$ for system (4) follows from a repetition of the proof given for Theorem 1 when it is noted that

$$K \cos^2 v - G \sin^2 v \quad \text{and} \quad [(K + G)/2] \sin 2v$$

are uniformly bounded on XSL by the summable function $2M(x)$. Such solutions need not be uniquely determined by the given initial conditions.

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[2] See [2, p. 349, Theorem 69.1].

[3] For use of this transformation in the study of nonlinear, and linear, differential systems, see [3].

THEOREM 2. *$u(x, \lambda) > 0$ at all points of XL.*

Proof. $u(a, \lambda) > 0$ for all λ on L. Suppose $u(c, d) = 0$. Since $u(a, d) > 0$, we let c be the first point of X for which $u(x, d) = 0$. From (4) we have

$$u(c - \epsilon, d) = u(a, d) \exp\left(\int_a^{c-\epsilon} [K + G] \sin 2v \, dt/2\right),$$

where $\epsilon > 0$ is arbitrary. Clearly the limit of the right side of this equation as ϵ goes to zero exists and is greater than zero. This limit, however, must be $u(c, d)$ since $u(x, \lambda)$ is continuous at (c, d). This contradicts the assumption that $u(c, d)$ vanishes, and thus yields the theorem.

In view of Theorem 2, all zeros of $y(x, \lambda)$ occur at points where $v(x, \lambda) = n\pi$, and those of $z(x, \lambda)$ occur where $v(x, \lambda) = (2n + 1)\pi/2$, where n is an integer or zero.

Let

$$U(x, \lambda) = u(x, \lambda)(\alpha^2 + \beta^2)^{1/2}, \quad \tau(x, \lambda) = \tan^{-1}[\alpha(x, \lambda)/\beta(x, \lambda)],$$

where $-\pi/2 \leq \tau(x, \lambda) \leq \pi/2$. Then

$$\phi(x, \lambda) = -U(x, \lambda) \sin(v - \tau). \tag{5}$$

Since α and β do not vanish simultaneously, $U(x, \lambda) > 0$ on XL. Boundary Condition (2)(b) becomes

$$U_a \sin(v_a - \tau_a) = U_b \sin(v_b - \tau_b), \tag{6}$$

where $U_a = U(a, \lambda)$, $v_a = v(a, \lambda)$, and, in general, $f_c = f(c, \lambda)$.

THEOREM 3. *Under the hypothesis that $\beta(x, \lambda)$ does not change signs on XL, the function $v(x, \lambda) - \tau(x, \lambda)$ is continuous on XL. In particular, then, $v_b - \tau_b$ and $v_a - \tau_a$ are continuous functions of λ on L.*

Proof. Since $\alpha(x, \lambda)$ and $\beta(x, \lambda)$ are continuous and do not vanish simultaneously, $\alpha(x, \lambda) \neq 0$ and of fixed sign in a neighborhood of a point where $\beta(x, \lambda) = 0$. When $\beta(x, \lambda) = 0$, $\tau(x, \lambda)$ has the value $\pi/2$ or $-\pi/2$ according as $\alpha(x, \lambda)$ is positive or negative at this point. Under the hypothesis on $\beta(x, \lambda)$, the ratio $\alpha(x, \lambda)/\beta(x, \lambda)$ does not change sign in a neighborhood of a point of vanishing for $\beta(x, \lambda)$; hence $\tau(x, \lambda)$, the inverse tangent of this ratio, is continuous at such point. The theorem follows from this continuity and the continuity of $v(x, \lambda)$, $\alpha(x, \lambda)$, and $\beta(x, \lambda)$.

COROLLARY. *If* $\beta(x, \lambda)$ *changed signs in passing through zero at* (c, d), *then* $\tau(c, \lambda)$ *and* $v(c, \lambda) - \tau(c, \lambda)$ *would have discontinuities of magnitude* π *at this point.*

THEOREM 4. $\sin(v_a - \tau_a)$ *does not vanish on* L.

Proof. Since

$$-\pi/2 < v_a < \pi/2 \quad \text{and} \quad -\pi/2 \leq \tau_a \leq \pi/2,$$

we have

$$-\pi < v_a - \tau_a < \pi.$$

Hence the vanishing of $\sin(v_a - \tau_a)$ would require

$$v_a = \tau_a \quad \text{or} \quad \tan^{-1}(\alpha/\beta) = \tan^{-1}(\gamma/\delta),$$

and hence $\alpha\delta - \beta\gamma = 0$. This, however, is contrary to the hypothesis $\Delta(\lambda) \neq 0$ on L.

DEFINITION 1. Let $m(f, g; \lambda)$ and $M(f, g; \lambda)$, respectively, be the greatest lower bound and least upper bound of $K[x, f(x), g(x); \lambda]$ and $-G[x, f(x), g(x); \lambda]$ on X for each fixed λ on L, where $f(x)$ and $g(x)$ are arbitrary[4] absolutely continuous functions on X. For each λ on L, let $m(\lambda)$ and $M(\lambda)$, respectively, be the greatest lower bound of the set $\{m(f, g; \lambda)\}$ and the least upper bound of the set $\{M(f, g; \lambda)\}$ obtained when $f(x)$ and $g(x)$ range over the class of absolutely continuous functions on X.

THEOREM 5. $m(\lambda)$ *is the greatest lower bound, and* $M(\lambda)$ *is the least upper bound, of the functions* $K(x, y, z; \lambda)$ *and* $-G(x, y, z; \lambda)$ *on* XS *for each fixed* λ *on* L.

Proof. Let $(p, q, r; \lambda)$ be any point of XSL. The functions

$$f(x) \equiv q, \quad g(x) \equiv r$$

are absolutely continuous on X. Hence

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[4] The class of functions used here may be narrowed considerably for any given system (1). In particular, known properties of the particular solution $y(x, \lambda)$, $z(x, \lambda)$ may be used to restrict the class. For example, bounds for these functions may be known and used. Actually, the case $f(x) = y(x, \lambda)$, $g(x) = z(x, \lambda)$, λ fixed, is of interest here, and we apply the hypotheses as nearly as possible to this case.

$$m(\lambda) \leq m(q,r;\lambda) \leq K(p,q,r;\lambda), \; -G(p,q,r;\lambda) \leq M(q,r;\lambda) \leq M(\lambda).$$

Also, let $f(x)$ and $g(x)$ be any two absolutely continuous functions on X, and let $x = p$ be a point of X. The point $(p, f(p), g(p); \lambda)$ belongs to XSL, and hence values of K and $-G$ at all such points lie between the greatest lower bound and the least upper bound of these functions over XSL. Hence

$$\text{glb}\{K, -G\} \leq m(\lambda) \leq M(\lambda) \leq \text{lub}\{K, -G\}$$

for each λ on L. The theorem follows from these inequalities and those given above.

THEOREM 6. *For each λ on L,*

$$-\pi + (b-a)m(\lambda) < v(b,\lambda) - \tau(b,\lambda) < \pi + (b-a)M(\lambda). \tag{9}$$

Proof. Integrate the first equation in (4) and subtract $\tau(b,\lambda)$ from each side to obtain

$$v(b,\lambda) - \tau(b,\lambda) = v(a,\lambda) - \tau(b,\lambda) + \int_a^b [K \cos^2 v - G \sin^2 v]dx$$

$$= v(a,\lambda) - \tau(b,\lambda) + \int_a^b \{[K - m(\lambda)] \cos^2 v$$

$$+ [-G - m(\lambda)] \sin^2 v\}dx + \int_a^b m(\lambda)dx,$$

where $\int_a^b m(\lambda)dx$ has been added and subtracted on the right side. Since

$$K - m(\lambda) \geq 0, \quad -G - m(\lambda) \geq 0, \text{ and } |v(a,\lambda) - \tau(b,\lambda)| < \pi,$$

we have

$$v(b,\lambda) - \tau(b,\lambda) > -\pi + m(\lambda)(b-a).$$

A similar procedure, which adds and subtracts $\int_a^b M(\lambda)dx$ and makes use of

$$K - M(\lambda) \leq 0, \; -G - M(\lambda) \leq 0,$$

shows that

$$v(b,\lambda) - \tau(b,\lambda) < \pi + (b-a)M(\lambda).$$

DEFINITION 2. If such integers exist, we let h be the smallest integer such that

$$M(\lambda) < (h - 3/2)\pi/(b-a)$$

for some λ on L, and let j be any integer such that

$$m(\lambda) > (j + 1/2)\pi/(b-a)$$

holds for at least one value of λ on L. We note that infinite values for $M(\lambda)$ and $m(\lambda)$ have not been excluded.

THEOREM 7. *The inequalities*

$$v(b, \lambda_1) - \tau(b, \lambda_1) < (h - 1/2)\pi,$$

$$v(b, \lambda_2) - \tau(b, \lambda_2) > (j - 1/2)\pi$$

hold for some λ_1 *and* λ_2 *on* L *whenever integers* h *and* j *exist.*

Proof. In keeping with the definitions of h and j, let λ_1 and λ_2 be chosen so that

$$M(\lambda_1) < (h - 3/2)\pi/(b-a), \quad m(\lambda_2) > (j + 1/2)\pi/(b-a).$$

It follows from Theorem 6 that

$$v(b, \lambda_1) - \tau(b, \lambda_1) < \pi + (b-a)M(\lambda_1) < \pi + (h - 3/2)\pi = (h - 1/2)\pi,$$

$$v(b, \lambda_2) - \tau(b, \lambda_2) > -\pi + (b-a)m(\lambda_2) > -\pi + (j + 1/2)\pi = (j - 1/2)\pi.$$

THEOREM 8.[5] *Let*

$$H(x, \lambda) = \frac{-G(x, y(x, \lambda), z(x, \lambda); \lambda)}{K(x, y(x, \lambda), z(x, \lambda); \lambda)},$$

$$\theta(\lambda) = \frac{[\alpha(b, \lambda)^2 + \beta(b, \lambda)^2]^2 [\delta(a, \lambda)^2 + H(a, \lambda)\gamma(a, \lambda)^2]}{\Delta(\lambda)^2 [\alpha(b, \lambda)^2 + H(b, \lambda)\beta(b, \lambda)^2]}.$$

Let λ_n *and* μ_n *be values of* λ *for which*

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[5] The paper [1] treats the linear case of system (1), (2). Lemma 5, p. 38, of that paper needs the additional hypotheses stated for Theorem 8, above, and the proof given for this lemma should be modified to take these into account.

$$v(b, \lambda_n) - \tau(b, \lambda_n) = (n - 1/2)\pi, \qquad v(b, \mu_n) - \tau(b, \mu_n) = (n + 1/2)\pi,$$

where n is a nonnegative whole number that is even or odd according as the nonvanishing continuous function $\sin[v(a, \lambda) - \tau(a, \lambda)]$ *is positive or negative. If either*

(a) $\theta(\mu_n) \geq 1$ *and* $H(x, \mu_n)$ *is monotonic increasing*[6] *on* X

or

(b) $\theta(\mu_n) \geq [H(a, \mu_n)]/H(b, \mu_n)$ *and* $H(x, \mu_n)$ *is monotonic decreasing*[6] *on* X,

then there is at least one characteristic number, η_n, *of the differential system* (1), (2) *between* λ_n *and* μ_n. *Furthermore, if* ρ_n *exists such that*

$$v(b, \rho_n) - \tau(b, \rho_n) = (n + 3/2)\pi,$$

and the foregoing conditions are satisfied, then there is at least one characteristic number between μ_n *and* ρ_n.

Proof. Let $y(x, \lambda)$, $z(x, \lambda)$ be a solution of (1), (2)(a) as described in Theorem 1. When this solution is substituted into Condition (2)(b) and transformation (3) applied, equation (6) results. It remains to show that this equation in λ has at least one root between λ_n and μ_n. Since $\sin(v_a - \tau_a)$ does not vanish on L, this requires that

$$q_1(\lambda) = \sin(v_b - \tau_b)$$

be equal to

$$q_2(\lambda) = [U_a/U_b]\sin(v_a - \tau_a)$$

for some λ between λ_n and μ_n. Since

$$q_1(\lambda_n)q_1(\mu_n) = -1, \; q_2(\lambda_n)q_2(\mu_n) > 0, \; q_1(\lambda_n)q_2(\lambda_n) < 0,$$

it follows that either

$$\text{(i)} \;\; q_1(\lambda_n) < 0 < q_2(\lambda_n) \quad \text{or} \quad \text{(ii)} \;\; q_2(\lambda_n) < 0 < q_1(\lambda_n).$$

[6]If this condition must be checked without explicit use of $y(x, \mu_n)$, $z(x, \mu_n)$, then bounds for these functions may be used to determine a class of absolutely continuous functions which would be used in $H(x, \mu_n)$ to test the validity of the hypotheses.

In case (i), we have $q_1(\mu_n) = 1$ and will therefore have $q_2(\mu_n) \leq q_1(\mu_n)$ if we show

$$[U_a/U_b]\,|\sin(v_a - \tau_a)| \leq 1.$$

The inequalities

$$q_1(\lambda_n) < q_2(\lambda_n), \quad q_1(\mu_n) \geq q_2(\mu_n)$$

show that the continuous functions $q_1(\lambda)$ and $q_2(\lambda)$ are equal for at least one $\lambda = \eta_n$ between λ_n and μ_n.

In case (ii), we have $q_1(\mu_n) = -1$ and will therefore have $q_1(\mu_n) \leq q_2(\mu_n)$ if we show

$$[U_a/U_b]\,|\sin(v_a - \tau_a)| \leq 1.$$

In this case, the inequalities

$$q_2(\lambda_n) < q_1(\lambda_n), \quad q_2(\mu_n) \geq q_1(\mu_n)$$

yield the existence of η_n between λ_n and μ_n such that $q_1(\eta_n) = q_2(\eta_n)$.

Hence the theorem follows when it is shown that

$$U_a\,|\sin(v_a - \tau_a)|\,/\,U_b \leq 1,$$

or

$$U_b^2 \geq U_a^2 \sin^2(v_a - \tau_a) \geq \Delta(\mu_n)^2$$

for $\lambda = \mu_n$. If the second equation of (4) is solved for $u(b, \mu_n)$, and the relation between U and u taken into account, the foregoing inequality which must be established to complete the proof of the theorem becomes

$$(10) \qquad e^{2\int_a^b (K+G)\sin v \cos v\, dx} \geq \Delta(\mu_n)^2/[\alpha_b^2 + \beta_b^2][\gamma_a^2 + \delta_a^2],$$

or

$$(11) \qquad 2I = 2\int_a^b (K+G)\sin v \cos v\, dx \geq \log\{\Delta(\mu_n)^2/[\alpha_b^2 + \beta_b^2][\gamma_a^2 + \delta_a^2]\}.$$

Using equation (4), we get

$$(12)\qquad \begin{aligned} 2I &= \int_{v_a}^{v_b} (K+G)\,d(\sin^2 v)/[K-(K+G)\sin^2 v] \\ &= -\int_{v_a}^{v_b} (K+G)\,d(\cos^2 v)/[(K+G)\cos^2 v - G] \\ &= \int_{v_a}^{v_b} (1-H)\,d(\sin^2 v)/[1-(1-H)\sin^2 v] \\ &= \int_{v_a}^{v_b} (1-1/H)\,d(\cos^2 v)/[1-(1-1/H)\cos^2 v]. \end{aligned}$$

Letting $\sin^2 v = w$, then we have $\cos^2 v = 1 - w$, and

$$2I = \int_{\sin^2 v_a}^{\sin^2 v_b} (1-H)\,dw/[1-(1-H)w]$$

$$= \int_{w=\sin^2 v_a}^{w=\sin^2 v_b} d[(1-H)w]/[1-(1-H)w] + \int_{H_a}^{H_b} w\,dH/[1-(1-H)w]$$

$$\geq \log\,[1-(1-H_a)\sin^2 v_a]/[1-(1-H_b)\sin^2 v_b],$$

since w, dH, $[1-(1-H)w]$ are nonnegative under hypothesis (a). Since

$$\sin^2 v_a = \gamma_a^2/(\gamma_a^2+\delta_a^2), \quad \sin^2 v_b = \beta_b^2/(\alpha_b^2+\beta_b^2),$$

we get

$$2I \geq \log[(\delta_a^2 + H_a\gamma_a^2)/(\alpha_b^2 + H_b\beta_b^2)][(\alpha_b^2+\beta_b^2)/(\gamma_a^2+\delta_a^2)]$$

$$\geq \log[\Delta(\mu_n)^2/(\alpha_b^2+\beta_b^2)(\gamma_a^2+\delta_a^2)],$$

by hypothesis (a). This is formula (11), and completes the proof of the theorem for the case where hypothesis (a) is met.

In formula (12), let $\cos^2 v = w$; then

$$2I = -\int_{\cos^2 v_a}^{\cos^2 v_b} (1-H)\,dw/[H+(1-H)w]$$

$$= \int_{\cos^2 v_a}^{\cos^2 v_b} (1-h)\,dw/[1-(1-h)w], \quad \text{where } h = 1/H$$

$$= \int_{\cos^2 v_a}^{\cos^2 v_b} d(1-h)w/[1-(1-h)w] + \int_{h_a}^{h_b} w\,dh/[1-(1-h)w].$$

Under hypothesis (b), $h = 1/H$ is monotonic increasing; hence dh, w, $[1-(1-h)w]$ are nonnegative. We thus have

$$2I \geq \log [1-(1-h_a)\cos^2 v_a]/[1-(1-h_b)\cos^2 v_b]$$

$$\geq \log [\gamma_a^2 + \delta_a^2/H_a][\alpha_b^2 + \beta_b^2]/[\beta_b^2 + \alpha_b^2/H_b][\gamma_a^2 + \delta_a^2]$$

$$\geq \log \{\Delta(\mu_n)^2/[\alpha_b^2 + \beta_b^2][\gamma_a^2 + \delta_a^2]\},$$

under Condition (b). This is formula (11) and completes the proof of the theorem for the case where hypothesis (b) is met.

The conclusion concerning ρ_n follows from

$$q_1(\rho_n)q_1(\mu_n) = -1, \quad q_2(\rho_n)q_2(\mu_n) > 0, \quad q_1(\rho_n)q_2(\rho_n) < 0$$

and

$$|q_1(\mu_n)| \geq |q_2(\mu_n)|.$$

THEOREM 9. *Under the hypotheses of Theorem* 8, *there exists at least one characteristic number* η_n *for the system* (1), (2) *such that* η_n *lies between* λ_n *and* μ_n, *and*

$$n\pi < v(b, \eta_n) - \tau(b, \eta_n) \leq (n + 1/2)\pi,$$

where n, λ_n, μ_n, *have the meanings described in Theorem* 8. *Also, there exists a characteristic number* η_{n+1} *between* μ_n *and* ρ_n *such that*

$$(n+1/2)\pi \leq v(b, \eta_{n+1}) - \tau(b, \eta_{n+1}) < (n+1)\pi.$$

The special case where $|q_2(\mu_n)| = 1$ *might make* $\eta_n = \eta_{n+1}$.

Proof. Since $v(b, \lambda) - \tau(b, \lambda)$ is continuous, it takes on the value $n\pi$ at least once when λ ranges from λ_n to μ_n. Let c_n be a value of λ for which

$$v(b, c_n) - \tau(b, c_n) = n\pi,$$

and note that

$$q_1(c_n) = 0, \quad q_2(c_n) q_2(\mu_n) > 0, \quad q_1(\mu_n) q_2(\mu_n) \geq 0, \quad |q_1(\mu_n)| \geq |q_2(\mu_n)|.$$

It follows from these inequalities and the continuity of $q_1(\lambda)$, $q_2(\lambda)$ that these functions are equal for at least one η_n between c_n and μ_n. The existence of η_{n+1} is established by a similar argument.

DEFINITION 3. By a *characteristic set*, S_n, for the system (1), (2) is meant the collection, $\{\eta_n\}$, of all characteristic numbers η_n for which

$$n\pi < v(b, \eta_n) - \tau(b, \eta_n) < (n+1)\pi.$$

THEOREM 10. *Let j and h be as described in Definition 2, and let j be greater than h. Under the hypotheses of Theorem 8, there exist at least $(j-h)/2$ or $(j-h+1)/2$, according as $j-h$ is even or odd, characteristic sets for system (1), (2).*

Proof. Since $v(b, \lambda) - \tau(b, \lambda)$ is continuous in λ on the interval from λ_1 to λ_2, where λ_1 and λ_2 are as described in Theorem 7, this function takes on the values

$$(h - 1/2)\pi, \ (h + 1/2)\pi, \ (h + 3/2)\pi, \cdots, (j - 1/2)\pi,$$

for values

$$c_0, c_1, \cdots, c_{j-h}$$

of λ on L. By Theorem 9, there is a characteristic number, η_i, between each pair of values, c_i and c_{i+1}. At most two of these can belong to the same characteristic set, this occurring only when the common value equals one of the c_i's. Hence there exist at least $(j-h)/2$, or $(j-h+1)/2$—according as $j-h$ is even or odd—characteristic sets.

THEOREM 11. *If η_n is any number in the characteristic set S_n, then $\phi(x, \eta_n)$ has at least n or $n+1$ zeros on $a \leq x \leq b$ according as $v_a - \tau_a$ is positive or negative on L. If $\alpha(x, \eta_n)/\beta(x, \eta_n)$ is nonincreasing on X, then $\phi(x, \eta_n)$ vanishes exactly n or $n+1$ times on X.*

Proof. We have $-\pi < v(a, \eta_n) - \tau(a, \eta_n) < \pi$ while $n\pi < v(b, \eta_n) - \tau(b, \eta_n)$, and $v(x, \eta_n) - \tau(x, \eta_n)$ is continuous on X. Hence this function takes on all positive integral multiples of π less than $n + 1$ at least once on (a, b). In case $v_a - \tau_a$ is negative, it also takes on the value zero. $\phi(x, \eta_n)$ vanishes when, and only when, $v(x, \eta_n) - \tau(x, \eta_n)$ is an integral or zero multiple of π.

If $\alpha(x, \eta_n)/\beta(x, \eta_n)$ is nonincreasing, $v(x, \eta_n) - \tau(x, \eta_n)$ is an actually increasing function of x, and hence takes on a given value only once on the interval X.

COROLLARY. *$y(x, \eta_n)$ vanishes at least $n - 1$ times on X while $z(x, \eta_n)$ vanishes at least n times on this interval. If $\alpha(x, \eta_n)/\beta(x, \eta_n)$ is nonincreasing, then $y(x, \eta_n)$ vanishes exactly $n - 1$, n, $n + 1$, or $n + 2$ times while $z(x, \eta_n)$ vanishes either n or $n + 1$ times on X.* (This corollary follows from the observation that $(n - 1/2)\pi < v(b, \eta_n) < (n + 3/2)\pi$.)

3. Example. The nonlinear system,

$$y' = \lambda(1 + 3x^2)[2 + \sin(y^2 + z^2)]z,$$

$$z' = -\lambda(1 + 3x^2)[2 + \sin(y^2 + z^2)]y,$$

has the family of solutions

$$y(x, \lambda) = A \sin[\lambda x(1 + x^2)(2 + \sin A^2) + B],$$

$$z(x, \lambda) = A \cos[\lambda x(1 + x^2)(2 + \sin A^2) + B],$$

where A and B are arbitrary constants, on

$$X: 0 \leq x \leq 1, \quad L: 0 < \lambda < \infty.$$

If boundary conditions

$$y(0, \lambda) = 0$$

$$z(0, \lambda) = z(1, \lambda) - y(1, \lambda)$$

are considered with this system, all hypotheses of Theorem 3, 7, 8, 9, 10, 11 are satisfied, where $m(\lambda) = \lambda$, $M(\lambda) = 12\lambda$, $h = 2$, and $j \geq 2$ may be taken as we please.

Characteristic numbers for this system are

$$\lambda_n = n\pi/(2 + \sin A^2) \qquad (n = 1, 2, \cdots),$$

where $A \neq 0$ may be arbitrarily assigned. Corresponding solutions are

$$y(x, \lambda_n) = A \sin n\pi x (1 + x^2),$$

$$z(x, \lambda_n) = A \cos n\pi x (1 + x^2).$$

It is noted that all λ on the interval $n\pi/3 \leq \lambda \leq n\pi$ belong to the characteristic set S_n which yields the above solution. Thus the characteristic sets contain continua, and these sets are not mutually exclusive. These properties reflect the nonlinearity of the system.

Theorem 8 uses hypotheses which, in effect, prevent amplitudes of oscillations in the solution functions from becoming too small as x takes on larger values. This property was important in establishing existence and oscillation theorems for the system. For stability investigations where it is desired that the amplitudes remain bounded as x increases, the hypotheses used in Theorem 8 would need to be changed to the extent of making $H(x, \mu_n)$ monotonic decreasing for Condition (a), and monotonic increasing for Condition (b). For the specific example given above, Conditions (a) and (b) both hold, so it is not surprising that the solutions given have bounded amplitudes for all values of x.

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