

Pacific Journal of Mathematics

**THE NUMBER OF SOLUTIONS OF CERTAIN TYPES OF
EQUATIONS IN A FINITE FIELD**

L. CARLITZ

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1. Using a very simple principle, Morgan Ward [3] indicated how one can obtain all solutions of the equation

$$(1) \quad y^m = f(x_1, \dots, x_r) \quad (y, x_i \in F),$$

where F is an arbitrary field, $f(x_1, \dots, x_r)$ is a homogeneous polynomial of degree n with coefficients in F , and $(m, n) = 1$. The same principle had been applied earlier to a special equation by Hua and Vandiver [2]. If this principle is applied in the case of a finite field F we readily obtain the total number of solutions of equations of the type (1). Somewhat more generally, let

$$f_i(x_i) = f_i(x_{i1}, \dots, x_{is_i}) \quad (i = 1, \dots, r)$$

denote r polynomials with coefficients in $GF(q)$, and assume

$$(2) \quad f_i(\lambda x_1, \dots, \lambda x_{s_i}) = \lambda^{m_i} f_i(x_1, \dots, x_{s_i}) \quad (\lambda \in GF(q));$$

assume also

$$(3) \quad (m, m_i, q - 1) = 1 \quad (i = 1, \dots, r).$$

We consider the equation

$$(4) \quad y^m = f_1(x_{11}, \dots, x_{1s_1}) + \dots + f_r(x_{r1}, \dots, x_{rs_r})$$

in $s_1 + \dots + s_r + 1$ unknowns.

Suppose first we have a solution of (4) with $y \neq 0$. Select integers h, k, l such that

$$(5) \quad hm + km_1 m_2 \dots m_r + l(q - 1) = 1, \quad (h, q - 1) = 1;$$

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this can be done in view of (3). Next put

$$(6) \quad y = \lambda^h, \quad x_{ij} = \lambda^{kM/m_i} z_{ij} \quad (M = m_1 m_2 \cdots m_r).$$

Substituting in (4) and using (2), we get

$$\lambda^{hm} = \lambda^{kM} \{f_1(z_1) + \cdots + f_r(z_r)\}.$$

Since $\lambda^{q-1} = 1$, it is clear from (5) that

$$(7) \quad \lambda = f_1(z_1) + \cdots + f_r(z_r).$$

Thus any solution (y, x_{ij}) of (4) with $y \neq 0$ can be obtained from (6) and (7) by assigning arbitrary values to z_{ij} such that the right member of (7) does not vanish. Let N denote the total number of solutions of (4) and let N_0 denote the number of solutions with $y = 0$. Thus there are $N - N_0$ sets z_{ij} for which $\lambda \neq 0$. Since in all there are $q^{s_1 + \cdots + s_r}$ sets z_{ij} it follows that

$$(8) \quad N = q^{s_1 + \cdots + s_r}.$$

This proves:

THEOREM. *Let the polynomials f_i satisfy (2) and (3). Then the total number of solutions of (4) is furnished by (8).*

2. In Theorem II of [2] Hua and Vandiver proved that the number of solutions of

$$(9) \quad c_1 x_1^{a_1} + c_2 x_2^{a_2} + \cdots + c_s x_s^{a_s} = 0$$

subject to the conditions

$$c_1 c_2 \cdots c_s x_1 x_2 \cdots x_s \neq 0, \quad (a_i, q-1) = k_i, \quad (k_i, k_j) = 1 \quad \text{for } i \neq j,$$

is equal to

$$(10) \quad \frac{q-1}{q} \{(q-1)^{s-1} + (-1)^s\}.$$

It is easy to show that (10) implies that the total number of solutions of (9) is equal to q^{s-1} , which agrees with (8). Conversely if N_s denotes the number of nonzero solutions of (9), and we assume that

$$(11) \quad (k_i, k_j) = 1 \quad (i, j = 1, \dots, s; i \neq j),$$

then using (8) we get

$$q^{s-1} = N_s + \binom{s}{1} N_{s-1} + \binom{s}{2} N_{s-2} + \dots + \binom{s}{s-1} N_1 + 1.$$

Hence (if we take $N_0 = 1$)

$$\begin{aligned} (q-1)^s &= \sum_{r=1}^s (-1)^{s-r} \binom{s}{r} q \sum_{t=0}^r \binom{r}{t} N_t + (-1)^s \\ &= q \sum_{r=0}^s (-1)^{s-r} \binom{s}{r} \sum_{t=0}^r \binom{r}{t} N_t - (-1)^s (q-1) \\ &= q \sum_{t=0}^s \binom{s}{t} N_t \sum_{r=t}^s (-1)^{s-r} \binom{s-t}{s-r} - (-1)^s (q-1) \\ &= qN_s - (-1)^s (q-1), \end{aligned}$$

and (10) follows at once. Thus if we assume (11) then (8) and (10) are equivalent.

If in place of (11) we assume only that

$$(12) \quad (k_1, k_2 k_3 \dots k_s) = 1,$$

the situation is somewhat different. As above let N_s denote the number of non-zero solutions of (9), and let M_{s-1} denote the total number of solutions x_2, \dots, x_s of

$$(13) \quad c_2 x_2^{a_2} + c_3 x_3^{a_3} + \dots + c_s x_s^{a_s} = 0.$$

Using (8) we now get

$$(14) \quad q^{s-1} = M_{s-1} + N_s + \binom{s-1}{1} N_{s-1} + \dots + \binom{s-1}{s-1} N_1,$$

which implies (with $M_0 = 1$)

$$(15) \quad (q-1)^{s-1} = \sum_{r=0}^{s-1} (-1)^{s-1-r} \binom{s-1}{r} M_r + N_s.$$

Thus making only the assumption (12) we see how the number of solutions of (13) can be expressed in terms of N_s and *vice versa*.

3. **Returning to equation (4)**, we see that a similar result can be obtained if we allow f_i to contain additional unknowns:

$$f_i(x_i; u_i) = f_i(x_{i1}, \dots, x_{is_i}; u_{i1}, \dots, u_{it_i}),$$

and assume that (2) holds only for the x 's. Then the number of solutions (y, x_{ij}, u_{hk}) of (4) becomes

$$q^{s_1 + \dots + s_r + t_1 + \dots + t_r}.$$

Similarly we may replace the left member of (4) by

$$\gamma_1^{a_1} \gamma_2^{a_2} \dots \gamma_s^{a_s} \quad (a_1, a_2, \dots, a_s) = m.$$

Then assuming (3) we again find that the number of solutions of the modified equation is equal to

$$q^{s_1 + \dots + s_r + s - 1}.$$

This kind of generalization lends itself well to equation (9). For example it is easy to show (see [1, Theorem 10]) that the total number of solutions of the equation

$$\sum_{i=1}^t c_i \prod_{j=1}^{k_i} x_{ij}^{a_{ij}} = 0,$$

subject to $(a_{i1}, \dots, a_{ik_i}, q-1) = d_i$, $(d_i, d_j) = 1$ for $i \neq j$, is equal to

$$q^{k_1 + \dots + k_t - 1}.$$

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