THE NUMBER OF SOLUTIONS OF CERTAIN TYPES OF EQUATIONS IN A FINITE FIELD

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1. Using a very simple principle, Morgan Ward [3] indicated how one can obtain all solutions of the equation

\[ y^m = f(x_1, \ldots, x_r) \quad (y, x_i \in F), \]

where \( F \) is an arbitrary field, \( f(x_1, \ldots, x_r) \) is a homogeneous polynomial of degree \( n \) with coefficients in \( F \), and \( (m, n) = 1 \). The same principle had been applied earlier to a special equation by Hua and Vandiver [2]. If this principle is applied in the case of a finite field \( F \) we readily obtain the total number of solutions of equations of the type (1). Somewhat more generally, let

\[ f_i(x_i) = f_i(x_{i1}, \ldots, x_{is_i}) \quad (i = 1, \ldots, r) \]

denote \( r \) polynomials with coefficients in \( GF(q) \), and assume

\[ f_i(\lambda x_1, \ldots, \lambda x_{s_i}) = \lambda^{m_i} f_i(x_1, \ldots, x_{s_i}) \quad (\lambda \in GF(q)); \]

assume also

\[ (m_i, q - 1) = 1 \quad (i = 1, \ldots, r). \]

We consider the equation

\[ y^m = f_1(x_{11}, \ldots, x_{1s_1}) + \cdots + f_r(x_{r1}, \ldots, x_{rs_r}) \]

in \( s_1 + \cdots + s_r + 1 \) unknowns.

Suppose first we have a solution of (4) with \( y \neq 0 \). Select integers \( h, k, l \) such that

\[ km + km_1 m_2 \cdots m_r + l(q - 1) = 1, \quad (h, q - 1) = 1; \]
this can be done in view of (3). Next put

$$\gamma = \lambda^h, \ x_{ij} = \lambda^{kM/m_i} z_{ij}$$

Substituting in (4) and using (2), we get

$$\lambda^{hm} = \lambda^{kM} \{ f_1(z_1) + \cdots + f_r(z_r) \}.$$  

Since $\lambda^{q-1} = 1$, it is clear from (5) that

$$\lambda = f_1(z_1) + \cdots + f_r(z_r).$$

Thus any solution $(\gamma, x_{ij})$ of (4) with $\gamma \neq 0$ can be obtained from (6) and (7) by assigning arbitrary values to $z_{ij}$ such that the right member of (7) does not vanish. Let $N$ denote the total number of solutions of (4) and let $N_0$ denote the number of solutions with $\gamma = 0$. Thus there are $N - N_0$ sets $z_{ij}$ for which $\lambda \neq 0$. Since in all there are $q^{s_1 + \cdots + s_r}$ sets $z_{ij}$ it follows that

$$N = q^{s_1 + \cdots + s_r}.$$  

This proves:

**Theorem.** Let the polynomials $f_i$ satisfy (2) and (3). Then the total number of solutions of (4) is furnished by (8).

2. In Theorem II of [2] Hua and Vandiver proved that the number of solutions of

$$c_1 x_1^{a_1} + c_2 x_2^{a_2} + \cdots + c_s x_s^{a_s} = 0$$

subject to the conditions

$$c_1 c_2 \cdots c_s x_1 x_2 \cdots x_s \neq 0, \ (a_i, q - 1) = k_i, \ (k_i, k_j) = 1 \text{ for } i \neq j,$$

is equal to

$$q^{\frac{q - 1}{q}} [(q - 1)^{s-1} + (-1)^s].$$

It is easy to show that (10) implies that the total number of solutions of (9) is equal to $q^{s-1}$, which agrees with (8). Conversely if $N_s$ denotes the number of nonzero solutions of (9), and we assume that
(11) \((k_i, k_j) = 1\) \((i, j = 1, \ldots, s; i \neq j)\),

then using (8) we get

\[ q^{s-1} = N_s + \binom{s}{1} N_{s-1} + \binom{s}{2} N_{s-2} + \cdots + \binom{s}{s-1} N_1 + 1. \]

Hence (if we take \(N_0 = 1\))

\[
(q - 1)^s = \sum_{r=1}^{s} (-1)^{s-r} \binom{s}{r} q^r \sum_{t=0}^{r} \binom{r}{t} N_t - (-1)^s (q - 1)
\]

\[
= q \sum_{t=0}^{s} \left( \sum_{r=t}^{s} (-1)^{s-r} \binom{s}{r} \binom{s-t}{s-r} \right) N_t - (-1)^s (q - 1)
\]

\[= qN_s - (-1)^s (q - 1), \]

and (10) follows at once. Thus if we assume (11) then (8) and (10) are equivalent.

If in place of (11) we assume only that

(12) \((k_1, k_2 k_3 \cdots k_s) = 1\),

the situation is somewhat different. As above let \(N_s\) denote the number of non-zero solutions of (9), and let \(M_{s-1}\) denote the total number of solutions \(x_2, \cdots, x_s\) of

(13) \(c_2 x_2^{a_2} + c_3 x_3^{a_3} + \cdots + c_s x_s^{a_s} = 0\).

Using (8) we now get

(14) \(q^{s-1} = M_{s-1} + N_s + \binom{s-1}{1} N_{s-1} + \cdots + \binom{s-1}{s-1} N_1\),

which implies (with \(M_0 = 1\))
Thus making only the assumption (12) we see how the number of solutions of (13) can be expressed in terms of $N_s$ and vice versa.

3. Returning to equation (4), we see that a similar result can be obtained if we allow $f_i$ to contain additional unknowns:

$$f_i(x_i; u_i) = f_i(x_i^1, \ldots, x_is_i; u_i^1, \ldots, u_i^r),$$

and assume that (2) holds only for the $x$'s. Then the number of solutions $(y, x_{ij}, u_{hk})$ of (4) becomes

$$q^{s_1 + \cdots + s_r + t_1 + \cdots + t_r}.$$

Similarly we may replace the left member of (4) by

$$y_1^{a_1} y_2^{a_2} \cdots y_s^{a_s}$$

and assume that (3) holds only for the $y$'s. Then assuming (3) we again find that the number of solutions of the modified equation is equal to

$$q^{s_1 + \cdots + s_r + s-1}.$$

This kind of generalization lends itself well to equation (9). For example it is easy to show (see [1, Theorem 10]) that the total number of solutions of the equation

$$\sum_{i=1}^t c_i \prod_{j=1}^{k_i} x_i^{a_{ij}} = 0,$$

subject to $(a_{i1}, \ldots, a_{ik_i}, q-1) = d_i$, $(d_i, d_j) = 1$ for $i \neq j$, is equal to

$$q^{k_1 + \cdots + k_t - 1}.$$

**Reference**


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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, c/o University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

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