CONSTRUCTIONS FOR POLES AND POLARS IN $n$-DIMENSIONS

ARTHUR PENTLAND DEMPSTER AND SEYMOUR SCHUSTER
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A. P. DEMPSTER AND S. SCHUSTER

1. Introduction. As far back as 1847, von Staudt [2, p. 131-136] introduced the notion of handling a symmetric polarity (that is, a nonnull polarity) by means of a self-polar simplex and an additional pair of corresponding elements. In projective space of two dimensions $(S_2)$ such a polarity is completely determined by a self-polar triangle $A_1A_2A_3$, a point $P$, and its polar line $p$. We write this polarity as $(A_1A_2A_3)(Pp)$. In $S_3$, the polarity is determined by a self-polar tetrahedron $A_1A_2A_3A_4$, a point $P$, and its polar plane $\pi$. We write it $(A_1A_2A_3A_4)(P\pi)$. In general, we have a polarity in $S_n$ determined by the self-polar simplex $A_1A_2\cdots A_{n+1}$, a point $P$, and its corresponding polar prime or hyperplane $\pi$. We write it $(A_1A_2\cdots A_{n+1})(P\pi)$.

Left unanswered by von Staudt and his followers is the following question: Given an arbitrary point $X$, how can we construct the polar prime $\chi$ of $X$? And, conversely, given the prime $\chi$, how do we actually find its pole, the point $X$?

2. Construction. The construction of the polar line $x$ of an arbitrary point $X$ for the polarity $(A_1A_2A_3)(Pp)$ in $S_2$ was given by Coxeter [1, 64]. We give a direct generalization of this to $n$ dimensions: to find the polar prime $\chi$ of an arbitrary point $X$ relative to $(A_1A_2\cdots A_{n+1})(P\pi)$.

Consider first the point $X$ not in any face of $A_1A_2\cdots A_{n+1}$. Let $\alpha_i$ denote face $A_1A_2\cdots A_{i-1}A_iA_{i+1}\cdots A_{n+1}$, and let

$$A_i' = PX \cdot \alpha_i, \quad P_i = XA_i \cdot \pi, \quad \text{and} \quad X^i = PA_i \cdot P_iA_i'. $$

In the plane $PXA_i$ we have pairs $P$, $P_i$ and $A_i$, $A_i'$ conjugate under the induced plane polarity. By Hesse's theorem in the plane [1, pp. 60-61], $X$ and $X^i$ are conjugate for the induced polarity, and hence for the given polarity. In this manner we determine $n + 1$ points $X^1, X^2, \ldots, X^{n+1}$ lying in $\chi$. The points $X^1, X^2, \ldots, X^n$ determine $\chi$ since otherwise they must lie in an $(n-2)$-flat which implies that the flat determined by $P, X^1, \ldots, X^n$ is of at most $(n-1)$ dimensions, which is impossible since the space contains $P, A_1, A_2, \ldots, A_n$. It

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follows that \( x \) is determined by any \( (n - 1) \) of the points \( X^i \). This completes the construction in \( S_n \) for general \( X \). This is illustrated for \( n = 3 \), and is easily seen to yield Coxeter's construction for \( n = 2 \).

A second approach is to reduce the question of finding \( x \) in \( S_n \) to two analogous constructions in \( (n - 1) \) dimensions, namely in any two faces \( \alpha_i \). Under the polarity induced in \( \alpha_i \) the point \( X_i = XA_i \cdot \alpha_i \) maps into an \( (n - 2) \)-flat \( x_i \) consisting of points conjugate to \( X \). For the general \( X \) considered, no two \( x_i \) coincide; hence, any two of them determine an \( (n - 1) \)-flat of points conjugate to \( X \). This can only be \( x \). Using this idea we can reduce the construction in \( S_n \) to \( 2^r \) analogous constructions in \( n - r \) dimensions, and at any stage of this induction on \( r \), we may use the first method to solve the question completely.

In particular, if \( n = 2 \) we can construct directly by the first method or use the construction for corresponding points in two involutions on the sides of \( A_1A_2A_3 \). If \( n = 3 \) we can use the first method, or carry out constructions in two faces of \( A_1A_2A_3A_4 \), or carry out constructions in four edges of \( A_1A_2A_3A_4 \).

Going back to \( n \) dimensions, suppose \( X \) is not of general position; that is, \( X \) lies in a face \( \alpha_i \). If \( X \) lies in \( r \) such faces we may name these \( \alpha_1, \ldots, \alpha_r \). Then \( x \) contains \( A_1, \ldots, A_r \). Considering the \( (n - r) \)-flat determined by simplex \( A_{r+1} \cdots A_{n+1} \), we see that the polarity induced in this space has \( A_{r+1} \cdots A_{n+1} \) as a self-polar simplex and \( X \) belongs to the space but is not on a face of \( A_{r+1} \cdots A_{n+1} \). Thus, we can use the first method to determine the polar prime \( x' \) of \( X \) in this space. Then \( A_1, \ldots, A_r, \) and \( x' \) generate an \( (n - 1) \)-flat of points conjugate to \( X \). This \( (n - 1) \)-flat is \( x \).

The problem of finding \( X \) when given \( x \) is solved by dualizing the foregoing procedures.
REFERENCES


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