

# Pacific Journal of Mathematics

## **SOME DETERMINANTS INVOLVING BERNOULLI AND EULER NUMBERS OF HIGHER ORDER**

FRANK R. OLSON

# SOME DETERMINANTS INVOLVING BERNOULLI AND EULER NUMBERS OF HIGHER ORDER

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**1. Introduction.** In this paper we evaluate certain determinants whose elements are the Bernoulli, Euler, and related numbers of higher order. In the notation of Nörlund [1, Chapter 6] these numbers may be defined as follows: the Bernoulli numbers of order  $n$  by

$$(1.1) \quad \left( \frac{t}{e^t - 1} \right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} B_v^{(n)},$$

the related “ $D$ ” numbers by

$$(1.2) \quad \left( \frac{t}{\sin t} \right)^n \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} D_{2v}^{(n)} \quad (D_{2v+1}^{(n)} = 0),$$

the Euler numbers of order  $n$  by

$$(1.3) \quad (\sec t)^n = \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} E_{2v}^{(n)} \quad (E_{2v+1}^{(n)} = 0),$$

and the “ $C$ ” numbers by

$$(1.4) \quad \left( \frac{2}{e^t + 1} \right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} \frac{C_v^{(n)}}{2^v}.$$

(By  $n$  we denote an arbitrary complex number. When  $n = 1$ , we omit the upper index in writing the numbers; for example,  $B_v^{(1)} = B_v$ .)

We evaluate determinants such as

$$|B_i^{(x_j)}| \quad (i, j = 0, 1, \dots, m)$$

for the Bernoulli numbers, and similar determinants for the other numbers. The

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proofs of these results follow from the evaluation of a determinant of a more general nature; see (3.4), below. Finally, a number of applications are given.

**2. Preliminaries and notation.** The numbers  $B_v^{(n)}$ ,  $D_{2v}^{(n)}$ ,  $E_{2v}^{(n)}$ , and  $C_v^{(n)}$  may be expressed as polynomials in  $n$  of degree  $v$  [1, Chapter 6]; in particular,

$$B_0^{(n)} = D_0^{(n)} = E_0^{(n)} = C_0^{(n)} = 1.$$

Although little is known about these polynomials, it will suffice for our purposes to give explicitly the values of the coefficients of  $n^v$  in each of the four cases.

Considering first the Bernoulli numbers, we use the recursion formula [1, p. 146]

$$(2.1) \quad B_v^{(n)} = -\frac{n}{v} \sum_{s=1}^v (-1)^s \binom{v}{s} B_s B_{v-s}^{(n)}.$$

Let

$$B_v^{(n)} = b_v n^v + b_{v-1} n^{v-1} + \dots + b_0,$$

$$B_{v-1}^{(n)} = b'_{v-1} n^{v-1} + b'_{v-2} n^{v-2} + \dots + b'_0,$$

and compare coefficients of  $n^v$  on both sides of (2.1). We find that

$$b_v = -\frac{1}{v} (-1) \binom{v}{1} B_1 b'_{v-1}.$$

But  $B_1 = -1/2$  and therefore  $b_v = -b'_{v-1}/2$ . Since  $B_0^{(n)} = 1$ , the preceding leads us recursively to

$$(2.2) \quad B_v^{(n)} = \left(-\frac{1}{2}\right)^v n^v + b_{v-1} n^{v-1} + \dots + b_0.$$

In a similar fashion the formula [1, p. 146]

$$(2.3) \quad C_{v+1}^{(n)} = -n \sum_{s=0}^v (-1)^s \binom{v}{s} C_s C_{v-s}^{(n)},$$

coupled with  $C_0^{(n)} = 1$ , permits us to write

$$(2.4) \quad C_v^{(n)} = (-1)^v n^v + c_{v-1} n^{v-1} + \dots + c_0.$$

As for the Euler numbers, we consider the symbolic formula [1, p. 124]

$$(2.5) \quad (E^{(n)} + 1)^{2v} + (E^{(n)} - 1)^{2v} = 2E_{2v}^{(n-1)}$$

in which, after expansion, exponents on the left side are degraded to subscripts. Hence we have

$$(2.6) \quad E_{2v}^{(n)} + \frac{(2v)(2v-1)}{1 \cdot 2} E_{2v-2}^{(n)} + \dots = E_{2v}^{(n-1)}.$$

Writing

$$E_{2v}^{(n)} = e_v n^v + e_{v-1} n^{v-1} + \dots + e_0,$$

and

$$E_{2v-2}^{(n)} = e'_{v-1} n^{v-1} + e'_{v-2} n^{v-2} + \dots + e'_0,$$

we see first that

$$E_{2v}^{(n)} - E_{2v}^{(n-1)} = v e_v n^{v-1} + \text{terms of lower degree}.$$

Hence comparing coefficients of  $n^{v-1}$  in (2.6) we have

$$e_v = - \frac{(2v)(2v-1)}{2v} e'_{v-1}.$$

Since  $E_0^{(n)} = 1$ , we obtain recursively

$$(2.7) \quad E_{2v}^{(n)} = \frac{(2v)!}{(-2)^v v!} n^v + e_{v-1} n^{v-1} + \dots + e_0.$$

Next, from [1, p. 129]

$$(2.8) \quad (D^{(n)} + 1)^{2v+1} - (D^{(n)} - 1)^{2v+1} = 2(2v+1)D_{2v}^{(n-1)},$$

we find that

$$(2.9) \quad D_{2v}^{(n)} = \left(-\frac{1}{6}\right)^v \frac{(2v)!}{v!} n^v + d_{n-1} n^{v-1} + \dots + d_0.$$

We shall employ the difference operator  $\Delta_d = \Delta$  for which

$$\Delta f(x) = f(x+d) - f(x) \quad \text{and} \quad \Delta^v = \Delta \cdot \Delta^{v-1}.$$

We recall that if

$$f(x) = a_v x^v + a_{v-1} x^{v-1} + \dots + a_0,$$

then

$$(2.10) \quad \Delta^v f(x) = a_v d^v v!$$

**3. Main results.** Let

$$(3.1) \quad f_n(x) = a_{n,n} x^n + a_{n,n-1} x^{n-1} + \dots + a_{n,0} \quad (a_{n,n} \neq 0),$$

and consider the determinant

$$(3.2) \quad |f_i(x_j)| \quad (i, j = 0, 1, \dots, m).$$

This may be written as the product of the two determinants

$$(3.3) \quad \begin{vmatrix} a_{0,0} & 0 & \dots & 0 \\ a_{1,0} & a_{1,1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{m,0} & a_{m,1} & \dots & a_{m,m} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_m \\ \dots & \dots & \dots & \dots \\ x_0^m & x_1^m & \dots & x_m^m \end{vmatrix}.$$

The first determinant in (3.3) reduces simply to the product of the elements on the main diagonal, and the second is the familiar Vandermond determinant. Hence

$$(3.4) \quad |f_i(x_j)| = \prod_{k=0}^m a_{k,k} \prod_{r>s} (x_r - x_s) \quad (r, s = 0, 1, \dots, m).$$

If we let

$$f_i(x_j) = B_i^{(x_j)},$$

then it follows from (3.4) and (2.2) that

$$(3.5) \quad |B_i^{(x_j)}| = \prod_{k=0}^m \left(-\frac{1}{2}\right)^k \prod_{r>s} (x_r - x_s) \quad (i, j, r, s = 0, 1, \dots, m).$$

Application of (3.4) to (2.4), (2.7), and (2.9) yields results of a similar nature for the  $C$ ,  $D$ , and  $E$  numbers. Consequently we have:

**THEOREM 1.** For  $i, j = 0, 1, \dots, m$ ,

$$(i) \quad |B_i^{(x_j)}| = \prod_{k=0}^m \left(-\frac{1}{2}\right)^k \prod_{r>s} (x_r - x_s),$$

$$(ii) \quad |C_i^{(x_j)}| = \prod_{k=0}^m (-1)^k \prod_{r>s} (x_r - x_s),$$

$$(iii) \quad |D_{2i}^{(x_j)}| = \prod_{k=0}^m \left(-\frac{1}{6}\right)^k \frac{(2k)!}{k!} \prod_{r>s} (x_r - x_s),$$

$$(iv) \quad |E_{2i}^{(x_j)}| = \prod_{k=0}^m \left(-\frac{1}{2}\right)^k \frac{(2k)!}{k!} \prod_{r>s} (x_r - x_s).$$

If we take  $x_j = a + jd$  then we obtain:

**COROLLARY 1.** For  $i, j = 0, 1, \dots, m$ ,  $a$  and  $d$  constants,

$$(i) \quad |B_i^{(a+jd)}| = \prod_{k=0}^m \left(-\frac{d}{2}\right)^k k!,$$

$$(ii) \quad |C_i^{(a+jd)}| = \prod_{k=0}^m (-d)^k k!,$$

$$(iii) \quad |D_{2i}^{(a+jd)}| = \prod_{k=0}^m \left(-\frac{d}{6}\right)^k (2k)!,$$

$$(iv) \quad |E_{2i}^{(a+jd)}| = \prod_{k=0}^m \left(-\frac{d}{2}\right)^k (2k)!$$

If we let

$$f_i(a + xd_i) = g_i(x),$$

$f_i(x)$  defined as in (3.1), then we can readily show by the above method that

$$(3.6) \quad |f_i(a + jd_i)| = \prod_{k=0}^m a_{k,k} d_k^k k! \quad (i, j = 0, 1, \dots, m).$$

Hence (3.6) implies

$$|B_i^{(a+jd_i)}| = \prod_{k=0}^m \left(-\frac{d_k}{2}\right)^k k!,$$

with like results for the other numbers.

We remark that the determinants of Corollary 1 may also be evaluated by a succession of column subtractions.

**4. Applications.** We consider first the determinant

$$(4.1) \quad |B_i^{(a+jd)}(x)| \quad (i, j = 0, 1, \dots, m; a, d \text{ constants}),$$

where  $B_i^{(n)}(x)$  is the Bernoulli polynomial of order  $n$  defined by [1, p.145]

$$\left(\frac{t}{e^t - 1}\right)^n e^{xt} = \sum_{r=0}^{\infty} \frac{t^r}{r!} B_r^{(n)}(x).$$

(For  $x = 0$ ,  $B_v^{(n)}(0) = B_v^{(n)}$ , the Bernoulli number of order  $n$ .) Also, by [1, p. 143 ],

$$B_v^{(n)}(x) = \sum_{s=0}^v \binom{v}{s} x^{v-s} B_s^{(n)}.$$

Consequently

$$(4.2) \quad |B_i^{(a+jd)}(x)| = \left| \sum_{s=0}^i \binom{i}{s} x^{i-s} B_s^{(a+jd)} \right|.$$

If we define

$$\binom{0}{0} = 1 \text{ and } \binom{i}{j} = 0 \text{ for } j > i,$$

then the right member of (4.2) may be written as the product of the two determinants;

$$\left| \binom{i}{j} x^{i-j} \right| \cdot |B_i^{(a+jd)}|.$$

The first determinant has value 1 and hence, by Corollary 1 (i),

$$(4.3) \quad |B_i^{(a+jd)}(x)| = \prod_{k=0}^m \left(-\frac{d}{2}\right)^k k!.$$

The Bernoulli polynomials may also be expressed in terms of the  $D$  numbers by [1, p. 130]

$$(4.4) \quad B_v^{(n)}(x) = \sum_{s=0}^{[v/2]} \binom{v}{2s} \left(x - \frac{n}{2}\right)^{v-2s} D_{2s}^{(n)} / 2^{2s}.$$

If in (4.3) we let  $x = hn$ ,  $h \neq 1/2$ , then

$$(4.5) \quad B_v^{(n)}(hn) = \sum_{s=0}^{[v/2]} \binom{v}{2s} \left(h - \frac{1}{2}\right)^{v-2s} n^{v-2s} D_{2s}^{(n)} / 2^{2s}.$$

Since  $D_{2n}^{(n)}$  may be written as a polynomial in  $n$  of degree  $s$ , and  $D_0^{(n)} = 1$ , it follows readily from (4.4) that, expressed as a polynomial in  $n$ ,

$$(4.6) \quad B_v^{(n)}(hn) = \left(h - \frac{1}{2}\right)^v n^v + \text{terms of lower degree}.$$

Consequently, using the same procedure that gave (3.4), we can show for  $a, d$  fixed constants,  $i, j = 0, 1, \dots, m$ , that

$$(4.7) \quad |B_i^{(a+jd)}(h(a+jd))| = \prod_{k=0}^m \left(h - \frac{1}{2}\right)^k d^k k!.$$

For  $h = 0$ , (4.7) reduces to the case of Corollary 1 (i). If  $h = 1/2$  and  $v$  is odd, then it follows from (4.4) that



$$B_v^{(n)}(n/2) = 0.$$

Therefore for  $m \geq 1$ , the value of the determinant in (4.6) is zero. However, if  $v$  is even, then

$$B_v^{(n)}(n/2) = D_v^{(n)}/2^{2v},$$

and

$$(4.7)' \quad \left| B_{2i}^{(a+jd)} \left( \frac{a+jd}{2} \right) \right| = |D_{2i}^{(a+jd)}/2^{2i}| = \prod_{k=0}^m \left( -\frac{d}{24} \right)^k (2k)!,$$

where in evaluating the second determinant we have applied Corollary 1(iii).

Finally, it is of interest to point out that [1, p. 4]

$$\Delta^v f(x) = \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} f(x+jd)$$

together with (2.2), (2.4), (2.7), (2.9), and (2.10) yield the recursion formulas

$$(4.8) \quad \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} B_v^{(a+jd)} = \left( -\frac{d}{2} \right)^v v!,$$

$$(4.9) \quad \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} C_v^{(a+jd)} = (-d)^v v!,$$

$$(4.10) \quad \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} E_{2v}^{(a+jd)} = \left( -\frac{d}{2} \right)^v (2v)!$$

and

$$(4.11) \quad \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} D_{2v}^{(a+jd)} = \left( -\frac{d}{6} \right)^v (2v)!.$$

**5. Some additional results.** The above methods may also be applied to the evaluation of determinants involving the classic orthogonal polynomials. We consider first the Laguerre polynomials defined by [2, p. 97]

$$(5.1) \quad L_n^{(\alpha)} = \prod_{v=0}^n \binom{n+\alpha}{n-v} \frac{(-x)^v}{v!} .$$

Setting  $\alpha = a + jd$  and writing (5.1) as a polynomial in  $j$  we have

$$L_n^{(a+jd)}(x) = j^n \frac{d^n}{n!} + \text{terms of lower degree} .$$

Consequently, as in § 3, we obtain

$$(5.2) \quad |L_i^{(a+jd)}(x)| = \prod_{k=0}^{m-1} d^k = d^{\frac{1}{2}m(m-1)} \quad (i, j = 0, 1, \dots, m-1) .$$

For the Jacobi polynomials defined by [2, p. 67]

$$(5.3) \quad P_n^{(\alpha, \beta)}(x) = \sum_{v=0}^n \binom{n+\alpha}{n-v} \binom{n+\beta}{v} \left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v}$$

we set  $\alpha = a + jd$  and hold  $\beta$  fixed. Then, as a polynomial in  $j$

$$P_n^{(a+jd, \beta)}(x) = j^n \frac{d^n}{2^n} \frac{(x+1)^n}{n!} + \text{terms of lower degree} .$$

Hence, we find

$$(5.4) \quad |P_i^{(a+jd, \beta)}(x)| = \left\{ \frac{(x+1)d}{2} \right\}^{\frac{1}{2}m(m-1)} \quad (i, j = 0, 1, \dots, m-1) .$$

Similarly

$$(5.5) \quad |P_i^{(a, b+je)}(x)| = \left\{ \frac{(x-1)e}{2} \right\}^{\frac{1}{2}m(m-1)} \quad (i, j = 0, 1, \dots, m-1) .$$

We consider next, as a polynomial in  $j$ ,

$$\begin{aligned} &P_n^{(a+jd, b+je)}(x) \\ &= j^n \sum_{v=0}^n \frac{\alpha^{n-v}}{(n-v)!} \frac{e^v}{v!} \left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v} + \text{terms of lower degree} \end{aligned}$$

$$= \frac{j^n}{n!} \left[ \frac{(d+e)x + d - e}{2} \right]^n + \text{terms of lower degree,}$$

which yields

$$(5.6) \quad |P_i^{(a+jd, b+je)}(x)| = \left[ \frac{(d+e)x + d - e}{2} \right]^{\frac{1}{2}m(m-1)}$$

( $i, j = 0, 1, \dots, m-1$ ).

Finally, for  $\alpha = \beta$ , the Jacobi polynomials reduce to the ultraspherical polynomials  $P^{(\alpha)}(x)$ . It follows from (5.6) that

$$(5.7) \quad |P_i^{(a+jd)}(x)| = (dx)^{\frac{1}{2}m(m-1)} \quad (i, j = 0, 1, \dots, m-1).$$

#### REFERENCES

1. N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924.
2. Gabor Szegő, *Orthogonal polynomials*, New York, 1939.

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