Pacific Journal of Mathematics

IN THIS ISSUE—

Vol. 5, No. 2 **October, 1955**

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, Alfred Horn at the University of California, Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, c/o University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

AN EXTENSION OF A THEOREM OF JORDAN AND VON NEUMANN

LEONARD M. BLUMENTHAL

1. Introduction. Let $\{E\}$ denote the class of generalized euclidean spaces E (that is, $E \subseteq \{E\}$ provided all finite dimensional subspaces of E are euclidean spaces). The problem of characterizing metrically the class $\{E\}$ with respect to the class $\{B\}$ of all Banach spaces has been solved in many different ways.[1] Fréchet's characteristic conditions [5]

$$(*) \qquad \frac{1}{2} \sum_{i,j=1}^{3} [\,\|p_i\|^2 + \|p_j\|^2 - \|p_i - p_j\|^2]x_i x_j \geq 0, \qquad (p_1, p_2, p_3 \in \mathbf{B})$$

was immediately weakened by Jordan and von Neumann [6] to

$$(**) \qquad \|p_1 + p_2\|^2 + \|p_1 - p_2\|^2 = 2(\|p_1\|^2 + \|p_2\|^2) \qquad (p_1, p_2 \in \mathbf{B}).$$

This relation has now become a kind of standard to which others repair by showing that it is implied by newly postulated conditions [3, 4, 10], and it has been, apparently, the motivation of work in which it does not enter directly [7, 9]. Perhaps the best possible result in this direction, however, is due to Aronszajn [1] who assumed merely that

$$\|(x+y)/2\| = \frac{1}{2}\,\phi(\|x\|, \|y\|, \|x-y\|) \qquad (x, y \in \mathbf{B}),$$

with ϕ *unrestricted* except for being nonnegative and $\phi(r, 0, r) = r$, $r \geq 0$.

These conditions, and others like them, are all equivalent in a Banach space, for each is necessary and sufficient to insure the euclidean character of all subspaces. In a more general environment, however, this is not the case, and so the desirability of making a comparative study of such conditions in more general spaces is suggested. In this note the larger environment is furnished by the class $\{M\}$ of complete, metrically convex and externally convex,

[1] This note deals exclusively with normed linear spaces over the field of reals.

Received November 1, 1953. Presented to the Society, September 5, 1953.
Pacific J. Math. 5 (1955), 161-167

metric spaces, of which the class of Banach spaces is a very special subclass.[2] After giving purely metric interpretations of those conditions that we shall discuss (in order that they might be meaningful in spaces of class $\{M\}$) we are chiefly concerned with showing that the Jordan-von Neumann relation (**) characterizes class $\{\mathbf{E}\}$ among the class $\{M\}$.[3] This is true, *a fortiori*, for Fréchet's condition (*) also, but an easy example shows that the inequality used in Schoenberg [**10**] is *not* so extensible.

2. Metrization of the Jordan-von Neumann relation and comparison with other four-point conditions. Writing (**) in the form

$$\|(p_1+p_2)/2\| = \frac{1}{2}[2\|p_1\|^2 + 2\|p_2\|^2 - \|p_1-p_2\|^2]^{1/2}$$

we see that the length $\|(p_1+p_2)/2\|$ of the median of the triangle with vertices θ, p_1, p_2 (θ denotes the null element of $\mathbf{B}$) is the same function of the lengths $\|p_1\|$, $\|p_2\|$, $\|p_1-p_2\|$ of the sides of the triangle that it is in euclidean space. Since any three elements x, y, z of $\mathbf{B}$ are superposable with θ, $p_1=y-x$, $p_2=z-x$ (the middle-element $(y+z)/2$ of y, z being carried into $(p_1+p_2)/2$, the middle-element of p_1, p_2) we have the following metric interpretation of (**): ($\dagger$) *every four elements p, q, r, s of* $\mathbf{B}$ *with q a middle-element of p, r (that is, $pq=qr=pr/2$) are congruently imbeddable in the euclidean plane E_2.*

In this formulation, the Jordan-von Neumann criterion is meaningful in every metric space and may, therefore, be compared with other so-called *four-point conditions* that antedated it.

A metric space has the *euclidean k-point property* provided each k-tuple of its elements is congruently contained in a euclidean space (and hence in an E_{k-1}). Observing that every metric space has the euclidean three-point property, W. A. Wilson [**11**] investigated in 1932 the consequences of assuming that a space has the euclidean *four-point* property. It follows from a result due to the writer [**2**, p. 131] that if M is any metric space whatever, and $M^{1/2}$ denotes the space obtained by taking the positive square root of the metric of M, then $M^{1/2}$ has the euclidean four-point property. Thus the special class $\{M^{1/2}\}$ of spaces with the euclidean four-point property has the same cardinality as the class of

[2] For definitions of these and other metric concepts used in this paper see [**2**].

[3] The abstract of [**8**] given in *Math. Rev.* vol. 13 (1952) p. 850 indicates a connection between that paper (which the writer has not seen) and the present note.

all metric spaces (of which it is a proper subclass) and consequently the same is true of the class of all spaces with the euclidean four-point property.

But none of the spaces $\{M^{1/2}\}$ is metrically convex, and Wilson proved that if a complete, metrically convex space has the euclidean four-point property, it is congruent with a subset of a generalized euclidean space. If also external convexity is assumed, then congruence with a generalized euclidean space results.

The *weak euclidean four-point property*, introduced by the writer in 1933, assumes the congruent imbedding in euclidean space (and hence in E_2) of only those quadruples that contain a linear triple (that is, a triple which is imbeddable in E_1), and it was shown that the weak euclidean four-point property suffices to obtain all of the results that Wilson had proved by use of the stronger assumption [2, pp. 123-128]. But the Jordan-von Neumann condition, as metrized in (†), restricts the class of quadruples assumed to be imbeddable in euclidean space even more than does the weak euclidean four-point property, and consequently is a weaker assumption. We shall refer to it as the *feeble euclidean four-point property.*

3. Equivalence in $\{M\}$ of the feeble and the weak euclidean four-point properties. We prove in this section that in complete, metrically convex and externally convex metric spaces, the feeble euclidean four-point property implies (and hence is equivalent to) the weak euclidean four-point property. Some elementary consequences of the feeble property in such a space are first set down.

I. *Middle-elements are unique*; for if $p, r \in M(p \neq r)$ and q_1, q_2 are middle-elements of p, r then

$$p, q_1, q_2, r \approx p', q_1', q_2', r'$$

where the "primed" points are in E_2 and "$\approx$" denotes the congruence relation. But then q_1' and q_2' are middle-points of p', r' and consequently

$$q_1' = q_2', \; q_1 q_2 = q_1' q_2' = 0, \;\; q_1 = q_2 .$$

II. *Each two distinct elements are joined by exactly one metric segment.* Since M is complete, metrically convex and metric, each two of its distinct points are joined by at least one metric segment. If $p, r \in M(p \neq r)$ and $S_{p,r}$, $S^*_{p,r}$ are two segments with end-elements p, r, suppose q^* belongs to the second segment and not to the first. Then $p \neq q^* \neq r$, and traversing $S^*_{p,r}$ from q^* to p a first point p^* of $S_{p,r}$ is encountered. Similarly, traversing $S^*_{p,r}$ from q^* to r a first point r^* of $S_{p,r}$ is obtained. The sub-segments S_{p^*,r^*}, $S^*_{p^*,r^*}$ have only their end-elements in common, but each obviously contains a middle-element of p^*, r^*, contrary to I.

III. *Segments admit unique prolongations.* Since M is externally convex, each segment may be prolonged beyond its end-elements. But if $S_{p,q}$ admits two prolongations beyond q, then clearly elements r, r^* of different prolongations exist $(r \neq r^*)$ such that q is a middle-element of p^*, r as well as a middle-element of p^*, r^* for some element p^* of $S_{p,q}$. The congruent imbedding in E_2 of p^*, q, r, r^* shows this to be impossible.

IV. *Each two distinct elements of M are on exactly one metric line.* Since M is metric, complete, metrically convex and externally convex, each two of its distinct points p, q are on at least one metric line $L(p,q)$ [2, p. 56]. It follows at once from II and III that $L(p,q)$ is unique.

THEOREM 3.1. *If p is a point and L a metric line of M, then $L+(p)$ is congruently imbeddable in E_2.*

Proof. If $p \in L$ then

$$L+(p) = L \approx E_1 \subset E_2,$$

by the definition of a metric line. Suppose $p \notin L$, select points r_0, r_1 on L with $r_0 r_1 = 1$, and let p', r_0', r_1' be points of E_2 such that $p, r_0, r_1 \approx p', r_0', r_1'$. Let L' denote the straight line of E_2 determined by r_0', r_1', and consider the one-to-one correspondence

$$\Gamma: p \leftrightarrow p',\ L(r_0, r_1) \approx L'(r_0', r_1'),$$

where the congruence of the two lines is the unique extension of the congruence $r_0, r_1 \approx r_0', r_1'$. We shall show that Γ is a congruence.

If $r_{1/2}$ denotes the unique middle-point of r_0, r_1, and $r_{1/2}' = \Gamma(r_{1/2})$, then $r_{1/2}'$ is the middle-point of r_0', r_1'. By the feeble euclidean four-point property

$$p, r_0, r_{1/2}, r_1 \approx \overline{p}, \overline{r}_0, \overline{r}_{1/2}, \overline{r}_1 .$$

with the "barred" points in E_2, and since $p', r_0', r_1' \approx p, r_0, r_1$, a motion of E_2 exists that carries p, r_0, r_1 into p', r_0', r_1', respectively. This motion evidently sends $\overline{r}_{1/2}$ into $r'_{1/2}$, and we have

$$p, r_0, r_{1/2}, r_1 \approx p', r'_0, r'_{1/2}, r'_1 ;$$

that is, $pr_{1/2} = p'r'_{1/2}$.

If $r_{3/4}$ denotes the middle-point of $r_{1/2}$, r_1, and $r'_{3/4} = \Gamma(r_{3/4})$, the feeble four-point property, applied to the quadruple p, $r_{1/2}$, $r_{3/4}$, r_1 gives $pr_{3/4} = p'r'_{3/4}$. Continuing in this manner, we obtain $pr_{i/2^n} = p'r'_{i/2^n}$ for each dyadically rational fraction $i/2^n$. Since the points $r_{i/2^n}$ are dense in $\operatorname{seg}[r_0, r_1]$, continuity of the metric (and continuity of the congruence $L(r_0, r_1) \approx L'(r'_0, r'_1)$) yields $px = p'x'$, $x' = \Gamma(x)$, for every $x \in \operatorname{seg}[r_0, r_1]$.

Let r_2 be a point of L such that r_1 is the middle-point of r_0, r_2. The feeble four-point property gives (in the manner employed above) $p, r_0, r_1, r_2 \approx p', r'_0, r'_1, r'_2$, where $r'_2 = \Gamma(r_2)$, and consequently $pr_2 = p'r'_2$. Then from $p, r_1, r_2 \approx p', r'_1, r'_2$ we obtain $px = p'x'$, $x \in \operatorname{seg}[r_1, r_2]$ in the same manner as described above for $\operatorname{seg}[r_0, r_1]$. It is clear that a continuation of the procedure establishes $px = p'x'$ for every x of L and $x' = \Gamma(x)$.

THEOREM 3.2. *In a complete, metrically convex and externally convex metric space* M, *the feeble and the weak euclidean four-point properties are equivalent.*

Proof. The weak property obviously implies the feeble one in any metric space. Suppose M has the feeble property, and p, q, r, $s \in M$ (pairwise distinct) with q, r, s congruent with a triple of E_1. Then the line $L(q, r)$ contains s, and $L(q, r) + (p)$ is congruently imbeddable in E_2. Hence p, q, r, s are imbeddable in E_2.

4. Extension of the Jordan-von Neumann theorem. The writer has shown [2, p. 127] that a complete, metrically convex and externally convex semimetric space with the weak euclidean four-point property has the euclidean k-point property for every positive integer k. It follows easily that such a space is generalized euclidean. Use of Theorem 3.2 now yields the following result:

THEOREM 4.1. *A complete, metrically convex and externally convex metric space with the feeble euclidean four-point property is generalized euclidean.*

This is the desired extension of the Jordan-von Neumann theorem for real normed linear spaces. For if **L** is such a space, and **L** satisfies the Jordan-von Neumann condition (**), then the Banach space that arises by completing

L in the Hausdorff manner is a complete, metrically convex and externally convex metric space with the feeble euclidean four-point property. According to Theorem 4.1, it is generalized euclidean and so an inner product is definable in it. Hence an inner product is definable in **L**, and the Jordan-von Neumann theorem for real normed linear spaces is obtained. Thus the metric essence of (**) determines the euclidean character of **L** *by use of the purely metric features of the space,* without regard, for example, for its very special properties due to linearity.

5. Concluding remarks. Condition (*) of Fréchet is equivalent to Wilson's euclidean four-point condition [**2**, p. 106] and consequently his theorem of 1935 had already been proved in more general form by Wilson in 1932.

A semimetric space is *ptolemaic* provided for any four of its elements p, q, r, s, the three products $pq \cdot rs$, $ps \cdot qr$, $pr \cdot qs$ of "opposite" distances satisfy the triangle inequality. Schoenberg [**10**] showed that in a real linear seminormed ptolemaic space, the semi-norm satisfies the triangle inequality (and so is actually a norm) and an inner product is definable which is related to the norm in the usual way.

Schoenberg's ptolemaic condition which (as a norm postulate in L has the form

$$||f|| \cdot ||g-h|| + ||g|| \cdot ||h-f|| \geq ||h|| \cdot ||f-g|| \qquad (f, g, h \in \mathbf{L})$$

is not extensible to the class $\{\mathbf{M}\}$. For if three pairwise distinct rays of E_2, with a common initial point, be metrized convexly (that is, if p, q are points of different rays, then $pq = e(p, o) + e(o, q)$, where $e(,)$ denotes euclidean distance and o is the common point of the rays, while $pq = e(p, q)$ if p, q belong to the same ray) the resulting space is easily shown to be metric, complete, convex and externally convex, and ptolemaic. But it is not, of course, generalized euclidean. It would be interesting to know whether or not this "tripod" is present in every such example.

References

1. N. Aronszajn, *Caractérisation métrique de l'espace de Hilbert, des espaces vectoriels et de certains groupes métriques*, C.R. Acad. Sci. Paris, **201** (1935), 811-813.

2. L.M. Blumenthal, *Theory and applications of distance geometry*, Oxford, at the Clarendon Press, 1953.

3. M.M. Day, *Some characterizations of inner product spaces*, Trans. Amer. Math. Soc. **62** (1947), 320-337.

4. F.A. Ficken, *Note on the existence of scalar products in normed linear spaces*, Ann. of Math. **45** (1944), 362-366.

5. M. Fréchet, *Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert*, Ann. of Math. **36** (1935), 705-718.

6. P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Ann. of Math. **36** (1935), 719-723.

7. M. Nagumo, *Characterisierung der allgemeine euklidischen Räume durch ein Postulat für Schwerpunkte*, Jap. Jour. of Math. **12** (1936), 123-128.

8. K. Ōhira, *On a certain complete, separable and metric space*, Mem. Fac. Sci. Kyūsyū Univ. A **6** (1951), 9-15.

9. H. Rubin and M. Stone, *Postulates for generalizations of Hilbert space*, Proc. Amer. Math. Soc. **4** (1953), 611-616.

10. I. J. Schoenberg, *A remark on M.M. Day's characterization of inner product spaces and a conjecture of L.M. Blumenthal*, Proc. Amer. Math. Soc. **3** (1952), 961-964.

11. W.A. Wilson, *A relation between metric and euclidean spaces*, Amer. Jour. Math. **54** (1932), 505-517.

UNIVERSITY OF MISSOURI

NOTE ON THE MULTIPLICATION FORMULAS FOR THE JACOBI ELLIPTIC FUNCTIONS

L. CARLITZ

1. Introduction. For t an odd integer it is well known [4, vol. 2, p. 197] that

$$sn\ tx = \frac{sn\ x \cdot G_1^{(t)}(z)}{G_0^{(t)}(z)} \qquad (z = sn^2 x), \tag{1.1}$$

where

$$\begin{aligned} G_0^{(t)} &= 1 + a_{01} z + a_{02} z^2 + \cdots + a_{0t'} z^{t'}, \\ G_1^{(t)} &= t + a_{11} z + a_{12} z^2 + \cdots + a_{1t'} z^{t'} \qquad (t' = (t^2-1)/2), \end{aligned} \tag{1.2}$$

and the a_{ij} are polynomials in $u = k^2$ with rational integral coefficients. If we define

$$\beta_m(t) = \beta_m(t, u)$$

by means of

$$\frac{sn\ tx}{t\ sn\ x} = \sum_{m=0}^{\infty} \beta_{2m}(t) \frac{x^{2m}}{(2m)!} \qquad (\beta_{2m+1}(t) = 0), \tag{1.3}$$

it follows from (1.1) and (1.2) that $t\beta_{2m}(t)$ is a polynomial in u with integral coefficients for all m and all odd t. We shall show that

$$\beta_{2m}(t) = H_m(t) - \sum_{\substack{p-1 \mid 2m \\ p \mid t}} \frac{1}{p} A_p^{2m/(p-1)}(u), \tag{1.4}$$

where $H_m(t) = H_m(t, u)$ denotes a polynomial in u with integral coefficients,

Received August 8, 1953.
Pacific J. Math. 5 (1955), 169-176

the summation in the right member is over all (odd) primes p such that $(p-1) \mid 2m$ and $p \mid t$; finally $A_p(u)$ is defined [4, vol. 1, p. 399] by means of

$$(1.5) \qquad sn\, x = sn(x, u) = \sum_{m=0}^{\infty} A_{2m+1}(u) \frac{x^{2m+1}}{(2m+1)!}.$$

so that $A_{2m+1}(u)$ is a polynomial in u with integral coefficients. We show also that

$$(1.6) \qquad t \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \beta_{m+s(p-1)}(t) A_p^{r-s}(u) \equiv 0 \pmod{(p^m, p^r)},$$

where p is an arbitrary odd prime and $r \geq 1$; by (1.6) we understand that the left member is a polynomial in u every coefficient of which is divisible by the indicated power of p.

The proof of these formulas depends upon the results of [2]; for a theorem analogous to (1.4), see [1].

2. Proof of (1.4). Put

$$(2.1) \qquad \frac{x}{sn\, x} = \sum_{m=0}^{\infty} \beta_{2m} \frac{x^{2m}}{(2m)!}.$$

Then β_{2m} is a polynomial in u with rational coefficients; indeed [2, Theorem 2],

$$(2.2) \qquad p\beta_{2m} \equiv \begin{cases} -A_p^{2m/(p-1)}(u) & ((p-1) \mid 2m) \\ 0 & ((p-1) \nmid 2m). \end{cases} \pmod{p}$$

In the next place, if we write

$$\frac{sn\, tx}{t\, sn\, x} = \frac{sn\, tx}{tx} \frac{x}{sn\, x},$$

and make use of (1.3), (1.5), and (2.1), it follows that

$$(2.3) \qquad \beta_{2m}(t) = \sum_{s=0}^{m} \binom{2m}{2s} \beta_{2m-2s} A_{2s+1}(u) \frac{t^{2s}}{2s+1}.$$

As already observed, $t\beta_{2m}(t)$ has integral coefficients; thus the denominator of $\beta_{2m}(t)$ is a divisor of t. Now let p denote a prime divisor of t, and assume $p^e \mid (2s+1)$, $e \geq 1$. Then

$$2s+1 \geq p^e \geq 3^e \geq e+2, \qquad 2s \geq e+1.$$

Thus not only is $t^{2s}/(2s+1)$ integral (mod p) but it is divisible by p. Since by (2.2) the denominator of β_{2m} contains p to at most the first power it therefore follows that the product

$$\beta_{2m-2s}\, t^{2s}/(2s+1) \tag{2.4}$$

is integral (mod p) when $p \mid (2s+1)$.

Suppose next that $p \nmid (2s+1)$, where $s \geq 1$. It is again clear that (2.4) is integral (mod p) since p occurs in the denominator of β_{2m-2s} at most once while it occurs in t^{2s} at least twice. Thus as a matter of fact (2.4) is divisible by p in this case.

It remains to consider the term $s = 0$ in (2.3). Clearly we have proved that

$$p\beta_{2m}(t) \equiv p\beta_{2m} \pmod{p}. \tag{2.5}$$

Comparing (2.5) with (2.2) we may state:

THEOREM 1. *If t is an arbitrary odd integer then* (1.4) *holds.*

We remark that the residue of $A_p(u)$ is determined [2, §6] by

$$\begin{aligned} A_p(u) &\equiv (-1)^{\frac{1}{2}(p-1)} F\left(\frac{1}{2}, \frac{1}{2}; 1; u\right) \\ &\equiv (-1)^{\frac{1}{2}(p-1)} \sum_{j=0}^{\frac{1}{2}(p-1)} \binom{\frac{1}{2}(p-1)}{j}^2 u^j \pmod{p}. \end{aligned} \tag{2.6}$$

Here F denotes the hypergeometric function.

3. Some corollaries. By means of Theorem 1 a number of further results are readily obtained. By H_{2m} will be understood an unspecified polynomial in u with integral coefficients.

Since β_{2m}, as defined by (2.1), is integral (mod 2) we have first:

THEOREM 2. *If t is divisible by the denominator of β_{2m}, then*

$$\beta_{2m}(t) = H_{2m} + \beta_{2m}. \tag{3.1}$$

If t is prime to the denominator of β_{2m}, then $\beta_{2m}(t)$ has integral coefficients.

THEOREM 3. *If t_1, t_2 are relatively prime and odd, then*

$$\beta_{2m}(t_1 t_2) = H_{2m} + \beta_{2m}(t_1) + \beta_{2m}(t_2). \tag{3.2}$$

If t is a power of a prime we get:

THEOREM 4. *If p is an odd prime and $r \geq 1$ we have*

$$\beta_{2m}(p^r) = H_{2m} + \beta_{2m}(p). \tag{3.3}$$

Using (3.2) and (3.3) we get also:

THEOREM 5. *The following identity holds:*

$$\beta_{2m}(t) = H_{2m} + \sum_{p \mid t} \beta_{2m}(p), \tag{3.4}$$

where the summation is over all prime divisors of t.

We have also:

THEOREM 6. *If a is an arbitrary integer, then the product*

$$a(a^m - 1)\beta_{2m}(t) \tag{3.5}$$

has integral coefficients.

4. A related result. It follows from (1.1) and (1.2) that, for t odd,

$$sn\ tx = \sum_{r=0}^{\infty} C_{2r+1} sn^{2r+1} x, \tag{4.1}$$

where the C_{2r+1} are polynomials in u with integral coefficients. Clearly we have

$$\beta_{2m}(t) = \frac{1}{t} \sum_{r=0}^{m} A_{2m}^{(2r)} C_{2r+1}, \tag{4.2}$$

where the $A_{2m}^{(2r)}$ are defined by

$$sn^{2r} x = \sum_{m=0}^{\infty} A_{2m}^{(2r)} \frac{x^{2m}}{(2m)!}, \tag{4.3}$$

and like the C's are polynomials with integral coefficients.

We shall now prove the following property of the C's.

THEOREM 7. *For t odd we have*

$$(2m+1) C_{2m+1} \equiv 0 \pmod{t} \qquad (m = 0, 1, 2, \cdots), \tag{4.4}$$

where (4.4) *indicates that every coefficient in* $(2m+1)C_{2m+1}$ *is divisible by* t.

Proof. Differentiating (4.1) with respect to x, we get

$$t \frac{cn\ tx\ dn\ tx}{cn\ x\ dn\ x} = \sum_{m=0}^{\infty} (2m+1) C_{2m+1} sn^{2m} x. \tag{4.5}$$

Now we have, in addition to (1.1),

$$\frac{cn\ tx}{cn\ t} = \frac{G_2^{(t)}(z)}{G_0^{(t)}(z)}, \quad \frac{dn\ tx}{dn\ x} = \frac{G_3^{(t)}(z)}{G_0^{(t)}(z)} \qquad (z = sn^2 x), \tag{4.6}$$

where G_2 and G_3 are polynomials in z of the same form as G_0. By means of (1.1) and (4.6) it is evident that (4.5) implies

$$t \sum_{m=0}^{\infty} H_m^{(t)} z^m = \sum_{m=0}^{\infty} (2m+1) C_{2m+1} z^m, \tag{4.7}$$

where the H_m are polynomials in u with integral coefficients. Clearly (4.4) is an immediate consequence of (4.7).

Kronecker [5, p. 439] has proved a similar result in connection with the transformation of prime order of $sn\ x$. For a result like Theorem 7 for the Weierstrass $\wp$-function, see [3].

Returning to (4.2) we recall [2, § 2] that

(4.8) $$A_{2m}^{(2r)} \equiv 0 \qquad (\mathrm{mod}\,(2r)!) \qquad (m = 0, 1, 2, \cdots).$$

We rewrite (4.2) in the form

(4.9) $$\beta_{2m}(t) = \sum_{r=0}^{m} \frac{(2r)!}{2r+1} \frac{A_{2m}^{(2r)}}{(2r)!} \frac{(2r+1)C_{2r+1}}{t}.$$

By (4.4) and (4.8) the last two fractions in the right member of (4.9) have integral coefficients; also $(2r)!/(2r+1)$ is integral unless $2r+1$ is prime. Consequently (4.9) becomes

(4.10) $$\beta_{2m}(t) = H_{2m} - \sum_{\substack{p-1 \mid 2m \\ p \mid t}} \frac{1}{p} A_{2m}^{(p-1)} \frac{pC_p}{t}.$$

Comparing (4.10) with (1.4) we get:

THEOREM 8. *If the prime p divides t, then*

(4.11) $$\frac{pC_p}{t} \equiv 1 \quad (\mathrm{mod}\, p).$$

Hence if $p^e \mid t$, $p^{e+1} \nmid t$ it follows that

(4.12) $$C_p \equiv \frac{t}{p} \qquad (\mathrm{mod}\, p^e).$$

5. Proof of (1.6). Again using (5.1) we have

(5.1) $$\frac{sn\, tx}{sn\, x} = \sum_{i=0}^{\infty} C_{2i+1}\, sn^{2i} x.$$

Now it is proved in [2, Theorem 4] that the coefficients $A_{2m}^{(2i)}$ defined by (4.3) satisfy

(5.2) $$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} A_p^{(r-s)b/(p-1)} A_{2m+sb}^{(2i)} \equiv 0 \quad (\mathrm{mod}\,(p^{2m}, p^{er})),$$

where $p^{e-1}(p-1) \mid b$. Hence using (1.3) and (5.1) we get:

THEOREM 9. *If* $p^{e-1}(p-1)\,|\,b$, *then*

$$(5.3)\qquad t\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}A_p^{(r-s)b/(p-1)}\beta_{2m+sb}(t)\equiv 0\pmod{(p^{2m},p^{er})},$$

For $b=p-1$, (5.3) evidently reduces to (1.6).

It is of some interest to compare Theorem 9 with the results of [2, § 7]. If we take $r=1$, (5.3) becomes

$$t\{\beta_{2m+b}(t)-A_p^{b/(p-1)}\beta_{2m}(t)\}\equiv 0\qquad \pmod{(p^{2m},\,p^{e})}.$$

If we put

$$\beta_{2m}(t)=\sum_i \beta_{2m,i}u^i$$

and recall that, by (2.6),

$$A_p(0)\equiv(-1)^{\frac{1}{2}(p-1)}\pmod{p}$$

we get exactly as in the proof of [2, Theorem 6].

THEOREM 10. *Let* $p^{e-1}(p-1)\,|\,b$ *and* $p^{j-1}\le i<p^j$. *Then*

$$(5.4)\qquad \beta_{2m+b,\,i}\equiv(-1)^{\frac{1}{2}b}\beta_{2m,\,i}\quad \pmod{(p^{2m},\,p^{e-j})}.$$

6. An elementary analogue of $\beta_{2m}(t)$. It may be of interest to say a word about the numbers $\phi_m(t)$ defined by

$$(6.1)\qquad \frac{e^{tx}-1}{t(e^x-1)}=\sum_{m=0}^{\infty}\phi_m(t)\frac{x^m}{m!},$$

where t is now an arbitrary integer. Clearly (6.1) implies that

$$t\phi_m(t)=S_m(t)=\sum_{s=0}^{t-1}s^m.$$

By a theorem of Staudt (see for example [6, p. 143]),

$$(6.2) \qquad \phi_m(t) = G + \sum_{p \mid t} \phi_m(p),$$

where G is an integer. Moreover,

$$(6.3) \qquad p\phi_m(p) \equiv \begin{cases} -1 & (p-1 \mid m) \\ 0 & (p-1 \nmid m). \end{cases} \pmod p$$

It follows [6, p. 153] that

$$(6.4) \qquad \phi_{2m}(t) = G - \sum_{\substack{p-1 \mid 2m \\ p \mid t}} \frac{1}{p}.$$

Thus Staudt's theorems (6.2) and (6.4) may be viewed as elementary analogues of (3.4) and (1.4).

Formulas like (6.2) and (6.4) hold also for the numbers $\psi_{2m}(t)$ occurring in

$$\frac{\sin tx}{t \sin x} = \sum_{m=0}^{\infty} \psi_{2m}(t) \frac{x^{2m}}{(2m)!} .$$

References

1. L. Carlitz, *An analogue of the Bernoulli polynomials*, Duke Math. J. **8** (1941), 405-412.

2. ———, *Congruences connected with the power series expansions of the Jacobi elliptic functions*, Duke Math. J. **20** (1953), 1-12.

3. J.W.S. Cassels, *A note on the division values of* $\wp(u)$, Proc. Cambridge Philos. Soc. **45** (1949), 167-172.

4. R. Fricke, *Die elliptischen Funktionen und ihre Anwendungen*, Leipzig and Berlin, vol. 1, 1916, and vol. 2, 1922.

5. L. Kronecker, *Zur Theorie der elliptischen Funktionen*, Werke, **4** (1929), 345-495.

6. J.V. Uspensky and M.A. Heaslet, *Elementary number theory*, New York, 1939.

DUKE UNIVERSITY

THE NUMBER OF SOLUTIONS OF CERTAIN TYPES OF EQUATIONS IN A FINITE FIELD

L. CARLITZ

1. Using a very simple principle, Morgan Ward [3] indicated how one can obtain all solutions of the equation

$$(1) \qquad y^m = f(x_1, \cdots, x_r) \qquad (y, x_i \in F),$$

where F is an arbitrary field, $f(x_1, \cdots, x_r)$ is a homogeneous polynomial of degree n with coefficients in F, and $(m, n) = 1$. The same principle had been applied earlier to a special equation by Hua and Vandiver [2]. If this principle is applied in the case of a finite field F we readily obtain the total number of solutions of equations of the type (1). Somewhat more generally, let

$$f_i(x_i) = f_i(x_{i1}, \cdots, x_{is_i}) \qquad (i = 1, \cdots, r)$$

denote r polynomials with coefficients in $GF(q)$, and assume

$$(2) \qquad f_i(\lambda x_1, \cdots, \lambda x_{s_i}) = \lambda^{m_i} f_i(x_1, \cdots, x_{s_i}) \qquad (\lambda \in GF(q));$$

assume also

$$(3) \qquad (m, m_i, q-1) = 1 \qquad (i = 1, \cdots, r).$$

We consider the equation

$$(4) \qquad y^m = f_1(x_{11}, \cdots, x_{1s_1}) + \cdots + f_r(x_{r1}, \cdots, x_{rs_r})$$

in $s_1 + \cdots + s_r + 1$ unknowns.

Suppose first we have a solution of (4) with $y \neq 0$. Select integers h, k, l such that

$$(5) \qquad hm + km_1 m_2 \cdots m_r + l(q-1) = 1, \qquad (h, q-1) = 1;$$

Received August 8, 1953.
Pacific J. Math. 5 (1955), 177-181

this can be done in view of (3). Next put

$$y = \lambda^h, \quad x_{ij} = \lambda^{kM/m_i} z_{ij} \qquad (M = m_1 m_2 \cdots m_r). \tag{6}$$

Substituting in (4) and using (2), we get

$$\lambda^{hm} = \lambda^{kM}\{f_1(z_1) + \cdots + f_r(z_r)\}.$$

Since $\lambda^{q-1} = 1$, it is clear from (5) that

$$\lambda = f_1(z_1) + \cdots + f_r(z_r). \tag{7}$$

Thus any solution (y, x_{ij}) of (4) with $y \neq 0$ can be obtained from (6) and (7) by assigning arbitrary values to z_{ij} such that the right member of (7) does not vanish. Let N denote the total number of solutions of (4) and let N_0 denote the number of solutions with $y = 0$. Thus there are $N - N_0$ sets z_{ij} for which $\lambda \neq 0$. Since in all there are $q^{s_1 + \cdots + s_r}$ sets z_{ij} it follows that

$$N = q^{s_1 + \cdots + s_r}. \tag{8}$$

This proves:

THEOREM. *Let the polynomials f_i satisfy* (2) *and* (3). *Then the total number of solutions of* (4) *is furnished by* (8).

2. In Theorem II of [2] Hua and Vandiver proved that the number of solutions of

$$c_1 x_1^{a_1} + c_2 x_2^{a_2} + \cdots + c_s x_s^{a_s} = 0 \tag{9}$$

subject to the conditions

$$c_1 c_2 \cdots c_s x_1 x_2 \cdots x_s \neq 0, \quad (a_i, q-1) = k_i, \quad (k_i, k_j) = 1 \text{ for } i \neq j,$$

is equal to

$$\frac{q-1}{q}\{(q-1)^{s-1} + (-1)^s\}. \tag{10}$$

It is easy to show that (10) implies that the total number of solutions of (9) is equal to q^{s-1}, which agrees with (8). Conversely if N_s denotes the number of nonzero solutions of (9), and we assume that

$$(k_i, k_j) = 1 \qquad (i, j = 1, \cdots, s;\ i \neq j), \tag{11}$$

then using (8) we get

$$q^{s-1} = N_s + \binom{s}{1} N_{s-1} + \binom{s}{2} N_{s-2} + \cdots + \binom{s}{s-1} N_1 + 1.$$

Hence (if we take $N_0 = 1$)

$$(q-1)^s = \sum_{r=1}^{s} (-1)^{s-r} \binom{s}{r} q \sum_{t=0}^{r} \binom{r}{t} N_t + (-1)^s$$

$$= q \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \sum_{t=0}^{r} \binom{r}{t} N_t - (-1)^s (q-1)$$

$$= q \sum_{t=0}^{s} \binom{s}{t} N_t \sum_{r=t}^{s} (-1)^{s-r} \binom{s-t}{s-r} - (-1)^s (q-1)$$

$$= qN_s - (-1)^s (q-1),$$

and (10) follows at once. Thus if we assume (11) then (8) and (10) are equivalent.

If in place of (11) we assume only that

$$(k_1, k_2 k_3 \cdots k_s) = 1, \tag{12}$$

the situation is somewhat different. As above let N_s denote the number of non-zero solutions of (9), and let M_{s-1} denote the total number of solutions $x_2, \cdots, x_s$ of

$$c_2 x_2^{a_2} + c_3 x_3^{a_3} + \cdots + c_s x_s^{a_s} = 0. \tag{13}$$

Using (8) we now get

$$q^{s-1} = M_{s-1} + N_s + \binom{s-1}{1} N_{s-1} + \cdots + \binom{s-1}{s-1} N_1, \tag{14}$$

which implies (with $M_0 = 1$)

$$(15)\qquad (q-1)^{s-1} = \sum_{r=0}^{s-1} (-1)^{s-1-r} \binom{s-1}{r} M_r + N_s .$$

Thus making only the assumption (12) we see how the number of solutions of (13) can be expressed in terms of N_s and *vice versa*.

3. Returning to equation (4), we see that a similar result can be obtained if we allow f_i to contain additional unknowns:

$$f_i(x_i\,; u_i) = f_i(x_{i1}, \cdots, x_{is_i}; u_{i1}, \cdots, u_{it_i}),$$

and assume that (2) holds only for the x's. Then the number of solutions (y, x_{ij}, u_{hk}) of (4) becomes

$$q^{s_1 + \cdots + s_r + t_1 + \cdots + t_r}.$$

Similarly we may replace the left member of (4) by

$$y_1^{a_1} y_2^{a_2} \cdots y_s^{a_s} \qquad\qquad (a_1, a_2, \cdots, a_s) = m.$$

Then assuming (3) we again find that the number of solutions of the modified equation is equal to

$$q^{s_1 + \cdots + s_r + s - 1}.$$

This kind of generalization lends itself well to equation (9). For example it is easy to show (see [**1**, Theorem 10]) that the total number of solutions of the equation

$$\sum_{i=1}^{t} c_i \prod_{j=1}^{k_i} x_{ij}^{a_{ij}} = 0,$$

subject to $(a_{i1}, \cdots, a_{ik_i}, q-1) = d_i$, $(d_i, d_j) = 1$ for $i \neq j$, is equal to

$$q^{k_1 + \cdots + k_t - 1}.$$

Reference

1. L. Carlitz, *The number of solutions of certain equations in a finite field*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 515-519.

2. L. K. Hua and H. S. Vandiver, *On the nature of the solutions of certain equations in a finite field*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 481-487.

3. Morgan Ward, *A class of soluble diophantine equations*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 113-114.

DUKE UNIVERSITY

THE GENERALIZED SIMPLEX METHOD FOR MINIMIZING A LINEAR FORM UNDER LINEAR INEQUALITY RESTRAINTS

GEORGE B. DANTZIG, ALEX ORDEN, PHILIP WOLFE

1. Background and summary. The determination of "optimum" solutions of systems of linear inequalities is assuming increasing importance as a tool for mathematical analysis of certain problems in economics, logistics, and the theory of games [**1**; **5**]. The solution of *large systems* is becoming more feasible with the advent of high-speed digital computers; however, as in the related problem of inversion of large matrices, there are difficulties which remain to be resolved connected with rank. This paper develops a theory for avoiding assumptions regarding rank of underlying matrices which has import in applications where little or nothing is known about the rank of the linear inequality system under consideration.

The simplex procedure is a finite iterative method which deals with problems involving linear inequalities in a manner closely analogous to the solution of linear equations or matrix inversion by Gaussian elimination. Like the latter it is useful in proving fundamental theorems on linear algebraic systems. For example, one form of the fundamental duality theorem associated with linear inequalities is easily shown as a direct consequence of solving the main problem. Other forms can be obtained by trivial manipulations (for a fuller discussion of these interrelations, see [**13**]); in particular, the duality theorem [**8**; **10**; **11**; **12**] leads directly to the Minmax theorem for zero-sum two-person games [**1d**] and to a computational method (pointed out informally by Herman Rubin and demonstrated by Robert Dorfman [**1a**]) which simultaneously yields optimal strategies for both players and also the value of the game.

The term "simplex" evolved from an early geometrical version in which (like in game theory) the variables were nonnegative and summed to unity. In that formulation a class of "solutions" was considered which lay in a simplex.

The generalized method given here was outlined earlier by the first of the authors (Dantzig) in a short footnote [**1b**] and then discussed somewhat more fully at the Symposium of Linear Inequalities in 1951. Its purpose, as we have

Received August 29, 1953.

Pacific J. Math. 5 (1955), 183-195

already indicated, is to remove the restrictive assumptions regarding the rank of the matrix of coefficients and constant elements without which a condition called "degeneracy" can occur.

Under degeneracy it is possible for the value of the solution to remain unchanged from one iteration to the next under the original simplex method. This causes the proof that no basis can be repeated to break down. In fact, for certain examples Alan Hoffman [**14**] and one of the authors (Wolfe) have shown that it was possible to repeat the basis and thus cycle forever with the value of the solution remaining unchanged and greater than the desired minimum. On the other hand, it is interesting to note that while most problems that arise from practical sources (in the authors' experience) have been degenerate, none have ever cycled [**9**].

The essential scheme for avoiding the assumptions on rank is to replace the original problem by a "perturbation" that satisfies these conditions. That such perturbations exist is, of course, intuitively evident; but the question remained to show how to make the perturbation in a simple way. For the special case of the transportation problem a simple method of producing a perturbation is found in [**1c**]. The second of the authors (Orden) has considered several types of perturbations for the general case. A. Charnes has extensively investigated this approach and his writing represents probably the best available published material in this regard [**2**; **3**; **4**].

It was noticed early in the development of these methods that the limit concept in which a set of perturbations tends in the limit to one of the solutions of the original problem was not essential to the proof. Accordingly, the third author (Wolfe) considered a purely algebraic approach which imbeds the original problem as a component of a *generalized matrix problem* and replaces the original nonnegative real variables by lexicographically ordered vectors. Because this approach gives a simple presentation of the theory, we adopt it here.

2. The generalized simplex method. As is well known, a system of linear inequalities by trivial substitution and augmentation of the variables can be replaced by an equivalent *system of linear equations in nonnegative variables*; hence, with no loss of generality, we shall consider the basic problem in the latter form throughout this paper. One may easily associate with such a system another system in which the constant terms are replaced by l-component constant row vectors and the real variables are replaced by real l-component variable row vectors. In the original system the real variables are nonnegative; in the generalized system we shall mean by a vector variable $\overline{x} > 0$ (in the *lexicographic sense*) that it has some nonzero components, the first of which is positive, and by $\bar{x} > \bar{y}$ that $\bar{x} - \bar{y} > 0$. It is easy to see that the first components of the vector variables of the generalized system satisfy a linear system in nonnegative variables in which the constant terms are the first components of the constant vectors.

Let $P = [P_0, P_1, \cdots, P_n]$ be a given matrix whose jth column, P_j, is a vector of $(m+1)$ components. Let M be a fixed matrix of *rank* $m+1$ consisting of $m+1$ l-component row vectors. The generalized matrix problem is concerned with finding a matrix $\widetilde{X}$ satisfying

$$(1) \qquad P\widetilde{X} = \sum_{0}^{n} P_j \bar{x}_j = M,$$

where $\bar{x}_j$ (the jth row of $\widetilde{X}$) is a row vector of l-components satisfying the conditions, in the lexicographic sense,

$$(2) \qquad \bar{x}_j \geq 0 \qquad (j = 1, 2, \cdots, n),$$

$$(3) \qquad \bar{x}_0 = \max.$$

where the relationship between max $\bar{x}_0$ and the minimization of a linear form will be developed in § 3.

Any set X of "variables" $(\bar{x}_0; \bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ satisfying (1) and (2) in the foregoing lexicographic sense will be referred to as a "feasible solution" (or more simply as a "solution")—a term derived from practical applications in which such a solution represents a situation which is physically realizable but not necessarily optimal. The first variable, $\bar{x}_0$, which will be called the "value" of the solution, is to be maximized; it is not constrained like the others to be nonnegative. In certain applications (as in § 3) it may happen that some of the other variables also are not restricted to be nonnegative. This leads to a slight variation in the method (see the discussion following Theorem 5).

Among the class of feasible solutions, the simplex method is particularly concerned with those called "basic." These have the properties, which we mention in passing, (a) that whenever any solution exists a basic solution also exists (Theorem 8), and (b) that whenever a maximizing solution exists and is unique it is basic solution, and whenever a maximizing solution is not unique there is a basic solution that has the same maximizing value (Theorem 6). A basic solution is one in which only $m+1$ variables (including $\bar{x}_0$) are considered in (1), the remainder being set equal to zero; that is, it is of the form

$$(4) \qquad BV = P_0 \bar{v}_0 + \sum_{i=1}^{m} P_{j_i} \bar{v}_i = M \qquad (\bar{v}_i \geq 0,\ j_i \neq 0),$$

where $B = [P_0, P_{j_1}, \cdots, P_{j_m}]$ is an $(m+1)$-rowed *square matrix* and V is a matrix of $m+1$ rows and l-columns whose ith row is denoted by $\overline{v}_i$ $(i=0, 1, \cdots, m)$.

It is clear from (4) that since M is of rank $m+1$ so are B and V. From this it readily follows that the $m+1$ columns of B constitute a *basis* in the space of vectors P_j, and the solution V is uniquely determined. Moreover, since the rank of V is $m+1$, none of the $m+1$ rows of V can vanish; that is, it is not possible that $\overline{v}_i = 0$. *Thus in a basic solution all variables associated with the vectors in the basis (except possibly $\overline{v}_0$) are positive; all others are zero.* The condition in (4) can now be strengthened to strict inequality

$$(5) \qquad \overline{v}_i > 0 \qquad (i = 1, \cdots, m).$$

Let β_i denote the ith *row* of B inverse:

$$(6) \qquad B^{-1} = [P_0, P_{j_1}, P_{j_2}, \cdots, P_{j_m}]^{-1} = [\beta_0', \beta_1', \cdots, \beta_m']'$$

where primed letters stand for transpose.

THEOREM 1. *A necessary and sufficient condition that a basic solution be a maximizing solution is*

$$(7) \qquad \beta_0 P_j \geq 0 \qquad (j = 1, \cdots, n).$$

THEOREM 2. *If a basic solution is optimal, then any other solution (basic or not) with the property that $\overline{x}_j = 0$ whenever $(\beta_0 P_j) > 0$ is also optimal; any solution with $\overline{x}_j > 0$ for some $(\beta_0 P_j) > 0$ is not optimal.*

Proofs. Let $\widetilde{X}$ represent *any* solution of (1), and V a basic solution with basis B; then multiplying both (1) and (4) through by β_0 and equating, one obtains, after noting from (6) that $\beta_0 P_0 = 1$ and $\beta_0 P_{j_i} = 0$,

$$(8) \qquad \overline{x}_0 + \sum_1^n (\beta_0 P_j)\overline{x}_j = \overline{v}_0;$$

whence, assuming $\beta_0 P_j \geq 0$, one obtains $\overline{x}_0 \leq \overline{v}_0$ (which establishes the sufficiency of Theorem 1); moreover the condition $\overline{x}_j = 0$ whenever $\beta_0 P_j > 0$ $(j \neq 0)$ implies the summation term of (8) vanishes and $\overline{x}_0 = \overline{v}_0$; whereas denial of this condition implies the summation term is positive if Theorem 1 is true (establishing Theorem 2).

In order to establish the necessity of (7) for Theorem 1, let Y_s be a column vector which expresses a vector P_s as a linear combination of the vectors in the basis:

$$(9) \qquad P_s = B(B^{-1}P_s) = BY_s = \sum_{i=0}^{m} P_{j_i} y_{is} \qquad (P_0 = P_{j_0}),$$

where it is evident from (6) that, by definition,

$$(10) \qquad y_{is} = \beta_i P_s \qquad (i = 0, 1, \cdots, m).$$

Consider a *class of solutions* which may be formed from (4) and (9), of the form

$$(11) \qquad B[V - Y_s \bar{\theta}] + P_s \bar{\theta} = M,$$

or more explicitly

$$(12) \qquad P_0[\bar{v}_0 - y_{0s}\bar{\theta}] + \sum_{i=1}^{m} P_{j_i}[\bar{v}_i - y_{is}\bar{\theta}] + P_s \bar{\theta} = M.$$

It is clear that, since $\bar{v}_i > 0$ for $i \geq 1$ has been established earlier (see (5)), a class of solutions with $\bar{\theta} > 0$ (that is, with $\bar{\theta}$ *strictly positive*) always exists such that the variables associated with P_s and P_{j_i} in (12) are nonnegative, hence admissible as a solution of (1). If $y_{0s} < 0$, then the values of these solutions are

$$(13) \qquad \bar{v}_0 - y_{0s}\bar{\theta} > \bar{v}_0 \qquad (y_{0s} < 0, \ \bar{\theta} > 0).$$

For a given increase in $\bar{\theta}$ the greatest increase in the value of the solution (that is, *direction of steepest ascent*) is obtained by choosing $s = j$ such that

$$(14) \qquad \beta_0 P_s = \min_j (\beta_0 P_j) < 0.$$

This establishes Theorem 3 (below) which is clearly only a restatement of the necessity of condition (7) of Theorem 1.

THEOREM 3. *There exists a class of solutions with values* $\bar{x}_0 > \bar{v}_0$, *if, for some* $j = s$,

$$(15) \qquad y_{0s} = \beta_0 P_s < 0.$$

THEOREM 4. *There exists a class of solutions with no upper bound for values* $\overline{x}_0$ *if, for some* s, $y_{0s} < 0$ *and* $y_{is} \leq 0$ *for all* i.

THEOREM 5. *There exists a new basic solution with value* $\overline{x}_0 > \overline{v}_0$, *(obtained by introducing* P_s *into the basis and dropping a unique* P_{j_r}*), if, for some* s, $y_{0s} < 0$ *and, for some* i, $y_{is} > 0$.

Proofs. From (12), if $y_{is} \leq 0$ for all i, then $\overline{\theta}$ can be arbitrarily large (that is, its first component can tend to $+\infty$) and the coefficients of P_{j_i} will remain nonnegative. The value of these solutions (13) will also be arbitrarily large provided that $y_{0s} < 0$ (establishing Theorem 4). In the event that some $y_{is} > 0$, the maximum value of $\overline{\theta}$ becomes

$$(16) \qquad \max \overline{\theta} = (1/y_{rs})\overline{v}_r = \min_{y_{is}>0} (1/y_{is})\overline{v}_i > 0 \qquad (y_{rs} > 0,\ i \neq 0),$$

where the minimum of the vectors (taken in the lexicographic sense) occurs for a *unique* $i = r$ (since the rank of V is $m+1$, no two rows of V can be proportional, whereas the assumption of nonuniqueness in (16) would imply two rows of V to be so—a contradiction). Setting $\overline{\theta} = \max \overline{\theta}$ in (12) yields a new basic solution since the coefficient of P_{j_r} vanishes. Thus a new basis has been formed consisting of $[P_0, P_{j_1}, \cdots, P_s, \cdots, P_{j_m}]$, where P_{j_r} is omitted and P_s is put in instead (Theorem 5).

The next section considers an application of the generalized simplex procedure in which the restriction $\overline{x}_j \geq 0$ is not imposed on all variables ($j = 1, 2, \cdots, n$). This leads to a slight modification of procedure: first, for all j for which $\overline{x}_j \geq 0$ is not required, both P_j and $-P_j$ should be considered as columns of P; secondly, if P_{j_i} is in the basis and the restriction $\overline{v}_i > 0$ is not required, then this term cannot impose a bound on $\overline{\theta}$; hence the corresponding i should be omitted from (16) in forming the minimum.

Starting with any basis $B = B^{(k)}$, one can determine a new basis $B^{(k+1)}$ by first determining the vector P_s to introduce into the basis by (14). If there exists no $\beta_0 P_s < 0$, then, by Theorem 1, the solution is optimal and $B^{(k)}$ is the *final basis*. If a P_s exists, then one forms $y_{is} = (\beta_i P_s)$ and determines the vector P_{j_r} to drop from the basis by (16) provided that there are $y_{is} > 0$. If there exist no $y_{is} > 0$, then, by Theorem 4, a class of solutions is obtained from (12) with no upper bound for $\overline{v}_0$ for arbitrary $\overline{\theta} > 0$. If P_{j_r} can be determined, then a new basis $B^{(k+1)}$ is formed dropping P_{j_r} and replacing it by P_s;

by (13), the value, $\bar{v}_0$, of this solution is strictly greater for $B^{(k+1)}$ than for $B^{(k)}$ since $\bar{\theta} > 0$ is chosen by (16). Thus one may proceed iteratively starting with the assumed initial basis and forming $k = 0, 1, 2, \cdots$ until the process stops because (a) an optimal solution has been obtained, or (b) a class of solutions with no finite upper bound has been obtained.

The number of different bases is finite, not exceeding the number of combinations of n things taken m at a time; associated with each basis B is a unique basic solution $V = B^{-1}M$—hence the number of distinct basic solutions is finite; finally, no basis can be repeated by the iterative procedure because contrariwise this would imply a repetition of the value $\bar{v}_0$, whereas by (13) *the values for successive basic solutions are strictly monotonically increasing—hence the number of iterations is finite.*

The $(k+1)$st iterate is closely related to the kth by simple transformations that constitute the computational algorithm [6; 7] *based on the method: thus* for $i = 0, 1, \cdots, m$ $(i \neq r)$,

$$(17.0) \qquad \bar{v}_i^{k+1} = \bar{v}_i^k + \eta_i \bar{v}_r^k; \qquad \bar{v}_r^{k+1} = \eta_r \bar{v}_r^k;$$

$$(17.1) \qquad \beta_i^{k+1} = \beta_i^k + \eta_i \beta_r^k; \qquad \beta_r^{k+1} = \eta_r \beta_r^k,$$

where the *superscripts* $k+1$ and k are introduced here to distinguish the successive solutions and bases, and where η_i are constants,

$$(18) \qquad \eta_i = -y_{is}/y_{rs} = -(\beta_i P_s)/(\beta_r P_s), \qquad (i \neq r)$$

$$\eta_r = 1/y_{rs} = 1/(\beta_r P_s).$$

Relation (17.0) is a consequence of (12) and (16); it is easy to verify that the matrix whose rows are defined by (17.1) satisfies the proper orthogonality properties for the inverse when multiplied on the right by the $(k+1)$st basis $[P_0, P_{j_1}, \cdots, P_s, \cdots, P_{j_m}]$. As a consequence of the iterative procedure we have established two theorems:

THEOREM 6. *If solutions exist and their values have a finite upper bound, then a maximizing solution exists which is a basic solution with the properties*

$$(19) \qquad BV = \sum_{i=0}^{m} P_{j_i} \bar{v}_i = M \qquad (P_{j_0} = P_0, \bar{v}_i > 0,\ i = 1, \cdots, m),$$

$$\beta_0 P_0 = 1,\ \beta_0 P_{j_i} = 0,\ \beta_0 P_j \geq 0 \qquad (j = 1, 2, \cdots, n),$$

$$\overline{v}_0 = \beta_0 M = \max \overline{x}_0 ,$$

where β_0 *is the* 1st *row of* B^{-1}.

THEOREM 7. *If solutions exist and their values have no finite upper bound, then a basis B and a vector* P_s *exist with the properties*

$$(20) \quad BV = \sum_{i=0}^{m} P_{j_i} \overline{v}_i = M \qquad (P_{j_0} = P_0,\ \overline{v}_i > 0,\ i = 1, \cdots, m),$$

$$\beta_0 P_s < 0,\ \beta_i P_s \leq 0,$$

$$\sum P_{j_i} [\overline{v}_i - (\beta_i P_s) \overline{\theta}] + P_s \overline{\theta} = M,$$

where the latter, with $\overline{\theta} \geq 0$ *arbitrary, forms a class of solutions with unbounded values* (β_i *is the* $(i + 1)$st *row of* B^{-1}).

Closely related to the methods of the next section, a constructive proof will now be given to:

THEOREM 8. *If any solution exists, then a basic solution exists.*

For this purpose adjust M and P so that the first nonzero component of each row of M

$$(20.1) \qquad \sum_{j=0}^{n} \begin{bmatrix} P_j \\ 0 \end{bmatrix} \overline{x}_j' + \sum_{i=1}^{m} \begin{bmatrix} U_i \\ 1 \end{bmatrix} \overline{x}_{n+i}' + \begin{bmatrix} \cdot \\ 1 \end{bmatrix} \overline{x}_{n+m+1}' = \begin{bmatrix} M & \cdot \\ \cdot & 1 \end{bmatrix}$$

$$(\overline{x}_j' \geq 0;\ j = 1, \cdots, n + m),$$

where $\overline{x}_j'$ has one more component than $\overline{x}_j$, and $\cdot$ represents the null vector. Noting that neither $\overline{x}_0'$ nor $\overline{x}_{n+m+1}'$ is required to be positive, one sees that an obvious basic solution is obtained using the variables $[\overline{x}_0', \overline{x}_{n+1}', \cdots, \overline{x}_{n+m+1}']$. It will be noted that the hypothesis of the theorem permits construction of a solution for which

$$\overline{x}_{n+i}' = 0 \qquad (i = 1, 2, \cdots, m).$$

Indeed, for $j \leq n$ set $\overline{x}_j' = (\overline{x}_j, 0) > 0$. However, it will be noted also that

$$\sum \bar{x}'_{n+i} = [\cdot\ 1]$$

so that

$$\max \bar{x}_{n+i} = [\cdot\ 1].$$

Accordingly, one may start with the basic solution for the augmented system, keeping the vectors corresponding to x'_0 and x'_{n+m+1} always in the basis[1], and use the simplex algorithm to maximize x'_{n+m+1}. Since, at the maximum,

$$\bar{x}'_{n+i} = 0 \qquad (i \neq m+1),$$

the corresponding vectors are not in the basis any longer (see (5)). By dropping the last component of this basic solution and by dropping x'_{n+m+1}, one is left with a basic solution to the original system.

3. Minimizing a linear form. The application of the generalized simplex method to the problem of minimizing a linear form subject to linear inequality, constraints consists in bordering the matrix of coefficients and constant terms of the given system by appropriate vectors. This can be done in many ways — the one selected is one which identifies the inverse of the basis as the additional components in a generalized matrix problem so that computationally no additional labor is required when the inverse is known.

The fundamental problem which we wish now to solve is to find a set $x = (x_0, x_1, \cdots, x_n)$ of real numbers satisfying the equations

$$(21) \qquad x_0 + \sum_1^n a_{0j} x_j = 0, \quad \sum_1^n a_{kj} x_j = b_k \qquad (b_k \geq 0;\ k = 2, 3, \cdots, m),$$

such that

$$(22) \qquad x_j \geq 0,$$

$$(23) \qquad x_0 = \max,$$

where without loss of generality one may assume $b_k \geq 0$. It will be noted that the subscript $k = 1$ has been omitted from (21). After some experimentation it has been found convenient[2] to augment the equations of (21) by a redundant equation formed by taking the negative sum of equations $k = 2, \cdots, m$. Thus

[1] To accomplish this omit $i = 0$ and $i = m+1$ in (16).

[2] Based on a recent suggestion of W. Orchard-Hays.

$$(24)\qquad \sum_1^n a_{1j}x_j = b_1 \qquad \left(a_{1j} = -\sum_{k=2}^m a_{kj},\ b_1 = -\sum_2^m b_k\right).$$

Consider the generalized problem of finding a set of vector "variables" (in the sense of §2) $(\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_n)$, and auxiliary variables $(\bar{x}_{n+1}, \bar{x}_{n+2}, \cdots, \bar{x}_{n+m})$ satisfying the matrix equations

$$(25)\quad \bar{x}_0 + \sum_1^n a_{0j}\bar{x}_j = (0, 1, 0, \cdots, 0),$$

$$\bar{x}_{n+k} + \sum_1^n a_{kj}\bar{x}_j = (b_k, 0, 0, \cdots, 1, \cdots, 0) \quad (b_1 \le 0;\ b_k \ge 0,\ k = 2, \cdots, m),$$

where the constant vectors have $l = m + 2$ components with unity in position $k + 2$, $\bar{x}_0$ and $\bar{x}_{n+1}$ are unrestricted as to sign and, for all other j,

$$(26)\qquad \bar{x}_j \ge 0 \qquad (j = 1, \cdots, n,\ n + 2, \cdots, n + m).$$

Adding equations $k = 1, \cdots, m$ in (25) and noting the definitions of a_{1j} and b_1 given in (24), one obtains

$$(27)\qquad \sum_1^m \bar{x}_{n+k} = (0, 0, 1, 1, \cdots, 1).$$

There is a close relationship between the solutions of (25) and those of (21) when $\bar{x}_{n+1} \ge 0$, for then the first components of $\bar{x}_j$, for $j = 0, \cdots, n$, satisfy (21). Indeed, by (27), if all $\bar{x}_{n+k} \ge 0$, the first component of all $\bar{x}_{n+k}$ must *vanish*; but the first component of the vector equations (25) reduces to (21) when the terms involving x_{n+k} are dropped. This proves the sufficiency of Theorem 9 (below).

THEOREM 9. *A necessary and sufficient condition for a solution of* (21) *to exist is for a solution of* (25) *to exist with* $\bar{x}_{n+1} \ge 0$.

THEOREM 10. *Maximizing solutions (or a class of solutions with unbounded values) of* (21) *are obtained from the* 1st *components of* $(\bar{x}_0, \cdots, \bar{x}_n)$ *of the corresponding type solution of* (25) *with* $\bar{x}_{n+1} \ge 0$.

Proofs. To prove necessity in Theorem 9, assume $(x_0, \cdots, x_n)$ satisfies

(21); then the set

$$\bar{x}_0 = (x_0, 1, 0, \cdots, 0), \tag{28}$$

$$\bar{x}_j = (x_j, 0, 0, \cdots, 0) \qquad (1 \le j \le n),$$

$$\bar{x}_{n+k} = (0, 0, \cdots, 1, \cdots, 0) \ge 0 \qquad (1 \le k \le m)$$

(where unity occurs in position $k+2$) satisfies (25). Because of the possibility of forming solutions of the type (28) from solutions of (21), it is easy to show that 1st components of maximizing solutions of (25) must be maximizing solutions of (28) (Theorem 10).

It will be noted that (25) satisfies the requirements for the generalized simplex process: first the right side considered as a matrix is of the form

$$M = [Q, U_0, U_1, \cdots, U_m],$$

where U_k is a unit column vector with unity in component $k+1$, and is of *rank* $m+1$ (the number of equations); second, an *initial basic solution* is available. Indeed, set $\bar{x}_0, \bar{x}_{n+1}, \bar{x}_{n+2}, \cdots, \bar{x}_{n+m}$ equal to the corresponding constant vectors in (25) where $\bar{x}_{n+k} \ge 0$ for $k = 2, \cdots, m$ because $b_k \ge 0$.

In applying the generalized simplex procedure, however, *both* $\bar{x}_0$ *and* $\bar{x}_{n+1}$ *are not restricted to be nonnegative.* Since $\bar{x}_{n+k} \ge 0$ for $k = 2, \cdots, m$, it follows that the values of the solutions, $\bar{x}_{n+1}$, of (27) have the right side of (27) as an upper bound.

To obtain a maximizing solution of (25), the *first phase* is to apply the generalized simplex procedure to maximize the variable $\bar{x}_{n+1}$ (with no restriction on $\bar{x}_0$). Since $\bar{x}_{n+1}$ has a finite upper bound, a basic solution will be produced after a finite number of changes of basis in which $\bar{x}_{n+1} \ge 0$, provided that $\max \bar{x}_{n+1} \ge 0$. If during the first phase x_{n+1} reaches a maximum less than zero, then, of course, by Theorem 9 there is *no solution* of (21) and the process terminates. If, in the iterative process, $\bar{x}_{n+1}$ becomes positive (even though not maximum), the *first phase*, which is the search for a solution of (21), is completed and the *second phase*, which is the search for an optimal solution, begins. Using the final basis of the first phase in the second phase, one sees that $\bar{x}_0$ is maximized under the additional constraint $\bar{x}_{n+1} \ge 0$.

Since the basic set of variables is taken in the initial order $(\bar{x}_0, \bar{x}_{n+1}, \cdots, \bar{x}_{n+m})$, and in the *first phase* the variable $\bar{x}_{n+1}$ is maximized, the *second row* *of the inverse of the basis*, β_1, is used to "select" the candidate P_s to introduce into the basis in order to increase $\bar{x}_{n+1}$ (see (14)); hence s is determined such that

$$\beta_1 P_s = \min_j (\beta_1 P_j) < 0. \tag{29}$$

However, in the *second phase*, since the variable to be maximized is $\bar{x}_0$ and the order of the basic set of variables is $(\bar{x}_0, \bar{x}_{n+1}, \cdots)$, then the *first row of the inverse of the basis*, β_0, is used; that is, one reverts back to (14). Application of the generalized simplex procedure in the second phase yields, after a finite number of changes in basis, either a solution with max $\bar{x}_0$ or a class of solutions of form (12) with no upper bound for $\bar{x}_0$. By Theorem 10 the first components of $\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_n$ form the corresponding solutions of the real variable problem.

The computational convenience of this setup is apparent. In the first place (as noted earlier), the right side of (21) considered as a matrix is of the form

$$M = [Q, U_0, U_1, \cdots, U_m],$$

where U_i is a unit column vector with unity in component $k+1$. In this case, by (4), the basic solution satisfies

$$V = B^{-1}M = [B^{-1}Q;\ B^{-1}].$$

This means (in this case) that of the $l = m+2$ components of the vector $\bar{v}_i$ *the last* $m+1$ *components of the vector variables* $\bar{v}_i$ *in the basic solution are identical with* β_i, *the corresponding row of the inverse*. In applications this fact is important because the last $m+1$ components of $\bar{v}_i$ are artificial in the sense that they belong to the perturbation and not to the original problem and it is desirable to obtain them with as little effort as possible. In the event that M has the foregoing special form, no additional computational effort is required when the inverse of the basis is known. Moreover, the columns of (25) corresponding to the $m+1$ variables $(\bar{x}_0, \bar{x}_{m+1}, \cdots, \bar{x}_{n+m})$ form the *initial identity basis* $(U_0, U_1, \cdots, U_m)$, so that the inverse of the initial basis is readily available as the identity matrix to initiate the first iteration.

REFERENCES

1. *Activity analysis of production and allocation*, T.C. Koopmans, Editor, John Wiley and Sons, New York, 1951.

(a) R. Dorfman, *Application of simplex method to a game theory problem,* Chapter XXII.

(b) G. Dantzig, *Maximization of a linear function of variables subject to linear inequalities,* Chapter XXI.

(c) G. Dantzig, *Application of the simplex method to a transportation problem,* Chapter XXIII.

(d) D. Gale, H. Kuhn, and A. Tucker, *Linear programming and the theory of games,* Chapter XIX.

2. A. Charnes, *Optimality and degeneracy in linear programming, Econometrica,* April 1952, pages 160-70.

3. A. Charnes, W. W. Cooper, and A. Henderson, *An introduction to linear programming,* John Wiley and Sons, New York, 1953.

4. A. Charnes and C. E. Lemke, *Computational problems of linear programming,* Proceeding of the Association of Computation Machinery, Pittsburgh, 1952, pages 97-98.

5. *Contributions to Theory of Games,* H. Kuhn and A. W. Tucker, Editors, Princeton University Press; Vol. I, 1950; Vol. II, 1953.

6. G. Dantzig, *Computational Algorithm of the Revised Simplex Method,* RAND P-394-1, October 1953.

7. G. Dantzig and W. Orchard-Hays, *The product form for the inverse in the simplex method,* RAND P-394-1, October 1953; also RAND P-460, November 1953.

8. G. Dantzig and A. Orden, *A duality theorem based on the simplex method,* Symposium on Linear Inequalities, USAF-Hq., SCOOP Publication No. 10, dated 1 April 1952, pages 51-55.

9. A. Hoffman, M. Mannos, D. Sokolowsky, and N. Wiegmann, *Computational experience in solving linear programs,* Journal of Social Industrial and Applied Mathematics, Vol. 1, No. 1, September 1953, pages 17-33.

10. T. S. Motzkin, *Two consequences of the transposition theorem of linear inequalities,* Econometrica, Vol. 19, No. 2, April 1951, pages 184-5.

11. ————, *Beiträge zur Theorie der Linearen Ungleichungen,* Dissertation, Basel, 1933; Jerusalem, 1935.

12. J. von Neumann, *Discussion of a maximization problem,* Institute for Advanced Study, 1947 Manuscript.

13. Alex Orden, *Solution of systems of linear inequalities on a digital computer,* Proceedings of the Association of Computing Machinery, Pittsburgh, 1952, pages 91-95.

14. A. J. Hoffman, *Cycling in the simplex algorithm,* National Bureau of Standards Report, No. 2974, December 16, 1953.

RAND CORPORATION

BURROUGHS CORPORATION

PRINCETON UNIVERSITY

CONSTRUCTIONS FOR POLES AND POLARS IN n-DIMENSIONS

A. P. DEMPSTER AND S. SCHUSTER

1. Introduction. As far back as 1847, von Staudt [2, p. 131-136] introduced the notion of handling a symmetric polarity (that is, a nonnull polarity) by means of a self-polar simplex and an additional pair of corresponding elements. In projective space of two dimensions (S_2) such a polarity is completely determined by a self-polar triangle $A_1A_2A_3$, a point P, and its polar line p. We write this polarity as $(A_1A_2A_3)(Pp)$. In S_3, the polarity is determined by a self-polar tetrahedron $A_1A_2A_3A_4$, a point P, and its polar plane π. We write it $(A_1A_2A_3A_4)(P\,\pi)$. In general, we have a polarity in S_n determined by the self-polar simplex $A_1A_2\cdots A_{n+1}$, a point P, and its corresponding polar prime or hyperplane π. We write it $(A_1A_2\cdots A_{n+1})(P\,\pi)$.

Left unanswered by von Staudt and his followers is the following question: Given an arbitrary point X, how can we construct the polar prime χ of X? And, conversely, given the prime χ, how do we actually find its pole, the point X?

2. Construction. The construction of the polar line x of an arbitrary point X for the polarity $(A_1A_2A_3)(Pp)$ in S_2 was given by Coxeter [1, 64]. We give a direct generalization of this to n dimensions: to find the polar prime χ of an arbitrary point X relative to $(A_1A_2\cdots A_{n+1})(P\,\pi)$.

Consider first the point X not in any face of $A_1A_2\cdots A_{n+1}$. Let α_i denote face $A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}$, and let

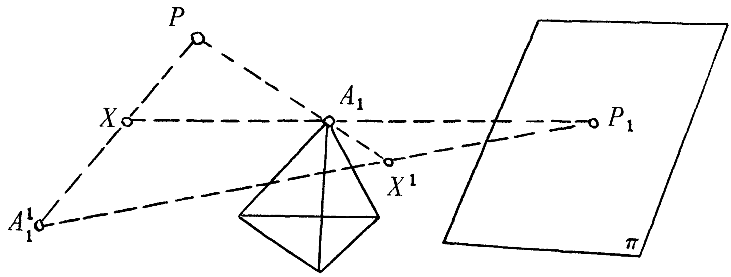
$$A_i' = PX\cdot\alpha_i,\quad P_i = XA_i\cdot\pi,\quad \text{and}\quad X^i = PA_i\cdot P_iA_i'.$$

In the plane PXA_i we have pairs P, P_i and A_i, A_i' conjugate under the induced plane polarity. By Hesse's theorem in the plane [1, pp. 60-61], X and X^i are conjugate for the induced polarity, and hence for the given polarity. In this manner we determine $n+1$ points $X^1, X^2, \cdots, X^{n+1}$ lying in χ. The points $X^1, X^2, \cdots, X^n$ determine χ since otherwise they must lie in an $(n-2)$-flat which implies that the flat determined by $P, X^1, \cdots, X^n$ is of at most $(n-1)$ dimensions, which is impossible since the space contains $P, A_1, A_2, \cdots, A_n$. It

Received August 1, 1953.

Pacific J. Math., 5 (1955), 197-199

follows that χ is determined by any $(n-1)$ of the points X^i. This completes the construction in S_n for general X. This is illustrated for $n=3$, and is easily seen to yield Coxeter's construction for $n=2$.



A second approach is to reduce the question of finding χ in S_n to two analogous constructions in $(n-1)$ dimensions, namely in any two faces α_i. Under the polarity induced in α_i the point $X_i = XA_i \cdot \alpha_i$ maps into an $(n-2)$-flat x_i consisting of points conjugate to X. For the general X considered, no two x_i coincide; hence, any two of them determine an $(n-1)$-flat of points conjugate to X. This can only be χ. Using this idea we can reduce the construction in S_n to 2^r analogous constructions in $n-r$ dimensions, and at any stage of this induction on r, we may use the first method to solve the question completely.

In particular, if $n=2$ we can construct directly by the first method or use the construction for corresponding points in two involutions on the sides of $A_1A_2A_3$. If $n=3$ we can use the first method, or carry out constructions in two faces of $A_1A_2A_3A_4$, or carry out constructions in four edges of $A_1A_2A_3A_4$.

Going back to n dimensions, suppose X is not of general position; that is, X lies in a face α_i. If X lies in r such faces we may name these $\alpha_1, \cdots, \alpha_r$. Then χ contains $A_1, \cdots, A_r$. Considering the $(n-r)$-flat determined by simplex $A_{r+1} \cdots A_{n+1}$, we see that the polarity induced in this space has $A_{r+1} \cdots A_{n+1}$ as a self-polar simplex and X belongs to the space but is not on a face of $A_{r+1} \cdots A_{n+1}$. Thus, we can use the first method to determine the polar prime χ' of X in this space. Then $A_1, \cdots, A_r$, and χ' generate an $(n-1)$-flat of points conjugate to X. This $(n-1)$-flat is χ.

The problem of finding X when given χ is solved by dualizing the foregoing procedures.

References

1. H. S. M. Coxeter, *The real projective plane*, New York, 1949.
2. C. G. C. von Staudt, *Geometrie der Lage*, Nuremberg, 1847.

UNIVERSITY OF TORONTO
POLYTECHNIC INSTITUTE OF BROOKLYN

POWER-TYPE ENDOMORPHISMS OF SOME CLASS 2 GROUPS

FRANKLIN HAIMO

1. Introduction. Abelian groups possess endomorphisms of the form $x \longrightarrow x^n$ for each integer n. In general, however, non-abelian groups do not possess such power endomorphisms. In an earlier note, it was possible to show [1] for a nilpotent group G with a uniform bound on the size of the classes of conjugates that there exists an integer $n \geq 2$ for which the mapping $x \longrightarrow x^n$ is an endomorphism of G into its center. We shall consider endomorphisms of some groups of class 2 which induce power endomorphisms on the factor-commutator groups. In particular, we shall show, under suitable uniform torsion conditions for the group of inner automorphisms, that such power-type endomorphisms form a ring-like structure. Let G be a group of class 2 for which Q, the commutator subgroup, has an exponent [2]. Then the relation [2] $(xy, u) = (x, u)(y, u)$ shows that $x \longrightarrow (x, u)$ is an endomorphism of G into Q for fixed $u \in G$. Let n be any integer such that $n(n-1)/2$ is a multiple of the exponent of Q. Then the mapping $x \longrightarrow x^n(x, u)$ is a trivial example of a power-type endomorphism. If G/Q has an exponent m, we shall show that the number of distinct endomorphisms of the form $x \longrightarrow x^j$, where x^j is in the center Z of G, divides m. In particular, a non-abelian group G of class 2 has 1 or p distinct central power endomorphisms if G/Q is an elementary p-group (an abelian group with a prime p as its exponent [2]).

2. Power-type endomorphisms. Let G be a group with center Z and commutator subgroup Q. We assume that $Q \subset Z$ so that [2] G is a group of class 2. Further, suppose that there exists a least positive integer N for which $x \in G$ implies $x^N \in Z$. This means that G/Z, a group isomorphic to the group of inner automorphisms of G, is a torsion abelian group with exponent N. An endomorphism α of G will be called a *power-type* endomorphism if there exists an integer $n = n(\alpha)$ for which $\alpha(x) \equiv x^n \bmod Q$ for every $x \in G$. α induces the power endomorphism

$$\alpha^*(xQ) = x^n Q$$

Received August 20, 1953. This research was supported in part by the USAF under contract No. AF18(600)-568 monitored by the Office of Scientific Research, Air Research and Development Command.

Pacific J. Math. 5 (1955), 201-213

on G/Q; and conversely, any extension of a power endomorphism of G/Q to an endomorphism of G must be a power-type endomorphism of G. For α, above, there exist elements

$$q(x) = q(x;\alpha) \in Q$$

such that $\alpha(x) = x^n q(x)$. It is easy to show that if m and n are two possible values for $n(\alpha)$ then $m \equiv n \bmod N$. We note that if N is taken to be the exponent for G/Q rather than for G/Z, then $n(\alpha)$ can be chosen least nonnegative, in fact, so that $0 \leq n(\alpha) < N$. We let $\mathfrak{P}$ denote the class of all power-type endomorphisms of a fixed group G of class 2. Let $\iota(x) = x$ for every $x \in G$ be the identity map on G. We have $\iota \in \mathfrak{P}$ with $n(\iota) = 1$. If e is the identity element of G, let $\nu(x) = e$ for every $x \in G$ be the trivial map of G. We have $\nu \in \mathfrak{P}$; in fact, any endomorphism of G which carries G into Q lies in $\mathfrak{P}$. Let the set of all such endomorphisms into the commutator subgroup be denoted by $\mathfrak{N}$. We have $\nu \in \mathfrak{N}$. If $\alpha \in \mathfrak{N}$ then $n(\alpha) = 0$, and conversely (for $\alpha \in \mathfrak{P}$).

Suppose that α and β are in $\mathfrak{P}$. Then

$$\begin{aligned}\alpha\beta(x) = \alpha[x^{n(\beta)} q(x;\beta)] &= [\alpha(x)]^{n(\beta)}\alpha[q(x;\beta)] \\ &= [x^{n(\alpha)} q(x;\alpha)]^{n(\beta)}\alpha[q(x;\beta)].\end{aligned}$$

Since $Q \subset Z$, we have

$$\alpha\beta(x) = x^{n(\alpha)n(\beta)}[q(x;\alpha)]^{n(\beta)}\alpha[q(x;\beta)].$$

This shows that $\alpha\beta \in \mathfrak{P}$ so that $\mathfrak{P}$ is closed under endomorphism composition. In fact,

$$n(\alpha\beta) \equiv n(\alpha)n(\beta) \bmod N.$$

This multiplication is associative. Suppose that $\alpha \in \mathfrak{P}$ and that $\gamma \in \mathfrak{N}$. Then it is easy to see that $\alpha\gamma$ and $\gamma\alpha \in \mathfrak{N}$, since Q is admissible under every endomorphism of G.

Let $\mathfrak{R}$ be the set of all elements of $\mathfrak{P}$ with the property that $\alpha \in \mathfrak{R}$ if and only if $N \mid n(\alpha)$. For endomorphisms α and β of G, we define a mapping $\alpha + \beta$, (not necessarily an endomorphism), by

$$(\alpha + \beta)(x) = \alpha(x)\beta(x)$$

for every $x \in G$. Then we have the following.

THEOREM 1. *If $\alpha \in \mathfrak{P}$, then $\alpha + \beta \in \mathfrak{P}$ for every $\beta \in \mathfrak{P}$ if and only if $\alpha \in \mathfrak{R}$. If $\alpha + \beta \in \mathfrak{P}$, then*

$$n(\alpha) + n(\beta) \equiv n(\alpha + \beta) \bmod N,$$

and

$$q(x; \alpha + \beta) = q(x; \alpha)q(x; \beta).$$

Proof. Suppose that $\alpha + \beta \in \mathfrak{P}$ for every $\beta \in \mathfrak{P}$. Choosing $\beta = \iota$, we have

$$(\alpha + \iota)(xy) = [(\alpha + \iota)(x)][(\alpha + \iota)(y)] = \alpha(x)x\alpha(y)y.$$

On the other hand,

$$(\alpha + \iota)(xy) = \alpha(xy)xy = \alpha(x)\alpha(y)xy,$$

so that $\alpha(y)x = x\alpha(y)$ for every $x,y \in G$. This places $\alpha(y) \in Z$; but

$$\alpha(y) = y^{n(\alpha)}q(y; \alpha)$$

where $q(y; \alpha) \in Q \subset Z$. Thus, $y^{n(\alpha)} \in Z$, for every $y \in G$, and $N \mid n(\alpha)$, placing $\alpha \in \mathfrak{R}$. Remaining details are immediate.

For elements of $\mathfrak{P}$, addition is commutative whenever one of the sums involved is in $\mathfrak{P}$, and if all the sums involved are in $\mathfrak{P}$, then addition is associative. A like statement can be made for the distributive law of multiplication over addition. $\mathfrak{R}$ is a ring with the two-sided ideal property in $\mathfrak{P}$ in that if $\alpha \in \mathfrak{P}$, $\beta \in \mathfrak{R}$, then $\alpha\beta$ and $\beta\alpha \in \mathfrak{R}$. $\mathfrak{N}$ likewise can be shown to be a ring which has the two-sided ideal property in $\mathfrak{P}$, therefore in $\mathfrak{R}$.

THEOREM 2. *Let G be a non-abelian group of class 2 for which the group of inner automorphisms J has the exponent N. If G/Q is aperiodic, then $\mathfrak{N}$ is a prime ideal in $\mathfrak{R}$.*

Proof. Suppose that $\alpha, \beta \in \mathfrak{R}$ and that $\alpha\beta \in \mathfrak{N}$. If $G = Q$, then $Q \subset Z$ implies that G is abelian. Hence we can find $x \in G$, $x \notin Q$ so that

$$\alpha\beta(x) = x^{n(\alpha)n(\beta)}q,$$

where both q and $\alpha\beta(x) \in Q$. Since G/Q is aperiodic, $n(\alpha)n(\beta) = 0$. We have really proved the prime ideal property of $\mathfrak{N}$ in $\mathfrak{P}$. The exponent on J, (isomorphic to G/Z) is required only to guarantee the existence of $\mathfrak{R}$. A related result is the following.

THEOREM 3. *Let G be a non-abelian group of class 2 for which G/Q is a p-group with exponent p^j. Then $\mathfrak{n}$ is a primary ideal in $\mathfrak{R}$. In particular, if G/Q is an elementary p-group* [2], *then $\mathfrak{n}$ is a prime ideal in $\mathfrak{R}$.*

Proof. The proof begins as for Theorem 2. Since G/Q has exponent p^j, the latter is a divisor of $n(\alpha)n(\beta)$. If $\alpha \notin \mathfrak{n}$, at least the first power of p would have to divide $n(\beta)$. For, G/Z has an exponent p^k where $1 \leq k \leq j$. Since $n(\beta^j) = [n(\beta)]^j$ we have $p^j \mid n(\beta^j)$ whence $\beta^j \in \mathfrak{n}$. The ring $\mathfrak{R}$ exists since G/Z has an exponent. If G/Q is elementary, then $j = k = 1$ so that $\mathfrak{n}$ is a prime ideal.

3. Additive inverses. An element α of $\mathfrak{P}$ is said to have an additive inverse $\alpha' \in \mathfrak{P}$ if $\alpha + \alpha' = \nu$. If such an additive inverse exists, it is unique, and

$$\alpha'(x) = x^{-n(\alpha)} q(x; \alpha)^{-1}.$$

A mapping with the structure of α' always exists, but it need not be, in general, an endomorphism, *ergo* not an additive inverse. If α' is an additive inverse of α, then α is the additive inverse of α'. We first prove the following.

LEMMA 1. *α has an additive inverse if and only if the $n(\alpha)$-powers of G form a commutative set.*

Proof. Whether the mapping α' is an endomorphism or not, we have

$$\alpha'(x) = [\alpha(x)]^{-1},$$

so that

$$\alpha'(xy) = \alpha'(y)\,\alpha'(x)$$

for every $x, y \in G$. Since $Q \subset Z$, the conclusion follows at once.

Let $\mathfrak{K}$ be the set of all $\alpha \in \mathfrak{P}$ with the property that $\operatorname{kern} \alpha \supset Q$.

LEMMA 2.

(a) *$\mathfrak{K}$ has the ideal property in $\mathfrak{P}$.*

(b) $\mathfrak{K} \supset \mathfrak{R} (\supset \mathfrak{n})$.

(c) *$\alpha \in \mathfrak{P}$ has an additive inverse if and only if $\alpha \in \mathfrak{K}$.*

(d) $\alpha \in \mathfrak{R}$ *and* $\beta \in \mathfrak{K}$ *implies that* $\alpha + \beta \in \mathfrak{K}$.

Proof. (a) and (d) are trivial. For $\alpha \in \mathfrak{P}$, we have

$$\alpha(x^{-1}y^{-1}xy) = x^{-n}y^{-n}x^{n}y^{n}$$

where $n = n(\alpha)$. If, further, $\alpha \in \mathfrak{R}$, then $x^n \in Z$ so that $\alpha(x, y) = e$, and (b) is established, since $(x, y) = x^{-1}y^{-1}xy$ is typical of the generators of Q. We have $\alpha \in \mathfrak{K}$ if and only if $\alpha(x, y) = e$, that is, if and only if $x^n y^n = y^n x^n$. Lemma 1 now enables us to prove (c).

For fixed $\gamma \in \mathfrak{K}$, we have $\gamma\alpha \in \mathfrak{K}$ for every $\alpha \in \mathfrak{P}$. Write $-\gamma\alpha$ for the additive inverse of $\gamma\alpha$; then $-\gamma\alpha \in \mathfrak{K}$. Let j_i be 0 or 1, and suppose that $\alpha_i \in \mathfrak{P}$, $i = 1, 2, \cdots, m$. A mapping

$$\sum_{i=1}^{m} (-1)^{j_i} \gamma \alpha_i = \sigma$$

is defined on G into G by

$$\sigma(x) = \prod_{i=1}^{m} x^{n(\gamma)(-1)^{j_i}n(\alpha_i)}[q(x;\gamma)]^{(-1)^{j_i}n(\alpha_i)}.$$

Call such a map a $\gamma - \Sigma$ *map*. It is clear that the sum of two $\gamma - \Sigma$ maps is a $\gamma - \Sigma$ map in the obvious way. The set of $\gamma - \Sigma$ maps is denoted by (γ) and will be called the *right principal ideal* generated by γ in $\mathfrak{P}$.

THEOREM 4. *If* $\gamma \in \mathfrak{K}$ *then* (γ) *is a ring, and* $(\gamma) \subset \mathfrak{K}$.

Proof. As we saw above, (γ) is closed under addition. $\gamma\nu = \nu$ so that (γ) has the zero element ν. If σ is defined as above, then

$$\sum_{i=1}^{m} (-1)^{j_i+1} \gamma \alpha_i = -\sigma \in (\gamma).$$

By its effect on $x \in G$ we see that $\sigma \in \mathfrak{P}$. Since $-\sigma$ exists, $(\gamma) \subset \mathfrak{K}$ by Lemma 2(c). Now $(\gamma\alpha)(\gamma\beta) = \gamma(\alpha\gamma\beta)$, so that (γ) is closed under multiplication, once we recall that the distributive law is valid whenever the sums involved are in $\mathfrak{P}$. A similar statement can be made for the associative laws, and we have proved that (γ) is a ring included in $\mathfrak{K}$.

THEOREM 5. *Let G be a non-abelian group of class 2, and let γ be in $\mathcal{K}$. If the ring (γ) has a right multiplicative identity or a left multiplicative identity, then it has a (unique) two-sided multiplicative identity.*

Proof. (γ) has a left (right) identity $\sigma \in (\gamma)$ if and only if $\sigma \in (\gamma)$ is a left (right) identity for the set of elements of (γ) of the form $\gamma\beta$. More, specifically, σ is a left identity if and only if $\sigma\gamma = \gamma$. A routine investigation shows that

$$\sigma\gamma(x) = x^{[n(\gamma)]^2 \sum_{i=1}^{m} (-1)^{j_i} n(\alpha_i)} q^{n(\gamma) \sum_{i=1}^{m} (-1)^{j_i} n(\alpha_i)}$$

where $q = q(x; \gamma)$. Let

$$u = n(\gamma) \sum_{i=1}^{m} (-1)^{j_i} n(\alpha_i) - 1.$$

Then $\sigma\gamma = \gamma$ if and only if

$$x^{n(\gamma)u} q^u = e$$

for every $x \in G$. Hence (1) $\gamma(x^u) = e$ for every $x \in G$, (2) $G/\ker\gamma$ has an exponent dividing u and (3) $\gamma(G)$ has an exponent dividing u are conditions each equivalent to (4) σ is a left identity of (γ). If (5) σ is a right identity of (γ), (6) $\gamma\sigma = \gamma$. But one can readily verify that (6) and (1) are equivalent, so that if σ is a right identity, it is also a left identity, whence (γ) would then have a unique two-sided identity.

If σ is a left identity, then $\sigma\gamma = \gamma$ and

$$\gamma\beta\sigma(x) = [\gamma(x)]^{n(\beta)} = \gamma\beta(x)$$

for every $x \in G$. Thus σ is also a right identity, and we have proved that every left identity is a right identity.

COROLLARY. *Let G be a non-abelian group of class 2 for which G/Q is an elementary p-group for an odd prime p. Let $\gamma \in \mathcal{K}$ have the properties* (a) *that $p \nmid n(\gamma) = n$ and* (b) *that there exists an integer m such that* (b_1) $mn \equiv 1 \bmod p$ *and* (b_2) $m-1$ *and* $n-1$ *are relatively prime. Then (γ) has an identity.*

Proof. $(m-1, n-1) = 1$ implies that $((m-1)n, n-1) = 1$ and that $(mn-1, n-1) = 1$ since $mn - 1 = (m-1)n + (n-1)$. Hence we can find an integer r such that

$$n(n-1)r \equiv m(m-1) \bmod (mn-1). \tag{7}$$

Form the mapping

$$\tau(x) = x^m [q(x;\gamma)]^r.$$

Since G is a group of class 2, we have [2] the identity

$$(xy)^t = x^t y^t z^{v(t)},$$

where

$$z = (y,x) = y^{-1}x^{-1}yx \text{ and } v(t) = t(t-1)/2.$$

Since γ is an endomorphism, we have

$$q(xy;\gamma) z^{v(n)} = q(x;\gamma) q(y;\gamma).$$

Hence

$$\tau(xy) = x^m y^m z^{v(m)} [q(x;\gamma)]^r [q(y;\gamma)]^r z^{-rv(n)}.$$

Let us write the exponent of z as $h/2$ where $h = m(m-1) - rn(n-1)$. By the choice of r we have $h \equiv 0 \bmod (mn-1)$. But $mn - 1 \equiv 0 \bmod p$, so that $h \equiv 0 \bmod p$. Since p is odd we obtain $h/2 \equiv 0 \bmod p$.

Since G/Q has the exponent p, $Q \subset Z$ implies that G/Z has an exponent t where $t \mid p$. Since G is non-abelian we have $t = p$. In [**1**], we proved that if G/Z has the exponent p then the mutual commutator group (G, Z_2) has an exponent t' which divides p. Here Z_2 is the second member of the ascending central series of G. Since G is of class 2 we have $Z_2 = G$, and $(G, Z_2) = Q$. If $t' = 1$, then G is abelian, a contradiction with hypothesis. Hence $t' = p$ and $z^{h/2} = e$, since $z \in Q$ and $p \mid (h/2)$. As a result, $\tau(xy)$ reduces to $\tau(x)\tau(y)$, so that τ is a power-type endomorphism with $n(\tau) = m$ and

$$q(x;\tau) = [q(x;\gamma)]^r.$$

Then

$$u = n(\gamma)n(\tau) - 1 = mn - 1.$$

Since p is the exponent of G/Q we have $x^u \in Q$ for every $x \in G$. But $\gamma \in \mathfrak{K}$ so that $\gamma(x^u) = e$. Using the theorem and (1) and (4) above, we see that $\gamma\tau$ is the required identity of (γ).

4. Some mappings into Q. Let $\mathfrak{E}$ be the set of all $\alpha \in \mathfrak{P}$ which are extensions both of the identity map on Q and of the identity map on G/Q. That is, $\alpha \in \mathfrak{E}$ if and only if $\alpha(x) = xq(x; \alpha)$ for every $x \in G$ and $\alpha(q) = q$ for every $q \in Q$. It can readily be verified that the elements of $\mathfrak{E}$ are automorphisms of G and that, under automorphism composition, they form an abelian group with unity ι. For $\alpha, \beta \in \mathfrak{E}$ and $x, y \in G$, it follows at once that

$$q(xy; \alpha) = q(x; \alpha)q(y; \alpha)$$

and that

$$q(x; \alpha\beta) = q(x; \alpha)q(x; \beta).$$

Let θ_x be a mapping defined on $\mathfrak{E}$ into Q such that $\theta_x(\alpha) = q(x; \alpha)$ for every $\alpha \in \mathfrak{E}$. It is immediate that the θ_x are homomorphisms. We can define an addition in the set $\mathfrak{J}$ of mappings θ_x by

$$(\theta_x + \theta_y)(\alpha) = \theta_x(\alpha)\,\theta_y(\alpha)$$

for every $\alpha \in \mathfrak{E}$. Likewise define mappings ϕ_α on G into Q by $\phi_\alpha(x) = q(x; \alpha)$. Here, too, in the set $\mathfrak{S}$ of mappings ϕ_α, mappings which are also homomorphisms, an addition is given by

$$(\phi_\alpha + \phi_\beta)(x) = \phi_\alpha(x)\,\phi_\beta(x)$$

for every $x \in G$. Let F be the set of elements of G which are the fixed points held in common by the elements of $\mathfrak{E}$. Then we obtain the following.

THEOREM 6.

(a) $\mathfrak{J} \cong G/F$.

(b) $\mathfrak{S} = \mathfrak{h}$ *and* $\mathfrak{h} \cong \mathfrak{E}$.

(c) $\mathfrak{h}$ *and* $\mathfrak{J}$ *are dual additive abelian groups in the sense that each can be represented faithfully as a set of homomorphisms on the other into* Q.

Proof. It is easy to verify that $\theta_x + \theta_y = \theta_{xy}$, and it follows that $\mathfrak{J}$ is an additive abelian group with unity θ_e. Let F_α be the subgroup of all $x \in G$ with $\alpha(x) = x$. For $\alpha \in \mathfrak{E}$, each F_α, and hence $F = \bigcap F_\alpha$, is a normal subgroup of G.

$\alpha \in \operatorname{kern} \theta_x$ if and only if $x \in F_\alpha$. $\theta_x = \theta_y$ if and only if $x \equiv y \bmod F$. The mapping θ on G into $\mathfrak{J}$ given by $\theta(x) = \theta_x$ is a homomorphism onto $\mathfrak{J}$ with kernel F. We have established (a).

ϕ_α is an endomorphism of G into Q with kern $\phi_\alpha = F_\alpha$. For $\gamma \in \mathfrak{N}$, let Γ be a mapping of G into G given by $\Gamma(x) = x\gamma(x)$. Since $\mathfrak{N} \subset \mathfrak{R} \subset \mathfrak{K}$, we have $\Gamma(q) = q\gamma(q) = q$ for every $q \in Q$, so that $\Gamma \in \mathfrak{E}$. Also, $\phi_\Gamma = \gamma$. Hence $\mathfrak{N} \subset \mathfrak{S}$. Trivially, $\mathfrak{S} \subset \mathfrak{N}$. The unity of $\mathfrak{N}$ as a group is ν which can be represented as ϕ_ι. The mapping ϕ given by $\phi(\alpha) = \phi_\alpha$ on $\mathfrak{E}$ onto $\mathfrak{S} = \mathfrak{N}$ turns out to be an isomorphism, whence (b).

The mappings c_x on $\mathfrak{N}$ into Q given by

$$c_x(\gamma) = \theta_x \phi^{-1}(\gamma)$$

for every $\gamma \in \mathfrak{N}$ are homomorphisms. $\gamma \in \operatorname{kern} c_x$ if and only if $x \in \operatorname{kern} \gamma$. We can introduce an addition into the set $\mathfrak{C}$ of mappings c_x by

$$(c_x + c_y)(\gamma) = c_x(\gamma)\, c_y(\gamma)$$

for every $\gamma \in \mathfrak{N}$. There is a homomorphism ψ of G onto $\mathfrak{C}$ with kernel equal to

$$U = \bigcap \operatorname{kern} \gamma,$$

where the cross-cut is taken over all $\gamma \in \mathfrak{N}$; and $\psi(x) = c_x$. A trivial argument shows that $U = F$. One can verify that the correspondence $\theta_x \leftrightarrow c_x$ is one-to-one and is an isomorphism of $\mathfrak{J}$ with $\mathfrak{C}$. Hence $\mathfrak{J}$ is represented faithfully as a set of homomorphisms on $\mathfrak{N}$ into Q.

Just as there are homomorphisms c_x on $\mathfrak{N}$ into Q, so there are homomorphisms b_α on $\mathfrak{J}$ into Q for each $\alpha \in \mathfrak{E}$, given by $b_\alpha(\theta_x) = \phi_\alpha(x)$. Here, kern b_α consists of all θ_x with $x \in F_\alpha$. The mapping b_α is single-valued; for $\theta_x = \theta_y$ if and only if there exists $r \in F$ with $y = xr$, and $\phi_\alpha(xr) = \phi_\alpha(x)$. We can introduce an addition into the set $\mathfrak{B}$ of such b_α by

$$(b_\alpha + b_\beta)(\theta_x) = \phi_\alpha(x)\, \phi_\beta(x).$$

Now $b_\alpha + b_\beta = b_{\alpha\beta}$, and, under this addition, $\mathfrak{B}$ becomes an abelian group with unity b_ι. The correspondence $b_\alpha \leftrightarrow \phi_\alpha$ is one-to-one and is an isomorphism of $\mathfrak{B}$ with $\mathfrak{N}$, so that $\mathfrak{N}$ is represented faithfully as a set of homomorphisms on $\mathfrak{J}$ into Q, and (c) is established.

Further, there is an isomorphism ω on $\mathfrak{E}$ onto $\mathfrak{B}$ given by $\omega(\alpha) = b_\alpha$. The mapping

$$\theta_x \omega^{-1} = \delta_x$$

is a homomorphism on $\mathfrak{B}$ into Q with kernel consisting of all b_α with $x \in F_\alpha$. For every $\alpha \in \mathfrak{E}$, let ζ_α be a mapping defined on $\mathfrak{C}$ into Q by

$$\zeta_\alpha(c_x) = \phi_\alpha(x).$$

It is clear that ζ_α is a homomorphism with kernel consisting of all c_x where $x \in \operatorname{kern} \phi_\alpha$. We summarize these results as follows.

COROLLARY.

$$\theta_x = \delta_x \omega = c_x \phi$$

on $\mathfrak{E}$ *into* Q, *and dually,*

$$\phi_\alpha = \zeta_\alpha \psi = b_\alpha \theta$$

on G *into* Q.

5. Some enumerations of mappings.

THEOREM 7. *The elements of* $\mathfrak{P}$ *are in one-to-one correspondence with the ordered pairs* (n, λ), *where* n *is an integer,* λ *is a mapping of* G *into* Q *and* n *and* λ *satisfy*

$$\text{(A)} \qquad \lambda(x)\lambda(y) = \lambda(xy) z^{v(n)}$$

for every $x, y \in G$, *where* $z = (y, x)$ *and* $v(n) = n(n-1)/2$.

Proof. If $\alpha \in \mathfrak{P}$, then $q(x; \alpha) = \lambda(x)$ and $n(\alpha) = n$. Conversely, if λ and n are given, and if (A) holds, define α on G into G by $\alpha(x) = x^n \lambda(x)$ for every $x \in G$. Condition (A) and the fact that

$$(xy)^n = x^n y^n z^{v(n)}$$

show that α is an endomorphism and is therefore in $\mathfrak{P}$.

COROLLARY. *If* Q *has the exponent* m, *and if* n *is an integer for which* $m \mid v(n)$, *then* $x \longrightarrow x^n$ *is a power endomorphism of* G.

Proof. If we let $\lambda(x) = e$ for every $x \in G$ then the pair (n, λ) satisfies (A) since, here, $z^{v(n)} = e$.

THEOREM 8. *For* $\alpha, \beta \in \mathfrak{P}$, *a necessary and sufficient condition that* $n(\alpha) = n(\beta)$ *is that there exists a* $\gamma = \gamma_{\alpha,\beta} \in \mathfrak{N}$ *such that* $\alpha = \beta + \gamma$.

Proof. Suppose that $n(\alpha) = n(\beta)$. Define a mapping γ by

$$\gamma(x) = q(x; \alpha)[q(x; \beta)]^{-1}.$$

We have

$$(\beta + \gamma)(x) = \beta(x)\gamma(x) = x^{n(\beta)}q(x; \beta)q(x; \alpha)[q(x; \beta)]^{-1}$$

$$= x^{n(\alpha)}q(x; \alpha) = \alpha(x),$$

so that $\beta + \gamma = \alpha$. Now

$$\gamma(xy) = q(xy; \alpha)[q(xy; \beta)]^{-1};$$

hence if we apply (A) to each of the q's and simplify, it turns out that $\gamma(xy) = \gamma(x)\gamma(y)$, so that γ is an endomorphism lying in $\mathfrak{N}$.

COROLLARY. *Let* M *be the cardinal of* $\mathfrak{N}$. *Then* $\mathfrak{P}$ *decomposes into partition classes, each of cardinal* M, *in such a way that* α *and* β *are in the same partition class if and only if* $n(\alpha) = n(\beta)$.

Examples of such partition classes are $\mathfrak{N}$ (where $n = 0$) and $\mathfrak{E}$ (where $n = 1$). Nontrivial $\mathfrak{E}$ and $\mathfrak{E} \cong \mathfrak{N}$ along with an exponent on Q imply, by the Corollary of Theorem 7, the existence of an infinite number of partition classes.

Let I_N denote the group of integers, modulo N.

THEOREM 9. *Let* G *be a group of class* 2 *with exponent* N *on* G/Z. *Then there exists a nontrivial mapping* τ *on* $\mathfrak{P}$ *into* I_N *which preserves addition and multiplication* (*whenever they are defined on* $\mathfrak{P}$). $\mathfrak{N} \subset \operatorname{kern} \tau$.

Proof. Let j_N denote the residue class, modulo N, to which the integer j belongs. Let $\tau(\alpha) = (n(\alpha))_N$. Then $\tau(\iota) = 1_N$, so that τ is nontrivial. The remaining statements are apparent. Note, however, that if N is the exponent of G/Q, then $\operatorname{kern} \tau = \mathfrak{N}$.

It should be noted that a well known lemma of Grün leads to nontrivial $\mathfrak{N}$ and hence to nontrivial elements of $\mathfrak{P}$. For, by this lemma, the mappings of the type $x \longrightarrow (x, u)$ for each fixed $u \in G$, $u \notin Z$ are in $\mathfrak{N}$ for groups of class 2.

Let G/Q have exponent n, so that G/Z has exponent $t \mid n$. By [1, Lemma,

p. 370], the mutual commutator group $(G, G) = Q$ has an exponent $k \mid t$. If t is odd, then $k \mid v(t)$, and $(xy)^t = x^t y^t$, whence $x \longrightarrow x^t$ is a central endomorphism of G. If t is even, then $x \longrightarrow x^{2t}$ is a central endomorphism. Since $x^n \in Q$, and since k is the exponent of Q, we have $x^{kn} = e$ for every $x \in G$. Now t is the exponent of G/Z, so that t must generate the ideal of exponents of central power endomorphisms of G in case t is odd. The central power endomorphisms are then all

$$x \longrightarrow x^{jt} \qquad (j = 0, 1, 2, \cdots (kn/t) - 1).$$

If kn is not the exponent of G but only an integral multiple thereof, then the number of distinct central power endomorphisms will be reduced (in proportion) to a submultiple of kn/t.

If t is even, then the generator t' of the ideal of exponents of central power endomorphisms of G must have the property $t \mid t' \mid 2t$. Hence $t' = t$ or $t' = 2t$. If $t' = t$ then the kn/t mappings $x \longrightarrow x^{jt}$ include all the central power endomorphisms (with possible repetitions). In fact, if k is odd, then $k \mid t/2$, and $t' = t$. If $t = t'$, then $k \mid v(t)$. It follows readily that $k \equiv 0 \mod 2^r$ implies $t \equiv 0 \mod 2^{r+1}$. Thus $k \equiv 0 \mod 2^r$ and $t \not\equiv 0 \mod 2^{r+1}$ imply $t' = 2t$. Whenever $t' = 2t$, there are, at most, $kn/2t$ central power endomorphisms of G. Since, in any event, a submultiple of kn/t or of $kn/2t$ is a submultiple of n, we have proved the following.

THEOREM 10. *Let G be a group of class 2 for which G/Q has exponent n. Then the number of central power endomorphisms of G divides n.*

The above is a generalization of the following: Let G be an abelian group with exponent n. Then there are precisely n power endomorphisms of G; for, $x^{n+m} = x^m$.

COROLLARY. *Let G be a non-abelian group of class 2 for which G/Q is an elementary* p*-group* [2] *for an odd prime* p. *Let G have at least one nontrivial element of order $\neq$* p. *Then G has precisely* p *central power endomorphisms. If* p $= 2$, *then G has only the trivial central power endomorphism.*

Proof. Since G is non-abelian we have $k \neq 1$, and $k \mid n = p$ implies $k = p$, so that $k \mid t \mid n$ leads to $t = p$. Likewise, $kn = p^2$. The exponent of G is not p, since there exists $y \in G$ with $y^p \neq e$. Hence the exponent of G must be p^2. If p is odd, then there are precisely $kn/t = p$ central power endomorphisms. The set of these endomorphisms is generated by the endomorphism $x \longrightarrow x^p$ under endomorphism composition. If $p = 2$ then $x \longrightarrow x^2$ is not an endomorphism; for, if it were, $(xy)^2 = x^2y^2$ would imply $yx = xy$, whence G would be abelian. Since $x^4 = e$, G has only the one trivial central power endomorphism, $x \longrightarrow x^4 = e$.

In a non-abelian group of class 2, as in the Corollary above, we can find an element of $\mathfrak{K}$ for which the corresponding right principal ideal does not have a unity. Let $\eta(x) = x^p$ so that $n(\eta) = p$. Since $k = p$ we have $\eta \in \mathfrak{K}$. If (η) had an identity, then there would exist mappings $\alpha_i \in \mathfrak{P}$, $i = 1, 2, \cdots, m$, with

$$p \sum n(\alpha_i) \equiv 1 \bmod p^2,$$

by the proof of Theorem 5, item (3), and the fact that p^2 is the exponent of $G \supset \eta(G)$. But the congruence $p\,\xi \equiv 1 \bmod p^2$ has no solution ξ.

References

1. F. Haimo, *Groups with a certain condition on conjugates*, Canadian J. Math., **4** (1952), 369-372.

2. H. Zassenhaus, *Gruppentheorie*, Leipzig and Berlin, 1937.

WASHINGTON UNIVERSITY
SAINT LOUIS, MISSOURI

ON GENERALIZED SUBHARMONIC FUNCTIONS

L. K. JACKSON

1. Introduction. In a previous paper [1], the notion of subharmonic functions was generalized in a manner corresponding to Beckenbach's [2] generalization of convex functions. This generalization was accomplished by replacing the dominating family of harmonic functions by a more general family of functions. In [1] the discussion was restricted to continuous subfunctions.

In the present paper we shall give some further properties of the dominating functions and extend the definition of subfunctions to permit upper semicontinuous subfunctions. We shall then show that results of J. W. Green [3] on approximately subharmonic functions extend to our subfunctions.

2. $\{F\}$-functions and sub-$\{F\}$ functions. Let D be a given plane domain and let $\{\gamma\}$ be a given family of contours bounding subdomains Γ of D such that $\overline{\Gamma} = \gamma + \Gamma \subset D$ where $\overline{\Gamma}$ indicates the closure of Γ. We assume that $\{\gamma\}$ contains all circles of radii less than a fixed number which lie, together with their interiors, in D. We shall use the Greek letter κ to represent a circle of $\{\gamma\}$ and K its interior. We shall use single small Roman letters to represent points in the plane. Let there be given a family of functions $\{F(x)\}$ which we shall call $\{F\}$-functions satisfying the following postulates.

POSTULATE 1. For any $\gamma \in \{\gamma\}$ and any continuous boundary value function $h(x)$ on γ, there is a unique $F(x; h; \gamma) \in \{F(x)\}$ such that

(a) $F(x; h; \gamma) = h(x)$ on γ,

(b) $F(x; h; \gamma)$ is continuous in $\overline{\Gamma}$.

POSTULATE 2. If $h_1(x)$ and $h_2(x)$ are continuous on γ and if $h_1(x) - h_2(x) \leq M$ on γ, $M \geq 0$, then

$$F(x; h_1; \gamma) - F(x; h_2; \gamma) \leq M$$

Received August 1, 1953. This research was supported in part by a Summer Research Fellowship granted by the Research Council of the University of Nebraska.

Pacific J. Math. 5 (1955), 215-228

in Γ; further, if the strict inequality holds at a point of γ, then the strict inequality holds throughout Γ.

POSTULATE 3. For any $\kappa \in \{\gamma\}$ and for any collection $\{h_\nu(x)\}$ of functions $h_\nu(x)$ which are continuous and uniformly bounded on κ, the functions $F(x; h_\nu; \kappa)$ are equicontinuous in K.

DEFINITION 1. The function $s(x)$ is defined to be sub-$\{F\}$ in D provided

(a) $s(x)$ is bounded on every closed subset of D,

(b) $s(x)$ is upper semicontinuous in D,

(c) $s(x) \leq F(x)$ on γ implies $s(x) \leq F(x)$ in Γ.

DEFINITION 2. The function $S(x)$ is defined to be super-$\{F\}$ in D provided

(a) $S(x)$ is bounded on every closed subset of D,

(b) $S(x)$ is lower semicontinuous in D,

(c) $S(x) \geq F(x)$ on γ implies $S(x) \geq F(x)$ in Γ.

Let Ω be a bounded connected open set comprised together with its boundary ω in D and let $g(x)$ be a bounded function defined on ω.

DEFINITION 3. The function $\phi(x)$ is an under-function (relative to $g(x)$) if $\phi(x)$ is continuous in $\overline{\Omega}$, sub-$\{F\}$ in Ω, and $\phi(x) \leq g(x)$ on ω.

DEFINITION 4. The function $\psi(x)$ is an over-function (relative to $g(x)$) if $\psi(x)$ is continuous in $\overline{\Omega}$, super-$\{F\}$ in Ω, and $\psi(x) \geq g(x)$ on ω.

POSTULATE 4. If Ω is any bounded connected open set comprised together with its boundary ω in D and $g(x)$ any bounded function defined on ω, then the associated families of over-functions and under-functions are both non-null.

POSTULATE 5. For any circle $\kappa \in \{\gamma\}$ and any real number M, there exist continuous functions $h_1(x)$ and $h_2(x)$, defined on κ, such that

$$F(x; h_1; \kappa) \geq M, \quad F(x; h_2; \kappa) \leq M \quad \text{in } \overline{K}.$$

POSTULATE 6. For any circle $\kappa \in \{\gamma\}$, if the functions $h_n(x)$ $(n = 0, 1, 2, \cdots)$, defined on κ, are continuous and uniformly bounded on κ, and

$$\lim_{n \to \infty} h_n(x) = h_0(x)$$

for all but at most a finite number of points of κ, then

$$\lim_{n \to \infty} F(x; h_n; \kappa) = F(x; h_0; \kappa)$$

at all points of K.

POSTULATE 7. For any circle $\kappa \in \{\gamma\}$, if the functions $h_n(x)$ $(n = 1, 2, \cdots)$, defined on κ, are continuous and uniformly bounded on κ and equicontinuous at a point $x_0 \in \kappa$, then the functions $F(x; h_n; \kappa)$ $(n = 1, 2, \cdots)$, defined in $\overline{K}$, are equicontinuous at x_0.

Our definition of sub-$\{F\}$ functions differs from the definition of subharmonic functions in that we have restricted our subfunctions to be bounded on closed subsets of D. This seems to be necessary since we do not have a Harnack theorem of the type that is available in the theory of harmonic functions.

3. Some theorems concerning the $\{F\}$-functions.

THEOREM 1. *If $\kappa \in \{\gamma\}$, N is any real number, and $x_0 \in K$, then there exists a continuous function $h(x)$ defined on κ such that $F(x_0; h; \kappa) = N$.*

Proof. By Postulate 5 there exist continuous functions $h_1(x)$ and $h_2(x)$ defined on κ such that

$$F(x; h_1; \kappa) > N, \quad F(x; h_2; \kappa) < N \text{ on } \overline{K}.$$

We define on κ

$$h_\lambda(x) = \lambda h_1(x) + (1 - \lambda) h_2(x), \qquad 0 < \lambda < 1.$$

Then for $0 < \lambda < 1$ we have

$$F(x_0; h_2; \kappa) < F(x_0; h_\lambda; \kappa) < F(x_0; h_1; \kappa).$$

Now set

$$\lambda_1 = \text{g.l.b. } [\lambda \mid F(x_0; h_\lambda; \kappa) \geq N]$$

and

$$\lambda_2 = \text{l.u.b. } [\lambda \mid F(x_0; h_\lambda; \kappa) \leq N].$$

Using Postulate 2 we see that $\lambda_1 = \lambda_2$ and

$$F(x_0; h_{\lambda_1}; \kappa) = N.$$

This result shows that Postulate 6 of [1] is actually a consequence of Postulates 1, 2, and 5 and may be omitted.

THEOREM 2. *If $f(x)$ is continuous in D and $x_0 \in D$, then given any $\epsilon > 0$, there exists $\kappa \in \{\gamma\}$ with center at x_0 and radius arbitrarily small such that*

$$|F(x; f; \kappa) - f(x)| < \epsilon \quad in\ \bar{K}.$$

Proof. If κ_1 is any circle of $\{\gamma\}$ with center at x_0, then by Theorem 1 there exists a continuous function $h(x)$ defined on κ_1 such that

$$F(x_0; h; \kappa_1) = f(x_0).$$

By continuity there exists a smaller concentric circle κ such that

$$|F(x; h; \kappa_1) - f(x)| < \epsilon/2 \quad \text{in } \bar{K}.$$

Let

$$h_1(x) = \max\,[F(x; h; \kappa_1),\ f(x)] \text{ on } \kappa,$$

$$h_2(x) = \min\,[F(x; h; \kappa_1),\ f(x)] \text{ on } \kappa.$$

Then in $\bar{K}$

$$F(x; h_2; \kappa) \le F(x; h; \kappa_1) \le F(x; h_1; \kappa),$$

$$F(x; h_2; \kappa) \le F(x; f; \kappa) \le F(x; h_1; \kappa),$$

$$F(x; h_1; \kappa) - F(x; h_2; \kappa) < \epsilon/2.$$

Therefore in $\bar{K}$ we have

$$|F(x; f; \kappa) - f(x)| \le |F(x; f; \kappa) - F(x; h; \kappa_1)| + |F(x; h; \kappa_1) - f(x)| < \epsilon.$$

THEOREM 3. *If $f(x)$ is bounded and upper semicontinuous on γ, then there exists a function $F(x; f; \gamma)$ such that*

(1) *$F(x; f; \gamma)$ is upper semicontinuous in $\bar{\Gamma}$ and continuous in Γ,*

(2) *$F(x; f; \gamma)$ is an $\{F\}$-function in Γ,*

(3) $F(x; f; \gamma) = f(x)$ *on* γ.

Proof. Let $F(x; f; \gamma)$ be the infimum in $\overline{\Gamma}$ of all over-functions in $\overline{\Gamma}$ with respect to the boundary function $f(x)$ on γ. Clearly $F(x; f; \gamma)$ is upper semicontinuous in $\overline{\Gamma}$. Theorems 11 and 12 of [**1**] show that $F(x; f; \gamma)$ is continuous and an $\{F\}$-function in Γ. Let $f_n(x)$ $(n = 1, 2, \cdots)$ be a monotone decreasing sequence of continuous functions converging to $f(x)$ on γ. Then $F(x; f_n; \gamma)$ is an over-function for each n and therefore $F(x; f; \gamma) = f(x)$ on γ.

Heretofore we have used the notation $F(x; h; \gamma)$ only for functions continuous in $\overline{\Gamma}$ but henceforth we shall use the same notation when $h(x)$ is bounded and upper semicontinuous on γ and $F(x; h; \gamma)$ is defined as in Theorem 3. No confusion should arise since, for $h(x)$ continuous on γ, the $F(x; h; \gamma)$ as defined by Theorem 3 is the unique $F(x; h; \gamma)$ of Postulate 1. Hence, if $s(x)$ is sub-$\{F\}$ in D and $\gamma \in \{\gamma\}$, there exists an $\{F\}$-function $F(x; s; \gamma)$ such that

$$s(x) = F(x; s; \gamma) \quad \text{on } \gamma$$

and

$$s(x) \leq F(x; s; \gamma) \quad \text{in } \Gamma.$$

THEOREM 4. *If* $h_1(x)$ *and* $h_2(x)$ *are bounded and upper semicontinuous on* γ *and* $h_1(x) - h_2(x) \leq M$ *on* γ, $M \geq 0$, *then*

$$F(x; h_1; \gamma) - F(x; h_2; \gamma) \leq M \quad on \ \Gamma.$$

Proof. Let $x_0 \in \Gamma$ and suppose that

$$F(x_0; h_1; \gamma) - F(x_0; h_2; \gamma) = M + \delta, \qquad \delta > 0.$$

By Postulate 4 and Theorem 3 there exists an over-function $\psi(x)$ with respect to $h_2(x)$ such that

$$0 \leq \psi(x_0) - F(x_0; h_2; \gamma) < \delta.$$

Then $F(x; \psi; \gamma)$ is also an over-function with respect to $h_2(x)$ and

$$0 \leq F(x_0; \psi; \gamma) - F(x_0; h_2; \gamma) < \delta.$$

Furthermore $F(x; \psi + M, \gamma)$ is an over-function with respect to $h_1(x)$; hence, by the preceding inequality and Postulate 2 we have

$$F(x_0;h_1;\gamma) - F(x_0;h_2;\gamma) \le F(x_0;\psi+M;\gamma) - F(x_0;h_2;\gamma)$$

$$\le F(x_0;\psi+M;\gamma) - F(x_0;\psi;\gamma) + F(x_0;\psi;\gamma) - F(x_0;h_2;\gamma) < M + \delta.$$

This is a contradiction and the theorem is proved.

THEOREM 5. *If the functions* $\{h_\nu(x)\}$ *are upper semicontinuous and uniformly bounded on* κ, *then the functions* $F(x;h_\nu;\kappa)$ *are equicontinuous in* K; *further the function*

$$u(x) \equiv \sup_\nu F(x;h_\nu;\kappa)$$

is continuous and sub-$\{F\}$ *in* K, *and*

$$v(x) \equiv \inf_\nu F(x;h_\nu;\kappa)$$

is continuous and super-$\{F\}$ *in* K.

The proof follows immediately from Postulate 3 and Theorem 4, and Lemma 1 and Theorem 11 of [1].

4. Some properties of sub-$\{F\}$ functions.

THEOREM 6. *A necessary and sufficient condition for the function* $s(x)$, *which is upper semicontinuous in* D *and bounded on every closed subset of* D, *to be sub-*$\{F\}$ *in* D *is that corresponding to each* $x_0 \in D$ *there exists a sequence of circles* κ_n *with centers at* x_0 *and radii* $r_n(x_0) \longrightarrow 0$ *such that*

$$s(x_0) \le F(x_0;s;\kappa_n)$$

for each n.

THEOREM 7. *If* $s_1(x), \cdots, s_n(x)$ *are sub-*$\{F\}$ *in* D, *then*

$$s(x) = \max[s_1(x), \cdots, s_n(x)]$$

is sub-$\{F\}$ *in* D.

THEOREM 8. *If* $s(x)$ *is sub-*$\{F\}$ *in* D *and* $\gamma \in \{\gamma\}$, *then*

$$s(x;\gamma) \equiv \begin{cases} s(x) & \text{for } x \in D - \overline{\Gamma} \\ F(x;s;\gamma) & \text{for } x \in \overline{\Gamma} \end{cases}$$

is sub-$\{F\}$ *in* D.

The proofs of these theorems parallel those given for continuous sub-$\{F\}$ functions in [1] and will be omitted.

THEOREM 9. *If* $s(x)$ *is sub-*$\{F\}$ *in* D, *then* $s(x)-M$, $M \geq 0$, *is sub-*$\{F\}$ *in* D.

Proof. Since $s(x)$ is upper semicontinuous in D and bounded on every closed subset of D, $s(x)-M$ has the same properties. Now let $x_0 \in D$ and $\kappa \in \{\gamma\}$ have its center at x_0. Then by Theorem 4

$$s(x_0) \leq F(x_0; s; \kappa) \leq M + F(x_0; s-M; \kappa),$$

hence,

$$s(x_0) - M \leq F(x_0; s-M; \kappa)$$

and by Theorem 6 $s(x)-M$ is sub-$\{F\}$ in D.

5. A Harnack theorem for the $\{F\}$-functions.

THEOREM 10. *If the decreasing sequence of sub-*$\{F\}$ *functions* $\{s_n(x)\}$ *is uniformly bounded on each closed subset of* D, *then*

$$\lim_{n\to\infty} s_n(x) = s(x)$$

is sub-$\{F\}$ *in* D.

Proof. Clearly $s(x)$ is upper semicontinuous and bounded on every closed subset of D; hence, to show that $s(x)$ is sub-$\{F\}$ in D it will be sufficient to show that it satisfies the Littlewood criterion of Theorem 6.

Let $x_0 \in D$ and let $\kappa \in \{\gamma\}$ have its center at x_0. By Theorem 4 we have

$$F(x; s_{n+1}; \kappa) \leq F(x; s_n; \kappa),$$

and

$$F(x; s; \kappa) \leq F(x; s_n; \kappa) \text{ in } \overline{K}$$

for each n. Since $s_n(x)$ is sub-$\{F\}$ in D for each n and the sequence $\{s_n(x)\}$ is decreasing, it follows that

$$s(x_0) \leq s_n(x_0) \leq F(x_0; s_n; \kappa).$$

Therefore,

$$s(x_0) \leq \lim_{n \to \infty} F(x_0; s_n; \kappa),$$

and we conclude the proof by showing that

$$\lim_{n \to \infty} F(x_0; s_n; \kappa) = F(x_0; s; \kappa).$$

Since

$$\lim_{n \to \infty} F(x_0; s_n; \kappa) \geq F(x_0; s; \kappa)$$

assume

$$\lim_{n \to \infty} F(x_0; s_n; \kappa) = F(x_0; s; \kappa) + \delta, \quad \delta > 0.$$

There exists an over-function $\psi(x)$ with respect to the boundary function $s(x)$ on κ such that

$$F(x_0; s; \kappa) \leq \psi(x_0) < F(x_0; s; \kappa) + \delta/2.$$

Since $\psi(x)$ is super-$\{F\}$ in K we have

$$F(x_0; s; \kappa) \leq F(x_0; \psi; \kappa) < F(x_0; s; \kappa) + \delta/2.$$

An application of Postulate 2 then gives

$$F(x_0; s; \kappa) < F(x_0; \psi + \delta/4; \kappa) < F(x_0; s; \kappa) + 3\delta/4.$$

Since $\psi(x) + \delta/4$ is continuous on κ and

$$s(x) < \psi(x) + \delta/4$$

on κ, it follows that for N sufficiently large we have

$$s_n(x) < \psi(x) + \delta/4$$

on κ for $n \geq N$. Then for $n \geq N$

$$F(x_0; s_n; \kappa) \leq F(x_0; \psi + \delta/4; \kappa) < F(x_0; s; \kappa) + 3\delta/4.$$

This is a contradiction, hence

$$\lim_{n \to \infty} F(x_0; s_n; \kappa) = F(x_0; s; \kappa),$$

and

$$s(x_0) \leq F(x_0; s; \kappa).$$

Consequently by Theorem 6 $s(x)$ is sub-$\{F\}$ in D.

As an immediate consequence of Theorem 5 and Theorem 10 we have the following Harnack type theorem for the $\{F\}$-functions:

THEOREM 11. *The limit of a uniformly bounded monotone decreasing sequence of $\{F\}$-functions is an $\{F\}$-function.*

Furthermore it is clear that if $f(x)$ is bounded and upper semicontinuous on κ, if $F(x; f; \kappa)$ is the $\{F\}$-function defined in Theorem 3, and if $\{f_n(x)\}$ $(n = 1, 2, \cdots)$ is any monotone decreasing sequence of continuous functions converging to $f(x)$ on κ, then

$$\lim_{n \to \infty} F(x; f_n; \kappa) = F(x; f; \kappa) \text{ in } \overline{K}.$$

6. Approximately sub-$\{F\}$ functions. D. H. Hyers and S. M. Ulam [**4**] have introduced the notion of approximately convex functions. A function $f(x)$ is said to be approximately convex provided

$$f(\lambda x + (1-\lambda)y) \leq \epsilon + \lambda f(x) + (1-\lambda) f(y),$$

for $0 \leq \lambda \leq 1$ and for a fixed $\epsilon > 0$. For $\epsilon = 0$ the definition is that of a convex function.

The notion of a subharmonic function may be thought of as an extension to two dimensions of the notion of a convex function in one dimension. Using this idea, Green [**3**] has defined an approximately subharmonic function as follows: a function $f(x)$ defined in a domain D is ϵ-subharmonic provided (a) it is upper semicontinuous, and (b) if $h(x)$ is a harmonic function in a domain D' interior to D, which is continuous on the boundary of D' and dominates $f(x)$ there on, then in D'

$$f(x) \leq \epsilon + h(x).$$

In an analogous way we define an approximately sub-$\{F\}$ function as follows:

DEFINITION 5. A function $g(x)$ is said to be ϵ-sub-$\{F\}$ in D provided

(a) $g(x)$ is bounded on every closed subset of D,

(b) $g(x)$ is upper semicontinuous in D,

(c) $g(x) \leq F(x)$ on the boundary of a subdomain D' of D implies $g(x) \leq \epsilon + F(x)$ in D'.

With this definition the theorem of Green for approximately subharmonic functions extends to approximately sub-$\{F\}$ functions.

THEOREM 12. *If $g(x)$ is ϵ-sub-$\{F\}$ in D, there exists a function $u(x)$,* sub-$\{F\}$ *in D, such that $u(x) \leq g(x) \leq \epsilon + u(x)$ in D.*

The proof of the theorem depends on the existence of a maximal sub-$\{F\}$ minorant for a continuous function. We shall give the proof of Theorem 12 after we have considered this question.

7. Maximal sub-$\{F\}$ minorants. The theorem given in this section has the same statement as the corresponding theorem for subharmonic functions and the proof is similar to the one given in [**3**].

THEOREM 13. *If $f(x)$ is continuous in a domain $R \subset D$ and has a sub-$\{F\}$ minorant in R, then it has a maximal sub-$\{F\}$ minorant $u(x)$. The function $u(x)$ is continuous in R and is an $\{F\}$-function where it is less than $f(x)$.*

Proof. Let S be the family of all functions sub-$\{F\}$ in R and dominated by $f(x)$. By hypothesis S is non-null. For $x \in R$ we define

$$u(x) = \sup_{s \in S} s(x).$$

We wish to show first that $u(x)$ is lower semicontinuous in R. Let $x_0 \in R$ and $\eta > 0$, then there exists $s(x) \in S$ such that

$$u(x_0) - \eta \leq s(x_0) \leq u(x_0) \leq f(x_0).$$

Then $s(x_0) - \eta < f(x_0) - \eta/2$ and, by the continuity of $f(x)$ and the upper semicontinuity of $s(x)$, there exists a circle κ_1 with center at x_0 such that

$$s(x) - \eta < f(x_0) - \eta/2 < f(x) \text{ in } \overline{K}_1.$$

By Theorem 2 we may choose a circle κ with center at x_0, with radius less than that of κ_1, and such that

$$F(x; f(x_0) - \eta/2; \kappa) < f(x) \text{ in } \overline{K}.$$

Then by Theorem 4 we have also

$$F(x; s - \eta; \kappa) < f(x) \text{ in } \overline{K}.$$

Now define

$$s^*(x; \kappa) = \begin{cases} s(x) - \eta & \text{for } x \text{ in } R - \overline{K} \\ F(x; s - \eta; \kappa) & \text{in } \overline{K}. \end{cases}$$

It follows from Theorem 8 and 9 that $s^*(x; \kappa) \in S$. Therefore,

$$\liminf_{x \to x_0} u(x) \geq \liminf_{x \to x_0} s^*(x; \kappa) = F(x_0; s - \eta; \kappa) \geq s(x_0) - \eta;$$

hence,

$$\liminf_{x \to x_0} u(x) \geq u(x_0) - 2\eta.$$

Since η is arbitrary this implies that

$$\liminf_{x \to x_0} u(x) \geq u(x_0)$$

and $u(x)$ is lower semicontinuous in R.

Now we designate by A the set of all $x \in R$ such that $u(x) \doteq f(x)$ and let $B = R \cap \text{comp } \overline{A}$. B is an open set and for the moment we assume that it is not void. Let $x \in B$, then since B is open, there exists a circle κ with center at x such that $\overline{K} \subset B$. Suppose that there exists an $s(x) \in S$ such that

$$F(x; s; \kappa) \geq f(x)$$

at some points of K. Then by Postulate 5 and Theorem 4 there would exist an $\eta \geq 0$ such that

$$F(x; s - \eta; \kappa) \leq f(x)$$

in $\overline{K}$ with the equality holding at some points of K. If for this $s(x)$, κ, and η we again define

$$s^*(x; \kappa) = \begin{cases} s(x) - \eta & \text{for } x \text{ in } R - \overline{K} \\ F(x; s - \eta; \kappa) & \text{in } \overline{K}. \end{cases}$$

then $s^*(x;\kappa) \in S$ and $s^*(s;\kappa) = f(x)$ at some points of K. Thus we would have $u(x) = f(x)$ at some points of K and this would contradict $\overline{K} \subset B$. Hence

$$F(x;s;\kappa) < f(x) \quad \text{in } \overline{K}$$

for every $s(x) \in S$. Hence for every $s(x) \in S$ let

$$s(x;\kappa) = \begin{cases} s(x) & \text{in } R - \overline{K} \\ F(x;s;\kappa) & \text{in } \overline{K}. \end{cases}$$

Then $s(x;\kappa) \in S$ and $s(x) \leq s(x;\kappa)$ in R, therefore,

$$u(x) = \sup_{s \in S} s(x) = \sup_{s \in S} s(x;\kappa).$$

We conclude by Theorem 5 that $u(x)$ is continuous and sub-$\{F\}$ in K and hence in B.

Now we define

$$u^*(x) = \begin{cases} f(x) & \text{for } x \in \overline{A} \\ u(x) & \text{for } x \in B. \end{cases}$$

Then clearly $u^*(x)$ is upper semicontinuous in R and $u(x) \leq u^*(x) \leq f(x)$ in R.

Next we show that $u^*(x)$ is sub-$\{F\}$ in R. We have already observed that $u(x)$ is sub-$\{F\}$ in B, hence $u^*(x)$ is sub-$\{F\}$ in B. Let $x_0 \in \overline{A}$ and let $\kappa \in \{\gamma\}$ have its center at x_0 and $\overline{K} \subset R$, then

$$s(x) \leq F(x;s;\kappa) \leq F(x;u^*;\kappa) \quad \text{in } \overline{K}$$

for every $s(x) \in S$. For $x \in A \cap K$

$$\sup_{s \in S} s(x) = f(x) \leq F(x;u^*;\kappa);$$

it follows by the continuity of $f(x)$ and $F(x;u^*;\kappa)$ in K that

$$u^*(x_0) = f(x_0) \leq F(x_0;u^*;\kappa).$$

By Theorem 6, $u^*(x)$ is sub-$\{F\}$ in R, therefore $u^*(x) \in S$ and $u^*(x) \leq u(x)$. This taken with the previous inequality shows that $u^*(x) \equiv u(x)$ and, being both upper semicontinuous and lower semicontinuous, $u(x)$ is continuous in R.

By using Theorem 2 one can easily see that $u(x)$ is an $\{F\}$-function in B.

In Theorem 16 of [1] it is shown that, if Ω is a bounded open set contained with its boundary ω in D and if for each $x \in \omega$ there is a circle κ such that $\overline{\Omega} \cap \overline{K} = x$, then Ω is a Dirichlet region for the $\{F\}$-functions. For such a region Ω we may construct barrier sub-$\{F\}$ functions as was done in Theorem 16 of [1] and thus obtain equality of $u(x)$ and $f(x)$ on the boundary ω of Ω. This would imply the continuity of $u(x)$ in $\overline{\Omega}$.

8. **Proof of Theorem 12.** Let $\Omega \subset D$ be a bounded Dirichlet region for the $\{F\}$-functions of the type mentioned in the previous paragraph. By Definition 5 the ϵ-sub-$\{F\}$ function $g(x)$ is bounded on $\overline{\Omega}$ and hence by Postulate 4 and Theorem 9 has a sub-$\{F\}$ minorant in Ω. Since $g(x)$ is upper semicontinuous in $\overline{\Omega}$, there is a decreasing sequence of continuous functions $\{f_n(x)\}$ converging to $g(x)$ in $\overline{\Omega}$. By Theorem 13 $f_n(x)$ has a maximal sub-$\{F\}$ minorant $u_n(x)$ in Ω. The sequence $\{u_n(x)\}$ is uniformly bounded and decreasing in $\overline{\Omega}$ and therefore by Theorem 10 converges to $u(x)$ which is sub-$\{F\}$ in Ω. Clearly $u(x)$ is the maximal sub-$\{F\}$ minorant of $g(x)$ in Ω.

For each $x \in \overline{\Omega}$, either $u_n(x) = f_n(x)$ or $u_n(x) < f_n(x)$. If $u_n(x) < f_n(x)$, let Ω' be the component containing x of the open subset of $\overline{\Omega}$ in which $u_n(x) < f_n(x)$. Then $u_n(x)$ is an $\{F\}$-function in Ω' and agrees with $f_n(x)$ on the boundary of Ω'. Hence $g(x) \leq u_n(x)$ on the boundary of Ω' and therefore $g(x) \leq \epsilon + u_n(x)$ in Ω'. Thus we have

$$g(x) \leq u_n(x) + \epsilon$$

in Ω and letting n become infinite

$$u(x) \leq g(x) \leq u(x) + \epsilon$$

in Ω.

This proves Theorem 12 for the above class of Dirichlet domains in D. Now consider a nested sequence of such bounded Dirichlet domains $\{\Omega_K\}$ exhausting D. Let $\{u_K(x)\}$ be the associated sequence of maximal sub-$\{F\}$ minorants of $g(x)$. This sequence is obviously decreasing and, since for $\kappa \geq N$

$$g(x) - \epsilon \leq u_K(x) \leq g(x)$$

on Ω_N, is uniformly bounded on each closed subset of D. Another application of Theorem 10 shows that the sequence $\{u_K(x)\}$ converges to a function which is sub-$\{F\}$ in D, is clearly the maximal sub-$\{F\}$ minorant of $g(x)$ in D, and satisfies the inequality of Theorem 12 in D. Theorem 12 is proved.

In a subsequent publication it will be shown that the solutions of certain elliptic partial differential equations satisfy the postulates of the $\{F\}$-functions.

REFERENCES

1. E. F. Beckenbach and L. K. Jackson, *Subfunctions of several variables*, Pacific J. Math. **3** (1953), 291-313.

2. E. F. Beckenbach, *Generalized convex functions*, Bull. Amer. Math. Soc. **43** (1937), 363-371.

3. J. W. Green, *Approximately subharmonic functions*, Proc. Amer. Math. Soc. **3** (1952), 829-833.

4. D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952), 821-828.

UNIVERSITY OF NEBRASKA

ON THE RENEWAL EQUATION

SAMUEL KARLIN

Introduction. Recently Chung and Pollard [3] considered the following problem: Let $X_i (i=1, 2, \cdots)$ denote independent identically distributed random variables having the distribution function $F(x)$ with mean

$$m = \int x \, dF(x) \qquad (0 < m)$$

and let

$$S_n = \sum_{k=1}^{n} X_k.$$

Define

$$u(\zeta) = \sum_{n=1}^{\infty} \Pr\{\zeta < S_n \leq \zeta + h\},$$

if X is not a lattice random variable then they show that $\lim_{\zeta \to \infty} u(\zeta) = h/m$. The above authors imposed the restriction that the distribution F possess an absolutely continuous part. T. E. Harris by written communication and independently D. Blackwell [2] show that this restriction was unnecessary. Of course, as can be verified directly, $u(\zeta)$ satisfies a renewal type equation

$$(*) \qquad u(\zeta) - \int_{-\infty}^{\infty} u(\zeta - t) \, dF(t) = \int_{\zeta}^{\zeta + h} dF(t) = g(\zeta).$$

The existence of solutions and the limiting behavior for bounded solutions of such renewal type equations which involve positive and negative values of t has not been treated.

Feller [**10**] and later Täcklind [**12**] have developed many Tauberian results for the cases where all the functions $u(\zeta)$, $F(\zeta)$ and $g(\zeta)$ considered are

Received July 22, 1953.
Pacific J. Math. 5 (1955), 229-257

zero for ζ negative. This reduces (*) to the classical renewal equation where Laplace transform methods can be exploited. Doob [6] and Blackwell [1] discussed the same type of renewal equation from the point of view of probability theory and appealed to the ergodic theory of Markoff chains.

In this work we shall show that most problems of the general renewal equation can be reduced to an application of the general Wiener theorem and the properties of slowly oscillating functions. Our methods are thoroughly analytic and apply to situations which do not necessarily correspond to probability models. Moreover, a complete analysis of (*) shall be given concerning existence and asymptotic behavior of solutions with results describing rates of convergence under suitable assumptions. Erdös, Pollard and Feller [7] and later Feller [9] in the study of recurrent events did apply the Wiener theorem to some discrete analogues of (*) and these examples have served to suggest to this writer this general unified approach. Most of the results of Täcklind who dealt with the classical renewal equation use deep methods of Fourier analysis. These results are illuminated and in many instances subsumed by our methods. Finally, in the course of revising this paper it has come to our attention that W. L. Smith very recently [11] independently has discussed the classical one-sided renewal equation from the point of view of Wiener's general Tauberian theorem. His treatment and this investigation supplement each other in many respects. We employ the basic properties of slowly oscillating functions while Smith uses Pitt's extension of the Wiener theorem.

Some fundamental differences appear between the general renewal equation (*) and the type of renewal equation studied in [8], [13] and [11]. For example, solutions to (*) need not exist and when they do exist there are, in general, infinitely many bounded and unbounded solutions. This complicates the analysis of the asymptotic behavior of solutions of (*). In fact, solutions $u(\zeta)$ can be found for certain examples which oscillate infinitely as $|\zeta| \longrightarrow \infty$. Even when we restrict ourselves to bounded solutions to (*), the abundance of such solutions necessitates a careful analysis which does not occur in the handling of the one-sided renewal equation. (See the beginning of § 3.)

In § 2 we present a complete treatment of the discrete renewal equation

$$(**) \qquad u_n - \sum_{k=\infty}^{\infty} a_{n-k} u_k = b_n .$$

In this case necessary and sufficient conditions are given to insure the existence of bounded solutions to (**). Asymptotic limit theorems for bounded solutions to (**) are obtained and appropriate conditions are indicated which yield results about the rates of convergence of such solutions as $n \longrightarrow \infty$.

The general equation (*) is treated in § 3 where the existence and limit theorems for bounded solutions of (*) are given. The Plancherel and Hausdorff-Young theorems are used to establish the existence of bounded solutions to (*). Limit theorems are analyzed and rates of convergence are obtained. Some applications are made to the classical renewal equation.

The relationship of Wiener's Tauberian theorem to ideal theory motivated the content of § 4. This last section indicates a new avenue of approach to the meaning of the renewal equation.

Finally, I wish to express my gratitude to James L. McGregor for his helpful discussions in the preparation of this manuscript.

2. Discrete renewal equation. This section is devoted to a complete analysis of the renewal equation

$$u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k = b_n . \tag{1}$$

The convolution of two sequences $\{x_n\}$ and $\{y_n\}$ is denoted by

$$x * y = \left\{ \sum_{k=-\infty}^{\infty} x_{n-k}\, y_k \right\}$$

This product operation is well defined whenever, for example, at least one of the sequences is an absolutely convergent series while the other sequence is uniformly bounded. Equation (1) can thus be written as

$$u - a * u = b , \tag{2}$$

We suppose hereafter, that the sequences $\{a_n\}$ and $\{b_n\}$ have the property that $a_n \geq 0$, $\sum a_n = 1$ and $\sum |b_n| < \infty$ and that u_n represents a solution of (2). In general, there exist many solutions of (2) which complicates the study of the asymptotic behavior of solutions $\{u_n\}$ of the renewal equations. We first investigate the general problem of the existence of solutions of (1). To this end, we introduce the linear operation T which can be applied to any sequence $\{c_n\}$ which forms an absolutely convergent series. Precisely, let

$$T\{c_n\} = \{(Tc)_n\}$$

where

$$(Tc)_n = \begin{cases} \sum_{i=n+1}^{\infty} c_i & n \geq 0 \\ -\sum_{-\infty}^{n} c_i & n < 0. \end{cases}$$

Let

$$\sigma_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

and define the linear functional $\phi_0(c) = \sum_{n=-\infty}^{\infty} c_n$. We note the following relation for future use

$$\phi_0(c)\,\sigma - \sigma * c = Tc. \tag{3}$$

The operation T can be repeated provided that the resulting sequence $\{Tc\}$ is an absolutely convergent series. If, for example, $\sum_{n=-\infty}^{\infty} |n^k c_n| < \infty$, then $T^k c$ is well defined. Moreover, we observe for later reference that if

$$\sum |n^k c_n| < \infty,$$

then

$$\lim_{|n| \to \infty} |n^k (Tc)_n| = 0.$$

We now impose two very fundamental assumptions.

ASSUMPTION A. *The greatest common divisor of the indices n where $a_n > 0$ is* 1.

ASSUMPTION B. *The series $\sum |na_n| < \infty$ and $\sum_{n=-\infty}^{\infty} na_n = m \neq 0$.* (For definiteness we take $m > 0$.)

Many of the following results can be extended to the case where the g.c.d. of the indices n where $a_n > 0$ is $d > 1$. We leave this task to the interested reader. However, Assumption B is indispensible for the validity of many of the subsequent results. Some results can be extended by suitable modifications to $m = \infty$.

An important tool to be used frequently is the following lemma.

LEMMA 1. *If Assumptions A and B are satisfied, then there exists a sequence* $\{r_n\}$ *with*

$$\sum |r_n| < \infty \text{ and } r * Ta = \delta$$

where $\delta = \{\delta_n^0\}$. (The sequence δ is the identity element with respect to the $*$ multiplication.)

Proof. For the sequence $\{a_n\}$ let $a(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta}$. The relation (3) implies for $0 < \theta < 2\pi$

$$\frac{1 - a(\theta)}{1 - e^{i\theta}} = \sum_{-\infty}^{\infty} (Ta)_n e^{in\theta} = Ta(\theta).$$

Assumption A implies that $Ta(\theta) \neq 0$ for $\theta \neq 0$ and $|\theta| < 2\pi$. Assumption B yields that $Ta(0) \neq 0$ and the fact that $\sum_{-\infty}^{\infty} |(Ta)| < \infty$. By virtue of Wiener's Tauberian theorem

$$\frac{1}{(Ta)(\theta)} = \sum_{-\infty}^{\infty} r_n e^{in\theta}$$

defines an absolutely convergent Fourier series. The conclusion of Lemma 1 is now evident from this last relation.

We now proceed to discuss the existence of solutions to (1) or (2).

THEOREM 1. *If Assumptions A and B are satisfied, then there exists a bounded solution of* (1). *Any two bounded solutions of* (1) *differ by a fixed constant.*

Proof. We seek a bounded solution of

$$u - a * u = b. \tag{2}$$

Multiplying formally (2) by σ and using (3) we obtain $u * (Ta) = \sigma * b$ and hence by Lemma 1

$$u = r * \sigma * b.$$

The sequence $r * \sigma * b$ is a bounded sequence and it is easily verified provides a solution for relation (2). To establish the second half of the theorem it is sufficient to show that

$$u - a * u = 0 \tag{4}$$

possess only constant bounded solutions. Let $(\Delta u) = (u_n - u_{n+1})$. It follows readily that (4) implies $(Ta) * \Delta u = 0$. Multiplication by r yields that $(\Delta u) = 0$ and hence the result sought for.

We now show that in general, nonbounded solutions of (4) and therefore of (1), can be found. This is illustrated by the following example. Although the example is special, the technique is general and the reader can easily construct many other such examples.

Let $a_1 = 1/2$, $a_2 = 1/2$ and $a_i = 0$ for $i \neq 1, 2$. Equation (4) becomes

$$u_n = \frac{1}{2} u_{n-1} + \frac{1}{2} u_{n-2} \qquad \text{all } n.$$

We can prescribe u_0 and u_1 arbitrarily and therefore we obtain a two-dimensional set of solutions. However, by virtue of Theorem 1 only a one-dimensional set of bounded solutions exists. Hence, unbounded solutions also exist. The unbounded solution oscillates infinitely as $n \longrightarrow -\infty$.

It is worth showing that a converse to Theorem 1 can be obtained.

THEOREM 2. *If*

$$\sum_{k=-\infty}^{\infty} b_k > 0, \ \Sigma |na_n| < \infty \ \text{ but } \sum_{-\infty}^{\infty} na_n = 0,$$

then there exists no bounded solutions to (2) *provided that*

$$\sum_{n=-\infty}^{\infty} b_n > 0, \ a_1 > 0 \ \text{ and } \ a_{-1} > 0.$$

Proof. Suppose to the contrary that $\{u_n\}$ is a bounded solution to (2). Let $\lambda = \overline{\lim}_{n \to \infty} u_n$ then there exists a subsequence $u_{n_i} \longrightarrow \lambda$. By virtue of a standard probability argument (see [10, p. 260]), it follows that $\lim_{n_i \to \infty} u_{n_i - k} = \lambda$ for each integer k. A similar subsequence m_i can be found such that $\lim_{m_i \to -\infty} u_{m_i - k} = u$, where $u = \overline{\lim}_{m \to -\infty} u_m$. As in Theorem 1 we obtain that $(Ta) * \Delta u = b$. Summing from m_i to n_i gives

$$\sum_{k=-\infty}^{\infty} (u_{n_i-k} - u_{m_i-k})(Ta)_k = \sum_{k=m_i}^{n_i} b_k.$$

Allowing $n_i \longrightarrow \infty$ and $m_i \longrightarrow -\infty$, it follows readily since $\sum |(Ta)_k| < \infty$ that

$$0 < \sum_{-\infty}^{\infty} b_k = (\lambda - u) \sum_k (Ta)_k = (\lambda - u) \sum_{n=-\infty}^{\infty} na_n = 0$$

a contradiction.

REMARK. Theorem 2 can be established using the weaker Assumption A in place of the hypotheses that $a_1 > 0$ and $a_{-1} > 0$. We omit the details

Having discussed the question of existence we now turn to investigate the asymptotic properties of bounded solutions to (2). Throughout the remainder of this section we assume that Assumptions A and B are satisfied. A useful result which we state here for later purposes is the following well known Abelian theorem.

LEMMA 2. *If* $\{r_n\}$ *is such that* $\sum_{n=-\infty}^{\infty} |r_n| < \infty$, $\{\omega_n\}$ *is bounded and* $\lim_{n\to\infty} \omega_n = 0$, *then*

$$\lim_{n\to\infty} \sum_{-\infty}^{\infty} \omega_{n-k}\, r_k = 0.$$

The following theorem is a simple Tauberian result for solutions of (2).

THEOREM 3. *If* u_n *is a bounded solution to* (1), *then* $\lim_{n\to\infty} u_n$ *and* $\lim_{u_n\to-\infty} u_n$ *exist.*

Proof. By Theorem 1, it is sufficient to prove the result for the special solution

$$u = r * b * \sigma.$$

For this special solution, we have

$$u_n = \sum_{k=-\infty}^{n} (r * b)_k.$$

Hence the limit exists, in fact,

$$\lim_{n \to -\infty} u_n = 0, \qquad \lim_{n \to \infty} u_n = \sum_{-\infty}^{\infty} (r * b)_k = \frac{\phi_0(b)}{m}$$

Q.E.D. To obtain more precise results let

$$v = u - \frac{\phi_0(b)}{m}$$

where $u = r * b * \sigma$ is the unique bounded solution for which $u_n \longrightarrow 0$ as $n \longrightarrow \infty$. From the proof of Theorem 3 it is clear that $v_n \longrightarrow 0$ as $|n| \longrightarrow \infty$. It is easy to show that

$$(5) \qquad Ta * v = -Tb + \frac{\phi_0(b)}{m} T^2 a$$

or

$$v = -r * \left[Tb - \frac{\phi_0(b)}{m} T^2 a \right].$$

Hence if we assume in addition to A and B that

$$\sum |(T^2 a)_n| < \infty \quad \text{and} \quad \sum |(Tb)_n| < \infty,$$

then it follows that $\sum |v_n| < \infty$. These new assumptions enable us to obtain further results about the rate of convergence of v_n and hence of u_n. To this end, we define the operation S on any sequence $\{t_n\}$, $St = \{nt_n\}$. The hypothesis

$$\sum |(T^2 a)_n| < \infty \quad \text{and} \quad \sum |(Tb)_n| < \infty$$

or the equivalent assumptions

$$\sum n^2 a_n < \infty \quad \text{and} \quad \sum |nb_n| < \infty,$$

respectively imply easily that STa defines an absolutely convergent series and $ST^2 a$ constitutes a bounded sequence which tends to zero as $|n| \longrightarrow \infty$. A direct calculation using (5) gives that

$$(6) \qquad S(Ta * v) - ST(a) * v = -STb + \frac{\phi_0(b)}{m} ST^2 a - STa * v.$$

The left side of (6) is identical componentwise with $Ta * Sv$. Multiply (6) by Ta, then with the aid of (5), we obtain

$$(7) \qquad Ta * Ta * Sv = -Ta * STb + \frac{\phi_0(b)}{m}\{Ta * ST^2a\}$$
$$-STa * \left[-Tb + \frac{\phi_0(b)}{m} T^2a\right].$$

On account of the hypothesis and Lemma 2, we find that the right side is a bounded sequence which tends to zero at $\pm\infty$. Employing Lemma 1, we conclude that Sv is bounded and $\lim_{n\to\infty} |nv_n| = 0$.

Although it might appear as if the relation (7) is rather fortuitous, a simple method to deduce the formula begins with the Fourier series relation

$$(8) \qquad Ta(\theta)v(\theta) = -Tb(\theta) + \frac{f_0(b)}{m} T^2a(\theta)$$

which is well defined and is an alternative way to express (5). Differentiation of (8) with multiplication by $Ta(\theta)$ and use of (8) gives a formal representation of (7). The preceding argument was in essence a justification of this differentiation process.

The preceding analysis extends with the aid of an induction argument. The details are omitted and we sum up the results in the following theorem.

THEOREM 4. *Let* $a_n \geq 0$, $\sum_{-\infty}^{\infty} a_n = 1$, *satisfying Assumptions A and B. Let* u_n *represent the unique bounded solution of* (2) *for which* $\lim_{n\to\infty} u_n = 0$ (see Theorem 3). *If*

$$\sum_{n=-\infty}^{\infty} |n^k b_n| < \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |n^{k+1} a_n| < \infty,$$

then

$$\sum_{n\geq 0} \left| n^{k-1}\left[u_n - \frac{\phi_0(b)}{m}\right]\right| + \sum_{n<0} |n^{k-1} u_n| < \infty$$

and

$$\lim_{n \to \infty} n^k \left[u_n - \frac{\phi_0(b)}{m} \right] = \lim_{n \to -\infty} n^k u_n = 0.$$

A first classical application of Theorem 1 can be obtained from the theory of Markov chains. Let E represent a recurrent state from an irreducible non-periodic chain. Let u_n represent the probability of starting from E and returning to E in n steps. Let a_n denote the probability that the first return occurs at the nth step $(n > 0)$. Put $u_0 = 1$, $u_{-n} = 0$ and $a_{-n} = 0$ (for $n \geq 0$), then

$$u_n - \sum_{k=0}^{n} a_{n-k} u_k = b_n$$

where $b_n = 0$ for $n \neq 0$ and $b_0 = 1$. Since E describes a recurrent state, $\sum a_i = 1$ and trivially $m = \sum_{i=0}^{\infty} i a_i > 0$. As an immediate consequence of Theorem 4, we infer that if

$$\sum_{0}^{\infty} n^{k+1} a_n < \infty, \text{ then } \sum_{n=1}^{\infty} n^{k-1} \left| u_n - \frac{1}{m} \right| < \infty \text{ and } \lim_{n \to \infty} n^k \left[u_n - \frac{1}{m} \right] = 0.$$

A second application deals with the following problem treated by K. L. Chung and J. Wolfowitz [4]. We generalize their result in obtaining stronger rates of convergence by assuming further conditions on the moments. Let X denote a random variable which assumes only integral values and define for all n

$$a_n = \Pr\{X = n\} \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Let $X_i (i = 1, 2, \cdots)$ denote an infinite sequence of independent events with the same distribution as X. Define

$$S_j = \sum_{i=1}^{j} X_i \text{ and } u_n = \sum_{j=1}^{\infty} \Pr\{S_j = n\} = \text{Expected number of sums where } S_j = n.$$

Let $m = E(x)$ be the expectation of X. Suppose the greatest common divisor of the indices n such that $a_n > 0$ is 1 and $0 < m < \infty$. Chung and Wolfowitz in [4] allow $m = \infty$, but the present method does not apply. The restriction on the greatest common divisor is not essential but the requirement that $m \neq 0$ is very crucial and in fact in the contrary case $u_n = \infty$ as is shown by Chung and Fuchs [5]. We obtain that if $\sum_{n=-\infty}^{\infty} |n^{k+1} a_n| < \infty$, then

$$\sum_{n\geq 0}\left|n^{k-1}\left[un-\frac{1}{m}\right]\right|+\sum_{n<0}|n^{k-1}a_n| \text{ and } \lim_{n\to\infty} n^k\left[u_n-\frac{1}{m}\right]=\lim_{n\to-\infty} n^k u_n=0.$$

Indeed, it follows from the definition of u_n that

$$u_n-\sum_{k=-\infty}^{\infty} a_{n-k}u_k=a_n.$$

It can be seen that the sequence u_n is uniformly bounded and $\lim_{n\to-\infty} u_n=0$ (see [5]). The conditions of Theorem 4 are met and the conclusion follows from the results of that theorem. Summing up, we have

COROLLARY. *Let X_i be identically distributed independent lattice random variables with distribution given by* $\Pr\{x=n\}=a_n$ *and* $u_n=\sum_{j=1}^{n}\Pr\{s_j=n\}$ *where* $s_j=\sum_{i=1}^{n}x_i$. *If the expected value of* $x=m>0$ *and* g.c.d. $n=1$, *then*

$$\sum_{n=-\infty}^{\infty}|n^{k+1}a_n|<\infty$$

implies

$$\sum_{n\geq 0}\left|n^{k-1}\left[u_n-\frac{1}{m}\right]\right|+\sum_{n<0}|n^{k-1}u_n|<\infty$$

while

$$\lim_{n\to\infty} n^k\left[u_n-\frac{1}{m}\right]=\lim_{n\to-\infty} n^k u_n=0.$$

3. Continuous renewal equation. This section is devoted to an analysis of the existence and asymptotic properties of solutions for each ξ of the relation

$$u(\xi)-\int_{-\infty}^{\infty}u(\xi-t)\,df(t)=g(\xi). \tag{9}$$

The convolution of two functions $x(t)$ and $y(t)$ is defined as

$$x*y=\int_{-\infty}^{\infty}x(t-\xi)y(\xi)\,d\xi$$

which exists if, say, x is integrable and y is bounded. We shall be concerned only with bounded solutions of (9). It is assumed that

$$df(t) \geq 0, \int_{-\infty}^{\infty} df = 1 \text{ and } \int |g| < \infty.$$

The following hypotheses are now imposed:

ASSUMPTION A$'$. The distribution f is a non-lattice distribution, that is, the points of increase of f do not concentrate at the multiples of a fixed value.

ASSUMPTION B$'$. $\int_{-\infty}^{\infty} |t|\,df(t) < \infty$ and $\int_{-\infty}^{\infty} t\,df(t) = m \neq 0$ (say $m > 0$)

These two assumptions constitute the continuous analogues of Assumptions A and B and hereafter we suppose these assumptions satisfied.

We introduce the operation T defined for any function of bounded total variation $h(t)$. Let

$$\sigma(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \text{ and } \phi_0(h) = \int_{-\infty}^{\infty} dh(t)$$

and

$$Th = \phi_0(h)\,\sigma - \sigma * h$$

or

$$(Th)(t) = \begin{cases} \int_t^{\infty} dh(t) & t \geq 0 \\ -\int_{-\infty}^{t} dh(t) & t < 0 \end{cases}$$

T is also defined for integrable functions $k(t)$ as follows:

$$Tk = Tk^*(t) \text{ where } k^*(t) = \int_{-\infty}^{t} k(\xi)\,d\xi.$$

Let

$$u_n(\xi) = n \int_{\xi}^{\xi + 1/n} u(t)\,dt$$

with g_n defined similarly. Equation (9) can be converted to

$$u_n(\xi) - \int_{-\infty}^{\infty} u_n(\xi - t)\, df(t) = g_n(\xi)$$

Since the derivative of u_n is essentially uniformly bounded, we obtain on integration by parts that

$$(10) \qquad Tf * u_n' = \int_{-\infty}^{\infty} u_n'(\xi - t)\, Tf(t)\, dt = g_n(\xi).$$

The finiteness of $\int_{-\infty}^{\infty} |t|\, df(t)$ is equivalent to the integrability of $Tf(t)$ and thus (10) is well defined. Integrating (10) from a to ξ gives

$$\int_{-\infty}^{\infty} [u_n(\xi - t) - u_n(a - t)]\, Tf(t)\, dt = \int_a^{\xi} g_n(t)\, dt.$$

Letting n go to ∞, we have almost everywhere

$$(11) \qquad \int_{-\infty}^{\infty} [u(\xi - t) - u(a - t)]\, Tf(t)\, dt = \int_a^{\xi} g(t)\, dt.$$

Since both the right and left hand sides of (11) are continuous this identity holds everywhere in ξ and a. Allowing $a \longrightarrow \infty$, we find from (11) that

$$\lim_{a \to -\infty} \int_{-\infty}^{\infty} u(a - t)\, Tf(t)\, dt = c.$$

Adding to any solution of (9) a constant produces a new solution u of (9). Therefore, we may suppose that $c = 0$. Thus,

$$(12) \qquad \int_{-\infty}^{\infty} u(\xi - t)\, Tf(t)\, dt = \int_{-\infty}^{\xi} g(t)\, dt.$$

We define for this u satisfying (12),

$$(12\text{-a}) \qquad v(\xi) = u(\xi) - \frac{\phi_0(g)}{m}\, \sigma(\xi),$$

It follows directly that

$$(13) \qquad v * Tf = -Tg + \frac{\phi_0(g)}{m}\, T^2 f.$$

We now present a series of lemmas needed in the sequel.

LEMMA 3. *Under the assumptions stated above, the Fourier transform*

$$(Tf)^*(\theta) = \int_{-\infty}^{\infty} e^{it\theta}\, Tf(t)\,dt$$

vanishes nowhere.

The proof is similar to that of Theorem 1, and is based on the identity

$$i\theta(Tf)^*(\theta) = -1 + \int_{-\infty}^{\infty} e^{it\theta}\, df(t).$$

LEMMA 4. *Any two bounded solutions of* (9) *differ by a constant.*

Proof. It is enough to show that the only bounded solution of

$$u(\xi) - \int_{-\infty}^{\infty} u(\xi - t)\, df(t) = 0$$

are constants. Using a reasoning similar to that of deducing (12), we get

$$\int_{-\infty}^{\infty} u(\xi - t)\, Tf(t)\,dt = c. \tag{13a}$$

By subtracting an appropriate constant from (13a), we have for $v = u - c'$ that $v * Tf = 0$. Lemma 3 and the general Wiener's Tauberian theorem yields that $v * r = 0$ for every integrable $r(t)$. It follows readily from this last fact that $v = 0$ almost everywhere or $u = c'$ a.e.

LEMMA 5. *If $r(t)$ is integrable and $w(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$, then*

$$\lim_{|\xi| \to \infty} \int_{-\infty}^{\infty} w(\xi - t)\, r(t)\,dt = 0.$$

This last Abelian theorem is well known and straightforward.

LEMMA 6. *If v is bounded and satisfies* (13), *then*

$$\lim_{|\xi| \to \infty} \int_{-\infty}^{\infty} r(\xi - t)\, v(t) = 0$$

for any integrable function r.

Proof. The hypothesis and the character of the operation T imply that

$$\lim_{|\xi|\to\infty} Tg(\xi) = \lim_{|\xi|\to\infty} T^2 f(\xi) = 0.$$

Consequently,

$$\lim_{|\xi|\to\infty} \int_{-\infty}^{\infty} v(\xi - t)\, Tf(t)\,dt = 0.$$

An application of the general Wiener Tauberian theorem leads to the conclusion of the lemma.

COROLLARY. *Under the assumptions of Lemma 6 we have*

$$\lim_{|x|\to\infty} \int_{x}^{x+\Delta} v(t)\,dt = 0.$$

Indeed, choose

$$r(\xi) = \begin{cases} \dfrac{1}{\Delta} & \text{for } 0 \le \xi \le \Delta \\ \\ 0 & \text{elsewhere} \end{cases}.$$

We now establish the fundamental asymptotic limit theorem for bounded solutions of (9). The basic Tauberian theorem used is the Wiener theorem coupled with the properties of slowly oscillating sequences.

THEOREM 5. *If u is a bounded solution of* (9), *and f has a decomposition $f = f_1 + f_2$ where f_1 is absolutely continuous and the total variation of $f_2 = \lambda < 1$, and* $\lim_{|\xi|\to\infty} g(\xi) = 0$, *then* $\lim_{t\to\infty} u(t)$ *and* $\lim_{t\to-\infty} u(t)$ *both exist. If* $\lim_{t\to-\infty} u(t) = 0$, *then* $\lim_{t\to\infty} u(t) = \phi(g)/m$.

Proof. It is enough to assume that v defined by (12-a) from u satisfies (13). This can be achieved if necessary by altering u by a fixed constant (see the discussion preceding Lemma 3). As before, we find that

$$\lim_{|\xi|\to\infty} \int_{-\infty}^{\infty} v(\xi - t)\, Tf(t)\,dt = 0 \tag{14}$$

It will now be shown that $v(t)$ is slowly oscillating as $|t| \longrightarrow \infty$ ($v(t)$ is said to be slowly oscillating (s.o.) if

$$\lim_{\substack{\xi\to\infty\\ \eta\to 0}} |v(\xi+\eta)-v(\xi)| = 0.$$

A similar definition applies at $t=-\infty$. The general Wiener theorem and the s.o. character of $v(t)$ implies the stronger conclusion over Lemma 6 that

$$\lim_{|t|\to\infty} v(t) = 0$$

which is our assertion. It thus remains to establish that $v(\xi)$ is s.o. and we confine our argument to the situation where $\xi\longrightarrow\infty$. A similar analysis applies at $-\infty$. Remembering that the convolution of an absolutely continuous distribution and any other distribution remains absolutely continuous, we obtain upon n fold iteration of (9) that

$$u(\xi) = \int_{-\infty}^{\infty} u(\xi-t)\,dk_1(t) + \int_{-\infty}^{\infty} u(\xi-t)\,dk_2(t) + \int_{-\infty}^{\infty} g(\xi-t)\,dk_3(t)$$

$$= I_1(\xi) + I_2(\xi) + I_3(\xi)$$

where k_1 is absolutely continuous, k_2 is the n fold convolution of f_2 with itself and $k_3(t)$ is of bounded total variation. Since $g(\xi)\longrightarrow 0$ as $|\xi|\longrightarrow\infty$ by Lemma 5

$$\lim_{|\xi|\to\infty} I_3(\xi) = 0.$$

Next, we observe that $|I_2(\xi)| \le \lambda^n c$ where c is the upper bound of u. Finally,

$$|I_1(\xi+\eta) - I_1(\xi)| \le \int |u(\xi-t)|\,|k_1'(t+\xi) - k_1'(t)|\,dt$$

$$\le c\int |k_1'(t+\eta) - k_1'(t)|\,dt \longrightarrow 0 \qquad \text{as } \eta\longrightarrow 0$$

by virtue of a well-known theorem of Lebesgue. Combining these estimates, we get that

$$\overline{\lim_{\substack{\xi\to\infty\\ \eta\to 0}}} |v(\xi+\eta)-v(\xi)| \le 2c\,\lambda^n$$

which by proper choice of n can be made as small as one pleases. This completes the proof.

REMARK. Theorem 5 is valid if we merely assume that some iterate of f has an absolutely continuous part.

COROLLARY. *Under the conditions of Lemma* 6, *if* $v(t)$ *is uniformly continuous for* $t > 0$ *and* $t < 0$, *then*

$$\lim_{|t|\to\infty} v(t) = 0.$$

Proof. The function $v(t)$ is s.o. from which the conclusion follows as in Theorem 5.

In many examples, we deal with a solution u of (9) which is by physical considerations bounded while in other cases boundedness for certain solutions has to be verified. Our next object is to give sufficient conditions so that we can establish the existence of bounded solutions of (9). From now on we assume that f is absolutely continuous and let

$$f(\xi) = \int_{-\infty}^{\xi} a(t)\,dt.$$

LEMMA 7. *If* u *is a solution of* (9) *which belongs to* L^p $(p \geq 1)$, $a \in L^1$ *also belongs to* $L^{p'}$ *where* p' *is the conjugate exponent to* p *and* g *is bounded, then* u *is bounded.*

Proof. Applying Hölder's inequality to (9) and an obvious change of variable, we obtain

$$|u(\xi)| \leq \left(\int_{-\infty}^{\infty} |u|^p\right)^{1/p} \left(\int_{-\infty}^{\infty} |a|^{p'}\right)^{1/p'} + c.$$

THEOREM 6. *If* $a(t)$ *belongs to* L^1 *and* L^2, $g(t)$ *is bounded,*

$$\int |x|^2\, a(x)\,dx < \infty$$

and

$$\int |x|\, g(x)\,dx < \infty,$$

then a bounded solution $u(t)$ *of* (9) *exists.*

Proof. The Fourier transform of any integrable $h(t)$ is denoted by $h^*(\theta)$. Consider the expression

$$w^*(\theta) = \frac{g^*(\theta) - (Ta)^*(\theta)\,[\phi(g)/m]}{1 - a^*(\theta)} \tag{14}$$

It will now be shown that (14) is the Fourier transform of a function in L^2. To this end, by the Riemann Lebesgue lemma $a^*(\theta) \longrightarrow 0$ as $|\theta| \longrightarrow \infty$ and $|a^*(\theta)| < 1$ for $\theta \neq 0$ with $a^*(\theta)$ continuous. Since the first moment of a exists, Ta is bounded and in L^1. Hence, Ta belongs to L^2 and $Ta^*(\theta) \in L^2$. A similar argument shows that $g^*(\theta) \in L^2$. Thus for $|\theta| \geq \alpha > 0$, $w^*(\theta)$ is in L^2 for any fixed positive constant α. But,

$$w^*(\theta) = \frac{\{g^*(\theta) - (Ta)^*(\theta)[\phi(g)/m]\}/i\theta}{(1 - a^*(\theta))/i\theta} \tag{15}$$

$$= \frac{-(Tg)^*(\theta) + [\phi(g)/m]\,(T^2a)^*(\theta)}{(Ta)^*\,(\theta)}$$

The existence of the second moment of a implies that $(T^2a)^*\,(\theta)$ is continuous. Analogously, $(Tg)^*\;(\theta)$ is continuous by virtue of $\int |x|\, g(x) < \infty$. Since $Ta^*(0) = m > 0$, we find that $w^*(\theta)$ is continuous in the neighborhood of zero and hence $w^*(\theta)$ is in L^2. Consequently, $w(t)$ in L^2 exists which is the Fourier transform of $w^*(\theta)$ and conversely. Moreover, (14) yields

$$w(\xi) - \int_{-\infty}^{\infty} w(t)\,a(\xi - t)\,dt = g(\xi) - Ta(\xi)\,\frac{\phi_0(g)}{m}$$

for almost all ξ.

As a convolution of two elements of L^2 the integral on the right is bounded and continuous. Hence the right side is bounded and remains unaltered, if w is changed on a set of measure zero.

As in Lemma 7, it follows that $w(\xi)$ is bounded. Putting

$$u(\xi) = w(\xi) + \frac{\phi_0(g)}{m}\,\sigma(\xi),$$

we find that u is bounded and satisfies (9).

REMARK. Theorem 6 can be established under the weaker conditions that

$$\int |x|^{1+\alpha} a(x) < \infty$$

and

$$\int |\xi|^{\alpha} g(x) < \infty$$

for some $\alpha > 0$. These assumptions are sufficient to imply the boundedness of $w^*(\theta)$ in the neighborhood of zero.

Other sufficient criteria can be obtained for the existence of bounded solutions to (9) involving use of the Hausdorff-Young inequalities in place of the Plancherel theorem.

THEOREM 7. *If $a(t)$ belongs to L^1 and L^p $(1 < p < 2)$,*

$$\int |t|^{1+\alpha} a(t)\,dt < \infty$$

with $\alpha > 0$,

$$\int |g^*(\theta)|^p\, d\theta < \infty$$

and g is bounded, then a bounded solution of (9) *exists.*

It is worth noting that the solutions u guaranteed by Theorems 6 and 7 have the property on account of Theorem 5 that $\lim_{|t|\to\infty} u(t)$ exist.

Our next objective is to find conditions which imply conclusions about the rate of convergence of $w(\xi)$ of Theorem 6 as $|\xi| \longrightarrow \infty$ and thus of $u(\xi)$. To this end, we differentiate (14) and 15), we get

$$w^{*\prime}(\theta) = \frac{a^{*\prime}(\theta)w^*(\theta) + g^{*\prime}(\theta) - Ta^{*\prime}(\theta)\,[\phi(g)/m]}{1 - a^*(\theta)} \tag{16}$$

$$w^{*\prime}(\theta) = \frac{Ta^{*\prime}(\theta)w^*(\theta) - Tg^{*\prime}(\theta) + [\phi(g)/m]\,(T^2a)^{*\prime}(\theta)}{Ta^*(\theta)} \tag{17}$$

Relation (17) can be derived from (16) by dividing numerator and denominator by $i\theta$ similar to the method of obtaining (15) from (14).

Under the assumptions that

$$\int |t|^3 a(t)dt < \infty$$

and

$$\int t^2 g(t)dt < \infty$$

with g bounded and monotone decreasing as $|t| \longrightarrow \infty$ we now show that $w^{*\prime}(\theta)$ belongs to L_2. Indeed, for $|\theta| \geq \alpha > 0$ we use (16) to estimate $w^{*\prime}(\theta)$ and we use (17) to analyze $w^{*\prime}(\theta)$ in the neighborhood of the origin.

For $\xi > 0$

$$\xi Ta(\xi) \leq \int_\xi^\infty ta(t)dt \leq c$$

and similarly $|\xi Ta(\xi)| \leq c$ for ξ negative. Also,

$$\int_{-\infty}^\infty t^2 Ta^2(t) \leq c\int_{-\infty}^\infty |t\, Ta(t)| \leq c' \int t^2\, a(t)dt < \infty$$

and

$$\int_{-\infty}^\infty t^2\, g^2(t) \leq c\int t^2 g(t) < \infty.$$

Since $g(t)$ is monotone decreasing as $|t| \longrightarrow \infty$, we obtain easily that $|tg(t)| \leq c$. As $a^{*\prime}(\theta)$ is the Fourier transform of $t\,a(t)$ in L^1 (except for a fixed constant factor) we know that $a^{*\prime}(\theta)$ is uniformly bounded. By Theorem 6, $w^*(\theta)$ is in L^2 and therefore $a^{*\prime}(\theta)\, w^*(\theta)$ is in L^2. $g^{*\prime}(\theta)$ is in L^2 by virtue of $t\,g(t) \in L^2$ and $Ta^{*\prime}(\theta)$ is in L^2 as a consequence of $t\, Ta(t)$ in L^2 which were established above. Since $|a^*(\theta)| < 1$ for $\theta \neq 0$ and tends to zero as $|\theta| \longrightarrow \infty$, we find, collecting all these cited facts, that $w^{*\prime}(\theta)$ is in L^2 for $|\theta| \geq \alpha > 0$. The assumptions of the existence of the third and second moments of a and g respectively yield as in the proof of Theorem 6 using (17) that $w^{*\prime}(\theta)$ is continuous at zero. Thus $w^{*\prime}(\theta)$ is square integrable throughout and as a result of standard Fourier analysis is the Fourier transform of $tw(t)$ in L^2. Relation (16) gives

$$(18) \qquad tw(t) - \int_{-\infty}^\infty a(t-\xi)\,\xi w(\xi)d\xi$$

$$= \int_{-\infty}^\infty w(t-\xi)\,\xi a(\xi) + tg(t) - \frac{\phi(g)}{m}\, tTa(t)$$

The fact that $tg(t)$ and $t(Ta)(t)$ are bounded imply by an argument completely analogous to the proof of Lemma 7 that $tw(t)$ is bounded. It follows as before that $tw(t)$ is s.o. (see Theorem 5). The relation (17) leads to

$$(19)\qquad \int_{-\infty}^{\infty} Ta(\xi - t)\, tw(t)\,dt$$

$$= \int_{-\infty}^{\infty} (\xi - t)\, Ta(\xi - t)\, w(t)\,dt - \xi\, Tg(\xi) + \frac{\phi(g)}{m}\, \xi\, T^2 a(\xi).$$

Since $w(t) \longrightarrow 0$, $\xi\, Tg(\xi) \longrightarrow 0$ and $\xi(T^2 a)(\xi) \longrightarrow 0$ as $|t| \longrightarrow \infty$, we obtain by Lemma 5 that the right side of (19) tends to zero as $|\xi| \longrightarrow \infty$. Combining the s.o. character of $tw(t)$, its boundedness and the Wiener Tauberian theorem leads to the conclusion that

$$\lim_{|t| \to \infty} tw(t) = 0.$$

Proceeding inductively we can obtain higher rates of convergence by imposing the requirement of the existence of higher moments using this same method. We sum up the discussion in the following theorem.

THEOREM 8. *Let*

$$\int_{-\infty}^{\infty} |t|^{n+2}\, a(t)\,dt < \infty$$

with a in L^1 *and* L^2. *Let* $g(t)$ *be bounded monotone decreasing for* $t \geq t_0 > 0$ *and nondecreasing for* $t \leq -t_0 < 0$ *with*

$$\int |t|^{n+1}\, g(t)\,dt < \infty,$$

then

$$\lim_{|t| \to \infty} t^n w(t) = 0$$

where

$$u = w + \frac{\phi_0(g)}{m}\, \sigma$$

is a solution of (9), *and* $w(t)$ *is the Fourier transform of* $w^*(\theta)$ (see (14)). (We recall that Lemma 4 shows that $u(t)$ as given above is the only bounded solution for which $u(t) \longrightarrow 0$ as $t \longrightarrow -\infty$.)

We now append some remarks about the classical renewal equation

$$u(x) = g(x) + \int_0^x u(x-\xi)\,df(\xi) \qquad x \geq 0. \tag{20}$$

The assumptions made are that

$$u \geq 0, \quad g \geq 0, \quad \int_0^\infty g(\xi)\,d\xi = b < \infty$$

and f is the distribution of a non-lattice random variable. The function $Tf(\xi)$ is introduced as before. If the first moment of f exists, then $Tf \in L^1$ and

$$m = \int_0^\infty x\,df(x) > 0.$$

Thus, we deduce as before that Tf possesses a Fourier transform which is never zero. Throughout the discussion of this case it is no longer necessary to assume any boundedness condition on $u(\xi)$, the nonnegativeness of u suffices to enable us to obtain all the results of Theorems 5-8.

To indicate the simplicity of our methods we now show how Wiener's Tauberian theorem can be used directly to establish a slight generalization of one of the fundamental results of Täcklind on the classical renewal equation. His procedure involves complicated estimates.

THEOREM 9. *Let* $\Phi(x)$ *denote a monotonic solution to the integral equation*

$$\Phi(x) = Q(x) + \int_0^x \Phi(x-y)\,df(y) \qquad x \geq 0 \tag{21}$$

where $\Phi(x)$ *is continuous and* $\Phi(0) = 0$, $Q(x)$ *is a distribution on* $(0, \infty)$ *with finite first moment and* f *is a non-lattice distribution continuous at zero with finite second moments, then*

$$\lim_{x \to \infty} \left[\Phi(x) - \frac{1}{m} x + \frac{\mu}{m} - \frac{a}{2m^2} \right] = 0$$

where

$$m = \int_0^\infty x\,df(x), \quad a = \int_0^\infty x^2\,df(x) \textit{ and } \mu = \int_0^\infty x\,dQ(x).$$

Proof. Define

$$\Psi(x) = \Phi(x) - \frac{1}{m}x,$$

then it follows from (21) that

$$\Psi(x) - \int_0^x \Psi(x-t)\,df = Q(x) - \frac{1}{m}\int_0^x Tf(\xi)\,d\xi. \tag{22}$$

Integrating (22) over the interval $(0, y)$ and then performing an integration by parts we obtain

$$\int_0^y Tf(y-t)\Phi(t)\,dt = -\int_0^y [1 - Q(\xi)]d + \frac{1}{m}\int_0^y T^2 f(\xi)\,d\xi. \tag{23}$$

By an elementary calculation as $y \longrightarrow \infty$ the limit of the right side tends to

$$\left(-\frac{\mu}{m} + \frac{\sigma^2}{2m^2}\right)\int_0^\infty Tf(t)\,dt.$$

We now collect the facts needed to employ the Wiener theorem. That the Fourier transform of Tf never vanishes has been shown previously. It is easy to show that $\Psi(t) = O(1)$ see [**12**]. Finally, we verify that $\Psi(t)$ is slowly decreasing (s.d.) that is,

$$\varliminf_{\substack{\xi \to \infty \\ \eta \to 0}} [\Psi(\xi + \eta) - \Psi(\xi)] \geq 0.$$

In fact,

$$\Psi(\xi + \eta) - \Psi(\xi) = \Phi(\xi + \eta) - \Phi(\xi) + \frac{\eta}{m} \geq \frac{\eta}{m}.$$

Thus,

$$\varliminf_{\substack{\xi \to \infty \\ \eta \to 0}} [\Psi(\xi + \eta) - \Psi(\xi)] \geq 0.$$

As Tf is nonnegative and $\Psi(\xi) \geq -C$ a sharp form of the Wiener theorem because of the (s.d.) character of Ψ implies that

$$\lim_{t\to\infty} \Psi(t) = -\frac{\mu}{m} + \frac{a}{2m^2}.$$

We continue with a brief examination of the example discussed in the introduction. Let X_i denote independent identically distributed non-lattice random variables with cumulative distribution f which has an absolutely continuous component. We assume the first moment exists and

$$\int x\,df = m > 0.$$

Put

$$s_j = \sum_{i=1}^{j} X_i \quad \text{and} \quad u(\xi) = \sum_{j=1}^{n} \Pr\{\xi \leq s_j \leq \xi + h\}$$

where h is a fixed positive number. The intuitive fact that $u(\xi)$ is bounded can be proved directly from probability considerations. We do not present the details. The function u is readily seen to satisfy the renewal equation (1).

$$u(x) - \int_{-\infty}^{\infty} u(x-\xi)\,df(\xi) = g(x) = \int_{x}^{x+h} df(\xi).$$

The hypothesis of the corollary to Theorem 5 can be shown to be satisfied by probability analysis and we obtain $\lim_{t\to\infty} u(t) = h/m$ and $\lim_{t\to-\infty} u(t) = 0$, the result obtained by Chung and Pollard by other methods [3]. We close this section by presenting some extensions of these results by imposing further conditions of the existence of higher moments of f to secure some results about the rate at which $u(t)$ converges.

THEOREM 10. *If X_i are independent identically distributed non-lattice random variables with density function $a(t)dt$ such that $a \in L_2$ and*

$$\int |t|^{n+2} a(t)\,dt < \infty$$

and

$$\int_{-\infty}^{\infty} ta(t)\,dt = m > 0, \quad u(\xi) = \sum_{j=1}^{\infty} \Pr\{\xi \leq s_j \leq \xi + h\}$$

where

$$s_j = \sum_{i=1}^{j} X_i ,$$

then

$$\lim_{t \to \infty} t^n \left| u(t) - \frac{h}{m} \right| = \lim_{t \to -\infty} t^n u(t) = 0.$$

Proof. This is an immediate consequence of Theorem 8.

4. Abstract renewal equation. The purpose of the subsequent analysis is to present an abstract approach to some of the fundamental ideas involved in the analysis of the renewal equation. Although some of the results are formal and simple, it is felt that this study sheds some light on the real nature of the renewal equation.

Let T denote a linear operator which can be viewed as a bounded operator from (m) into (m) or from (l) into (l). The spaces (m) and (l) designate the Banach spaces of bounded sequences and absolutely convergent series respectively. Suppose furthermore that the operator T is of norm one viewed in either space. Let $a_r \geq 0$, $\sum a_r = 1$ $(r = 0, \pm 1, \cdots)$ and we assume that the g.c.d. of the indices r for which $a_r > 0$ is 1. Suppose also that $\sum |n| \, a_n$ exists with $\sum n a_n = m \neq 0$. If $a_r = 0$ for $r < 0$, then automatically $\sum n a_n$ is not zero provided $a_1 \neq 1$. In this case we consider the operator $\sum_{r=0}^{\infty} a_r T^r$ where $T^0 = I$. This operator is linear and has norm bounded by 1 as $\| T^r \| \leq 1$ and $\sum a_r = 1$. If T^{-1} exists and is of norm 1, then we can deal with the general case where a_r is given not necessarily zero for both r positive and negative. We consider then the operator $\sum_{n=-\infty}^{\infty} a_n T^n$. As a generalization of the renewal equation, we set

$$(+) \qquad Su = \left[I - \sum_{r=-\infty}^{\infty} a_n T_n \right] u = v.$$

It is given that the operator S applied to u produces the element v. In many examples, u is a bounded sequence, that is, an element of (m) while v is an element in (l). Put,

$$r_n = \sum_{i=n+1}^{\infty} a_i \text{ for } n \geq 0 \text{ and } r_n = - \sum_{i=-\infty}^{n} a_i \text{ for } n < 0,$$

then $\sum |r_n| < \infty$ as $\sum |na_n| < \infty$. It is important to note on account of $\sum |r_n| < \infty$, the series

$$\sum_{n=-\infty}^{\infty} r_n T^n$$

defines a bounded linear operator which can be viewed acting either on (m) or (l) into itself. By a summation by parts, we obtain that

$$\left(\sum_{n=-\infty}^{\infty} r_n T_n \right) (I - T) u = v.$$

Since $\sum_{n=-\infty}^{\infty} r_n s^n$ with $|s| = 1$ has an absolute convergent reciprocal (Wiener's theorem is used here analogously to the analysis of section 1), we secure that $(\sum r_n T^n)^{-1}$ exists as a bounded operator over (m) and (l) and that

$$(I - T) u = (\sum r_n T^n)^{-1} v.$$

Since $v \in (l)$ we conclude that $(I - T) u \in (l)$ although u itself might only be an element of (m). This *represents the basic abstract conclusion* obtained from $(+)$. Further results are obtained by specializing T. A particular example is obtained by (m) = the set of all bounded sequences $u = \{u_n\}$ $n = 0, \pm 1, \pm 2, \cdots$, where T is the shift operator which moves each component one unit to the right. Whence, $(+)$ reduces to

$$(I - \sum a_n T^n) u = \left\{ u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k \right\}.$$

If all the hypothesis on a_n are met and $v_n \in (l)$, then the abstract theorem tells us that $(I - T) u \in (l)$ or

$$\sum_{n=-\infty}^{\infty} |u_n - u_{n+1}| < \infty.$$

This implies that both $\lim_{n\to\infty} u_n$ and $\lim_{n\to-\infty} u_n$ exist. Similar results are valid for the circumstance where T^{-1} does not exist. Then we deal only with the case where $a_n = 0$ for $n < 0$. Considering the same shift operator leads to

$$\left(I - \sum_{n=0}^{\infty} a_n T^n \right) u = u_n - \sum_{k=0}^{n} a_{n-k} u_k$$

and we deduce as above that $\lim_{n\to\infty} u_n$ exists.

We turn now to examine some continuous analogues of (+). Let $T(t)$ denote for $\infty \geq t \geq 0$ a strongly continuous semi-group of operators acting either on the space of bounded functions (M) or integrable functions (L) with $\|T(t)\| \leq 1$. Let A denote the infinitesimal generator of $T(t)$ and let $df(t)$ define a non-lattice distribution with finite first moment on $[0,\infty]$. If u belongs to (M) we consider

$$u(t) - \left[\int_0^\infty T(t)\, df(t)\right] u = v$$

where v belongs to (L). The linear operator

$$\int_0^\infty T(t) df(t)$$

is well defined either over (M) or (L) into itself. Put $r(t) = 1 - f(t)$, then $r \in L$ and the Fourier transform of r never vanishes. Since r is monotonic decreasing and in L it can be easily shown that

$$\left[\int_0^\infty r(t) T(t) dt\right] u$$

belongs to the domain of the infinitesimal generator A and

$$A \int_0^\infty r(t) T(t) dt\; u = v.$$

Formally, we also obtain upon commutating A and the integral operator

$$\left[\int_0^\infty r(t) T(t) dt\right] Au = v.$$

We note that if

$$\int_0^\infty r(t) T(t) dt$$

is multiplied by any other operator of the form

$$\int_0^\infty s(t) T(t) dt,$$

we obtain the operator

$$\int_0^\infty p(t)T(t)dt$$

where

$$p(t) = \int_0^t r(t-\xi)s(\xi)d\xi.$$

Since the Fourier transform of r does not vanish, then Wiener's theorem in a formal sense, furnishes an inverse to

$$\int_0^\infty r(t)T(t)dt$$

which takes $v \in L$ into L. Thus, Au belongs to L. Specializing $T(t)$ to the translation semi-group $T(t)\,u(x) = u(x-t)$, then $Au = du(x)/dx$ whenever the derivative exists and belongs to the proper space. The fact that $Au \in L$ yields $\int |du/dx|$ exists from which we infer that $\lim_{t\to\infty} u(t)$ exists. Thus we obtain the limit behavior of Theorem 5 for the one-sided case. The justification of these last formal considerations is very difficult and can only be carried through in certain cases as is shown in § 2. The full renewal equation is generalized by taking $T(t)$ a group and proceeding as above.

References

1. D. Blackwell, *A renewal theorem*, Duke Math. J., **15** (1948), 145-150.

2. ———, *Extension of a renewal theorem*, Pacific J. Math. **3** (1953), 315-332.

3. K. L. Chung and H. Pollard, *An extension of renewal theory*, Proc. Amer. Math. Soc., **3** (1952), 303-309.

4. K. L. Chung and J. Wolfowitz, *On a limit theorem in renewal theory*, Ann. of Math. **55-56** (1952), 1-6.

5. K. L. Chung and W. H. J. Fuchs, *On the distribution of values of sums of random variables*, Memoirs of the Amer. Math. Soc., No. 6.

6. J. L. Doob, *Renewal theory from the point of view of the theory of probability*, Trans. Amer. Math. Soc., **63** (1948), 422-438.

7. P. Erdös, H. Pollard and W. Feller, *A property of power series with positive coefficients*, Bull. Amer. Math. Soc., **55** (1949), 201-204.

8. W. Feller, *On the integral equation of renewal theory*, Ann. Math. Stat., **12** (1941), 243-267.

9. ———, *Fluctuation theory of recurrent events*, Trans. Amer. Math. Soc., **67** (1949), 98-119.

10. W. Feller, *An introduction to the theory of probability and its applications*, J. Wiley and Sons, 1950.

11. W. L. Smith, *Asymptotic renewal theorems*, Proc. of the Royal Soc. of Edinburgh, LXIV (1953-54), 9-48.

12. S. Täcklind, *Fourieranalytische Behandlung vom Erneurungsproblem*, Skandinavisk Aktuarietidskrift, 28 (1945), 68-105.

CALIFORNIA INSTITUTE OF TECHNOLOGY

SOME DETERMINANTS INVOLVING BERNOULLI AND EULER NUMBERS OF HIGHER ORDER

FRANK R. OLSON

1. Introduction. In this paper we evaluate certain determinants whose elements are the Bernoulli, Euler, and related numbers of higher order. In the notation of Nörlund [**1**, Chapter 6] these numbers may be defined as follows: the Bernoulli numbers of order n by

$$(1.1) \qquad \left(\frac{t}{e^t - 1}\right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} B_v^{(n)},$$

the related "D" numbers by

$$(1.2) \qquad \left(\frac{t}{\sin t}\right)^n \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} D_{2v}^{(n)} \qquad (D_{2v+1}^{(n)} = 0),$$

the Euler numbers of order n by

$$(1.3) \qquad (\sec t)^n = \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v}}{(2v)!} E_{2v}^{(n)} \qquad (E_{2v+1}^{(n)} = 0),$$

and the "C" numbers by

$$(1.4) \qquad \left(\frac{2}{e^t + 1}\right)^n = \sum_{v=0}^{\infty} \frac{t^v}{v!} \frac{C_v^{(n)}}{2^v}.$$

(By n we denote an arbitrary complex number. When $n = 1$, we omit the upper index in writing the numbers; for example, $B_v^{(1)} = B_v$.)

We evaluate determinants such as

$$|B_i^{(x_j)}| \qquad (i, j = 0, 1, \cdots, m)$$

for the Bernoulli numbers, and similar determinants for the other numbers. The

Received July 29, 1953.
Pacific J. Math. 5 (1955), 259-268

proofs of these results follow from the evaluation of a determinant of a more general nature; see (3.4), below. Finally, a number of applications are given.

2. Preliminaries and notation. The numbers $B_v^{(n)}$, $D_{2v}^{(n)}$, $E_{2v}^{(n)}$, and $C_v^{(n)}$ may be expressed as polynomials in n of degree v [**1**, Chapter 6]; in particular,

$$B_0^{(n)} = D_0^{(n)} = E_0^{(n)} = C_0^{(n)} = 1.$$

Although little is known about these polynomials, it will suffice for our purposes to give explicitly the values of the coefficients of n^v in each of the four cases.

Considering first the Bernoulli numbers, we use the recursion formula [**1**, p. 146]

$$B_v^{(n)} = -\frac{n}{v}\sum_{s=1}^{v} (-1)^s \binom{v}{s} B_s B_{v-s}^{(n)} . \tag{2.1}$$

Let

$$B_v^{(n)} = b_v n^v + b_{v-1} n^{v-1} + \cdots + b_0 ,$$

$$B_{v-1}^{(n)} = b'_{v-1} n^{v-1} + b'_{v-2} n^{v-2} + \cdots + b'_0 ,$$

and compare coefficients of n^v on both sides of (2.1). We find that

$$b_v = -\frac{1}{v}(-1)\binom{v}{1} B_1 b'_{v-1} .$$

But $B_1 = -1/2$ and therefore $b_v = -b'_{v-1}/2$. Since $B_0^{(n)} = 1$, the preceding leads us recursively to

$$B_v^{(n)} = \left(-\frac{1}{2}\right)^v n^v + b_{v-1} n^{v-1} + \cdots + b_0 . \tag{2.2}$$

In a similar fashion the formula [**1**, p. 146]

$$C_{v+1}^{(n)} = -n \sum_{s=0}^{v} (-1)^s \binom{v}{s} C_s C_{v-s}^{(n)} , \tag{2.3}$$

coupled with $C_0^{(n)} = 1$, permits us to write

$$C_v^{(n)} = (-1)^v n^v + c_{v-1} n^{v-1} + \cdots + c_0 . \tag{2.4}$$

As for the Euler numbers, we consider the symbolic formula [1, p. 124]

$$(E^{(n)} + 1)^{2v} + (E^{(n)} - 1)^{2v} = 2E_{2v}^{(n-1)} \tag{2.5}$$

in which, after expansion, exponents on the left side are degraded to subscripts. Hence we have

$$E_{2v}^{(n)} + \frac{(2v)(2v-1)}{1 \cdot 2} E_{2v-2}^{(n)} + \cdots = E_{2v}^{(n-1)} . \tag{2.6}$$

Writing

$$E_{2v}^{(n)} = e_v n^v + e_{v-1} n^{v-1} + \cdots + e_0 ,$$

and

$$E_{2v-2}^{(n)} = e'_{v-1} n^{v-1} + e'_{v-2n} n^{v-2} + \cdots + e'_0 ,$$

we see first that

$$E_{2v}^{(n)} - E_{2v}^{(n-1)} = v e_v n^{v-1} + \text{terms of lower degree} .$$

Hence comparing coefficients of n^{v-1} in (2.6) we have

$$e_v = -\frac{(2v)(2v-1)}{2v} e'_{v-1} .$$

Since $E_0^{(n)} = 1$, we obtain recursively

$$E_{2v}^{(n)} = \frac{(2v)!}{(-2)^v v!} n^v + e_{v-1} n^{v-1} + \cdots + e_0 . \tag{2.7}$$

Next, from [1, p. 129]

$$(D^{(n)} + 1)^{2v+1} - (D^{(n)} - 1)^{2v+1} = 2(2v+1) D_{2v}^{(n-1)} , \tag{2.8}$$

we find that

(2.9) $$D_{2v}^{(n)} = \left(-\frac{1}{6}\right)^v \frac{(2v)!}{v!} n^v + d_{n-1} n^{v-1} + \cdots + d_0 .$$

We shall employ the difference operator $\Delta_d = \Delta$ for which

$$\Delta f(x) = f(x+d) - f(x) \text{ and } \Delta^v = \Delta \cdot \Delta^{v-1}.$$

We recall that if

$$f(x) = a_v x^v + a_{v-1} x^{v-1} + \cdots + a_0 ,$$

then

(2.10) $$\Delta^v f(x) = a_v d^v v!$$

3. Main results. Let

(3.1) $$f_n(x) = a_{n,n} x^n + a_{n,n-1} x^{n-1} + \cdots + a_{n,0} \quad (a_{n,n} \neq 0),$$

and consider the determinant

(3.2) $$|f_i(x_j)| \qquad (i, j = 0, 1, \cdots, m).$$

This may be written as the product of the two determinants

(3.3) $$\begin{vmatrix} a_{0,0} & 0 & \cdots 0 \\ a_{1,0} & a_{1,1} & \cdots 0 \\ \cdots & \cdots & \cdots \\ a_{m,0} & a_{m,1} & \cdots a_{m,m} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots 1 \\ x_0 & x_1 & \cdots x_m \\ \cdots & \cdots & \cdots \\ x_0^m & x_1^m & \cdots x_m^m \end{vmatrix} .$$

The first determinant in (3.3) reduces simply to the product of the elements on the main diagonal, and the second is the familiar Vandermond determinant. Hence

(3.4) $$|f_i(x_j)| = \prod_{k=0}^{m} a_{k,k} \prod_{r>s} (x_r - x_s) \qquad (r, s = 0, 1, \cdots, m).$$

If we let

$$f_i(x_j) = B_i^{(x_j)},$$

then it follows from (3.4) and (2.2) that

$$|B_i^{(x_j)}| = \prod_{k=0}^{m} \left(-\frac{1}{2}\right)^k \prod_{r>s} (x_r - x_s) \qquad (i, j, r, s = 0, 1, \cdots, m). \tag{3.5}$$

Application of (3.4) to (2.4), (2.7), and (2.9) yields results of a similar nature for the C, D, and E numbers. Consequently we have:

THEOREM 1. *For $i, j = 0, 1, \cdots, m$,*

(i) $$|B_i^{(x_j)}| = \prod_{k=0}^{m} \left(-\frac{1}{2}\right)^k \prod_{r>s} (x_r - x_s),$$

(ii) $$|C_i^{(x_j)}| = \prod_{k=0}^{m} (-1)^k \prod_{r>s} (x_r - x_s),$$

(iii) $$|D_{2i}^{(x_j)}| = \prod_{k=0}^{m} \left(-\frac{1}{6}\right)^k \frac{(2k)!}{k!} \prod_{r>s} (x_r - x_s),$$

(iv) $$|E_{2i}^{(x_j)}| = \prod_{k=0}^{m} \left(-\frac{1}{2}\right)^k \frac{(2k)!}{k!} \prod_{r>s} (x_r - x_s).$$

If we take $x_j = a + jd$ then we obtain:

COROLLARY 1. *For $i, j = 0, 1, \cdots, m$, a and d constants,*

(i) $$|B_i^{(a+jd)}| = \prod_{k=0}^{m} \left(-\frac{d}{2}\right)^k k!,$$

(ii) $$|C_i^{(a+jd)}| = \prod_{k=0}^{m} (-d)^k k!,$$

(iii) $$|D_{2i}^{(a+jd)}| = \prod_{k=0}^{m} \left(-\frac{d}{6}\right)^k (2k)!,$$

(iv) $$|E_{2i}^{(a+jd)}| = \prod_{k=0}^{m} \left(-\frac{d}{2}\right)^k (2k)!$$

If we let

$$f_i(a + xd_i) = g_i(x),$$

$f_i(x)$ defined as in (3.1), then we can readily show by the above method that

$$(3.6) \qquad |f_i(a + jd_i)| = \prod_{k=0}^{m} a_{k,k} d_k^k k! \qquad (i, j = 0, 1, \cdots, m).$$

Hence (3.6) implies

$$|B_i^{(a+jd_i)}| = \prod_{k=0}^{m} \left(-\frac{d_k}{2}\right)^k k!,$$

with like results for the other numbers.

We remark that the determinants of Corollary 1 may also be evaluated by a succession of column subtractions.

4. Applications. We consider first the determinant

$$(4.1) \qquad |B_i^{(a+jd)}(x)| \qquad (i, j = 0, 1, \cdots, m;\ a, d \text{ constants}),$$

where $B_i^{(n)}(x)$ is the Bernoulli polynomial of order n defined by [**1**, p. 145]

$$\left(\frac{t}{e^t - 1}\right)^n e^{xt} = \sum_{r=0}^{\infty} \frac{t^v}{v!} B_v^{(n)}(x).$$

(For $x = 0$, $B_v^{(n)}(0) = B_v^{(n)}$, the Bernoulli number of order n.) Also, by [**1**, p. 143],

$$B_v^{(n)}(x) = \sum_{s=0}^{v} \binom{v}{s} x^{v-s} B_s^{(n)}.$$

Consequently

$$(4.2) \qquad |B_i^{(a+jd)}(x)| = \left| \sum_{s=0}^{i} \binom{i}{s} x^{i-s} B_s^{(a+jd)} \right|.$$

If we define

$$\binom{0}{0} = 1 \text{ and } \binom{i}{j} = 0 \text{ for } j > i,$$

then the right member of (4.2) may be written as the product of the two determinants;

$$\left|\binom{i}{j} x^{i-j}\right| \cdot |B_i^{(a+jd)}|.$$

The first determinant has value 1 and hence, by Corollary 1(i),

$$(4.3) \qquad |B_i^{(a+jd)}(x)| = \prod_{k=0}^{m} \left(-\frac{d}{2}\right)^k k!.$$

The Bernoulli polynomials may also be expressed in terms of the D numbers by [1, p. 130]

$$(4.4) \qquad B_v^{(n)}(x) = \sum_{s=0}^{[v/2]} \binom{v}{2s}\left(x - \frac{n}{2}\right)^{v-2s} D_{2s}^{(n)}/2^{2s}.$$

If in (4.3) we let $x = hn$, $h \neq 1/2$, then

$$(4.5) \qquad B_v^{(n)}(hn) = \sum_{s=0}^{[v/2]} \binom{v}{2s}\left(h - \frac{1}{2}\right)^{v-2s} n^{v-2s} D_{2n}^{(n)}/2^{2s}.$$

Since $D_{2n}^{(n)}$ may be written as a polynomial in n of degree s, and $D_0^{(n)} = 1$, it follows readily from (4.4) that, expressed as a polynomial in n,

$$(4.6) \qquad B_v^{(n)}(hn) = \left(h - \frac{1}{2}\right)^v n^v + \text{terms of lower degree}.$$

Consequently, using the same procedure that gave (3.4), we can show for a, d fixed constants, $i, j = 0, 1, \cdots, m$, that

$$(4.7) \qquad |B_i^{(a+jd)}(h(a + jd))| = \prod_{k=0}^{m} \left(h - \frac{1}{2}\right)^k d^k k!.$$

For $h = 0$, (4.7) reduces to the case of Corollary 1(i). If $h = 1/2$ and v is odd, then it follows from (4.4) that

$$B_v^{(n)}(n/2) = 0.$$

Therefore for $m \geq 1$, the value of the determinant in (4.6) is zero. However, if v is even, then

$$B_v^{(n)}(n/2) = D_v^{(n)}/2^{2v},$$

and

$$(4.7)' \qquad \left| B_{2i}^{(a+jd)}\left(\frac{a+jd}{2}\right) \right| = | D_{2i}^{(a+jd)}/2^{2i} | = \prod_{k=0}^{m} \left(-\frac{d}{24}\right)^k (2k)!,$$

where in evaluating the second determinant we have applied Corollary 1(iii).

Finally, it is of interest to point out that [1, p. 4]

$$\Delta^v f(x) = \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} f(x+jd)$$

together with (2.2), (2.4), (2.7), (2.9), and (2.10) yield the recursion formulas

$$(4.8) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} B_v^{(a+jd)} = \left(-\frac{d}{2}\right)^v v!,$$

$$(4.9) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} C_v^{(a+jd)} = (-d)^v v!,$$

$$(4.10) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} E_{2v}^{(a+jd)} = \left(-\frac{d}{2}\right)^v (2v)!$$

and

$$(4.11) \qquad \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} D_{2v}^{(a+jd)} = \left(-\frac{d}{6}\right)^v (2v)!.$$

5. Some additional results. The above methods may also be applied to the evaluation of determinants involving the classic orthogonal polynomials. We consider first the Laguerre polynomials defined by [2, p. 97]

$$(5.1) \qquad L_n^{(\alpha)} = \prod_{v=0}^{n} \binom{n+\alpha}{n-v} \frac{(-x)^v}{v!} .$$

Setting $\alpha = a + jd$ and writing (5.1) as a polynomial in j we have

$$L_n^{(a+jd)}(x) = j^n \frac{d^n}{n!} + \text{terms of lower degree}.$$

Consequently, as in § 3, we obtain

$$(5.2) \qquad |L_i^{(a+jd)}(x)| = \prod_{k=0}^{m-1} d^k = d^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

For the Jacobi polynomials defined by [2, p. 67]

$$(5.3) \qquad P_n^{(\alpha,\beta)}(x) = \sum_{v=0}^{n} \binom{n+\alpha}{n-v}\binom{n+\beta}{v}\left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v}$$

we set $\alpha = a + jd$ and hold β fixed. Then, as a polynomial in j

$$P_n^{(a+jd,\beta)}(x) = j^n \frac{d^n}{2^n} \frac{(x+1)^n}{n!} + \text{terms of lower degree}.$$

Hence, we find

$$(5.4) \qquad |P_i^{(a+jd,\beta)}(x)| = \left\{\frac{(x+1)d}{2}\right\}^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

Similarly

$$(5.5) \qquad |P_i^{(\alpha,b+je)}(x)| = \left\{\frac{(x-1)e}{2}\right\}^{\frac{1}{2}m(m-1)} \qquad (i, j = 0, 1, \cdots, m-1).$$

We consider next, as a polynomial in j,

$$P_n^{(a+jd,b+je)}(x)$$
$$= j^n \sum_{v=0}^{n} \frac{\alpha^{n-v}}{(n-v)!} \frac{e^v}{v!} \left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v} + \text{terms of lower degree}$$

$$= \frac{j^n}{n!}\left[\frac{(d+e)x+d-e}{2}\right]^n + \text{terms of lower degree},$$

which yields

$$(5.6) \qquad |P_i^{(a+jd,\,b+je)}(x)| = \left[\frac{(d+e)x+d-e}{2}\right]^{\frac{1}{2}m(m-1)}$$

$$(i,\ j = 0,\ 1, \cdots, m-1).$$

Finally, for $\alpha = \beta$, the Jacobi polynomials reduce to the ultraspherical polynomials $P^{(\alpha)}(x)$. It follows from (5.6) that

$$(5.7) \qquad |P_i^{(a+jd)}(x)| = (dx)^{\frac{1}{2}m(m-1)} \qquad (i,\ j = 0, 1, \cdots, m-1).$$

REFERENCES

1. N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924.
2. Gabor Szegö, *Orthogonal polynomials*, New York, 1939.

DUKE UNIVERSITY

THE ADJOINT SEMI-GROUP

R. S. PHILLIPS

Introduction. The purpose of this paper is to develop a general theory for the adjoint semi-group of operators which fits into the framework of the present theory of semi-groups. To each semi-group of linear bounded operators $[T(s)]$ defined on a Banach space $\mathfrak{X}$ to itself and possessing suitable continuity properties, we shall assign an adjoint semi-group with like continuity properties, defined on an "adjoint" Banach space $\mathfrak{X}^+$ which is in general a proper subspace of the adjoint space $\mathfrak{X}^*$. The usefulness of the adjoint semi-group has already been demonstrated by W. Feller [3] in his treatise on the parabolic differential equation.[1]

In our theory of the adjoint semi-group, the choice of the subspace $\mathfrak{X}^+ \subset \mathfrak{X}^*$ is decisive. We have been led to $\mathfrak{X}^+$ by two independent considerations. In the first place $\mathfrak{X}^+$ is the largest domain over which the ordinary adjoint $T^*(s)$ has suitable continuity properties. It should be noted, however, that a rather extensive theory of semi-groups has been developed by W. Feller [4] which has no such continuity requirements. The more compelling reason for our choice of $\mathfrak{X}^+$ has to do with the infinitesimal generator. In most applications of the theory of semi-groups one starts with an infinitesimal generator A and it is desired to establish the existence of a semi-group of operators generated by A. It is natural to expect the behavior of the semi-group operators $T(s)$ to be uniquely determined on the domain of A (in symbols $\mathfrak{D}(A)$); and since $T(s)$ is required to be bounded, there will exist a unique extension to the smallest closed subspace containing $\mathfrak{D}(A)$, namely $\overline{\mathfrak{D}(A)}$. *Further extensions are not uniquely determined by A and should not be associated with the operator A.* A reasonable approach to the adjoint semi-group would be to require that its infinitesimal generator be the adjoint A^* of the infinitesimal generator A of the original semi-group. In accordance with the above remarks, the proper domain for the adjoint semi-group

[1] It is remarkable that Feller actually obtained the entire adjoint semi-group without employing a precise notion for the adjoint to an unbounded operator such as the infinitesimal generator. For without this, the general formulation loses much of its significance.

Received August 30, 1953. This paper was written under the sponsorship of the Office of Ordnance Research, U.S. Army, under Contract DA-04-495-ORD-406.

Pacific J. Math. 5 (1955), 269-283

would then be $\overline{\mathfrak{D}(A^*)}$. Now $\mathfrak{X}^+$ is precisely $\overline{\mathfrak{D}(A^*)}$; however the infinitesimal generator A^+ of the adjoint semi-group turns out to be the maximal restriction of A^* with domain and range in $\overline{\mathfrak{D}(A^*)} = \mathfrak{X}^+$.

As in the ordinary theory of adjoint spaces, it is possible to develop an entire hierarchy of "adjoint" spaces for a given semi-group of operators.[2] However it can happen that the second "adjoint" is equal to the original space (under the natural mapping); in this case nothing new is achieved by going beyond the first "adjoint." This situation occurs not only when $\mathfrak{X}$ is reflexive in the usual sense but, more generally, when the resolvent of A is weakly compact (as in the case of most nonsingular problems of mathematical physics).

1. The adjoint transformation. We take $\mathfrak{X}$ and $\mathfrak{Y}$ to be Banach spaces over the real (or complex) scaler field. The transformation $y = T(x)$ is taken to be linear with domain $\mathfrak{D} \subset \mathfrak{X}$ and range $\mathfrak{R} \subset \mathfrak{Y}$, and it is assumed that $\mathfrak{D}$ is a linear subspace of $\mathfrak{X}$.

DEFINITION 1. Let $y = T(x)$ be defined on a domain $\mathfrak{D}$ dense in $\mathfrak{X}$ to $\mathfrak{Y}$, and let $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ be the adjoint spaces to $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. The *adjoint transformation* T^* of T is defined as follows: Its domain $\mathfrak{D}(T^*)$ consists of the set of all $y^* \in \mathfrak{Y}^*$ for which there exists an $x^* \in \mathfrak{X}^*$ such that $y^*[T(x)] = x^*(x)$ for all $x \in \mathfrak{D}$; for such a y^* we define $T^*(y^*) = x^*$.

It is clear that the density of $\mathfrak{D}$ in $\mathfrak{X}$ is required in order that T^* be single-valued. Further it is easy to show that T^* is a closed linear transformation on $\mathfrak{D}(T^*)$ to $\mathfrak{X}^*$. On the other hand the second adjoint is not always well defined since $\mathfrak{D}(T^*)$ is in general not dense in $\mathfrak{Y}^*$. In this connection we have:

THEOREM 1.1. *If T is a closed linear transformation with domain $\mathfrak{D}$ dense in $\mathfrak{X}$, then $\mathfrak{D}(T^*)$ is weakly* dense in $\mathfrak{Y}^*$. In particular, if $\mathfrak{Y}$ is reflexive then $\mathfrak{D}(T^*)$ is strongly dense in $\mathfrak{Y}^*$.*

Proof. If $\mathfrak{D}(T^*)$ were not weakly* dense in $\mathfrak{Y}^*$, then the weak* closure of $\mathfrak{D}(T^*)$ would be regularly closed [**1**] so that there would exist a $y_0 \in \mathfrak{Y}$, $y_0 \neq 0$, such that $y^*(y_0) = 0$ for all $y^* \in \mathfrak{D}(T^*)$. Now $(0, y_0)$ does not belong to the graph $\mathfrak{G}$ of T, and $\mathfrak{G}$ is a closed linear subspace of $\mathfrak{X} \oplus \mathfrak{Y}$. Hence by a theorem

---

[2] For example if $X = C_0(-\infty, \infty)$, the space of continuous functions $f(\xi)$ on $(-\infty, \infty)$ such that $\lim_{|\xi| \to 0} f(\xi) = 0$ and $\|f\| = \sup |f(\xi)|$, and if $A(f) = f'$, $D(A) = [f;\ f$ continuously differentiable, f and $f' \in C_0]$, then $X^+ = L_1(-\infty, \infty)$, $(X^+)^+ =$ space of all functions $f(\xi)$ uniformly continuous and bounded on $(-\infty, \infty)$ with $\|f\| = \sup |f(\xi)|$, and so on.

due to H. Hahn [5, Theorem 2.9.4], there exists an

$$(x_0^*, y_0^*) \in (\mathfrak{X} \oplus \mathfrak{Y})^* = \mathfrak{X}^* \oplus \mathfrak{Y}^*$$

such that

$$x_0^*(x) + y_0^*[T(x)] = 0 \quad \text{for all } x \in \mathfrak{D} \text{ and } x_0^*(0) + y_0^*(y_0) \neq 0.$$

It follows that

$$y_0^* \in \mathfrak{D}(T^*), \quad T^*(y_0^*) = -x_0^*, \text{ and yet } y_0^*(y_0) \neq 0,$$

which is impossible. In case $\mathfrak{Y}$ is reflexive we conclude that $\mathfrak{D}(T^*)$ is weakly dense and hence strongly dense in $\mathfrak{Y}^*$ (the latter conclusion follows from the above-mentioned Hahn theorem).

We turn now to the relation between a transformation, its adjoint, and their inverses.

THEOREM 1.2. *Let T be a linear transformation with $\overline{\mathfrak{D}} = \mathfrak{X}$. Then $(T^*)^{-1}$ exists if and only if $\overline{\mathfrak{R}} = \mathfrak{Y}$. More generally, $\overline{\mathfrak{R}}$ consists of the set of all points y such that $T^*(y^*) = 0$ implies $y^*(y) = 0$.*

Proof. If $T^*(y_0^*) = 0$, then

$$[T^*(y_0^*)](x) = y_0^*[T(x)] = 0$$

for all $x \in \mathfrak{D}$, and hence $y_0^*(\overline{\mathfrak{R}}) = 0$. In particular, $\overline{\mathfrak{R}} = \mathfrak{Y}$ implies that $y_0^* = 0$, and hence that T^* has an inverse. On the other hand if $y_0 \notin \overline{\mathfrak{R}}$, then by the Hahn theorem there exists a functional $y_0^* \in \mathfrak{Y}^*$ such that $y_0^*(y_0) = 1$ and $y_0^*(\overline{\mathfrak{R}}) = 0$. Thus $y_0^*[T(x)] = 0$ for all $x \in \mathfrak{D}$; it follows that $y_0^* \in \mathfrak{D}(T^*)$ and $T^*(y_0^*) = 0$; whereas $y_0^*(y_0) \neq 0$. In particular we see that if $\overline{\mathfrak{R}} \neq \mathfrak{Y}$, then T^* cannot have an inverse.

THEOREM 1.3. *Let T be a linear transformation with $\overline{\mathfrak{D}} = \mathfrak{X}$.* If *$\mathfrak{R}(T^*)$ is weakly* dense in $\mathfrak{X}^*$, then T has an inverse.*

Proof. Suppose that T has no inverse; then there is an $x_0 \neq 0$ such that $T(x_0) = 0$. Consequently

$$[T^*(y^*)](x_0) = y^*[T(x_0)] = 0$$

for all $y^* \in \mathfrak{D}(T^*)$, and this shows that the weak* closure of $\mathfrak{R}(T^*)$ is a proper subspace of $\mathfrak{X}^*$, contrary to assumption.

THEOREM 1.4. *Let T be a linear transformation with an inverse and such that $\overline{\mathfrak{D}} = \mathfrak{X}$ and $\overline{\mathfrak{R}} = \mathfrak{Y}$. Then $(T^*)^{-1} = (T^{-1})^*$; further T^{-1} is bounded if and only if $(T^*)^{-1}$ is bounded on $\mathfrak{X}^*$.*

Proof. In the first place $(T^{-1})^*$ exists because $\mathfrak{R} = \mathfrak{D}(T^{-1})$ is dense in $\mathfrak{Y}$, and $(T^*)^{-1}$ exists by Theorem 1.2. If $y \in \mathfrak{R}$ and $y^* \in \mathfrak{D}(T^*)$, then

$$y^*(y) = y^*\{T[T^{-1}(y)]\} = [T^*(y^*)][T^{-1}(y)].$$

This implies that $\mathfrak{R}(T^*) \subset \mathfrak{D}[(T^{-1})^*]$ and

$$(T^{-1})^*[T^*(y^*)] = y^*$$

for all $y^* \in \mathfrak{D}(T^*)$. Thus $(T^{-1})^*$ is an extension of $(T^*)^{-1}$. On the other hand if $x \in \mathfrak{D}$, then

$$x^*(x) = x^*\{T^{-1}[T(x)]\} = [(T^{-1})^*(x^*)][T(x)],$$

for all $x^* \in \mathfrak{D}[(T^{-1})^*]$. It follows that $\mathfrak{R}(T^*) \supset \mathfrak{D}[(T^{-1})^*]$. Therefore

$$\mathfrak{D}[(T^{-1})^*] = \mathfrak{R}(T^*) = \mathfrak{D}[(T^*)^{-1}],$$

and hence $(T^{-1})^* = (T^*)^{-1}$. If, in addition, T^{-1} is bounded, then it is clear that $(T^{-1})^*$ is also bounded. Conversely if $(T^*)^{-1}$ is bounded on $\mathfrak{X}^*$, then for all $x \in \mathfrak{R}$ and $x^* \in \mathfrak{X}^*$ we have

$$|x^*[T^{-1}(x)]| = |[(T^{-1})^*(x^*)](x)| \le ||(T^*)^{-1}|| \, ||x^*|| \, ||x||.$$

It follows that T^{-1} is bounded.

If T is a linear operator with both domain and range in $\mathfrak{X}$, $\overline{\mathfrak{D}} = \mathfrak{X}$, then the adjoint transformation T^* has its domain and range in $\mathfrak{X}^*$. It is easy to show for an arbitrary bounded operator B on $\mathfrak{X}$ to itself, that

$$(B + T)^* = B^* + T^* \quad \text{and} \quad \mathfrak{D}[(B + T)^*] = \mathfrak{D}(T^*).$$

We are especially interested in the combination $\lambda I - T$, where I is the identity operator and λ is a real (or complex) number. If $\lambda I - T$ has a bounded inverse with domain dense in $\mathfrak{X}$, then λ is said to belong to $\rho(T)$, the resolvent set of T, and

$$(\lambda I - T)^{-1} \equiv R(\lambda; T)$$

is called the resolvent of T.

THEOREM 1.5. *If T is a linear operator with $\overline{\mathfrak{D}} = \mathfrak{X}$ and $\mathfrak{R} \subset \mathfrak{X}$, then*

$$\rho(T) = \rho(T^*) \text{ and } [R(\lambda; T)]^* = R(\lambda; T^*).$$

Proof. If $\lambda \in \rho(T)$, then, according to Theorem 1.4, $\lambda \in \rho(T^*)$ and

$$[R(\lambda; T)]^* = R(\lambda; T^*).$$

On the other hand if $\lambda \in \rho(T^*)$, then Theorem 1.3 shows that T has an inverse, Theorem 1.2 shows that $\overline{\mathfrak{R}} = \mathfrak{X}$, and Theorem 1.4 then implies that $\lambda \in \rho(T)$.

2. The adjoint semi-group. We now apply the previous results to semi-groups of linear bounded operators (cf. [5]). Let $\mathfrak{E}(\mathfrak{X})$ be the Banach algebra of endomorphism of $\mathfrak{X}$, and let $[T(s)]$ be a one-parameter family of operators in $\mathfrak{E}(\mathfrak{X})$ defined for $s \in [0, \infty)$ and satisfying:

(i) $T(s_1 + s_2) = T(s_1)T(s_2)$ for all $s_1, s_2 \geq 0$, $T(0) = I$;

(ii) for each $x \in \mathfrak{X}$, $T(s)x$ is continuous for $s > 0$;

(iii) $\int_0^1 ||T(\sigma)x||d\sigma < \infty$ for each $x \in \mathfrak{X}$.

If T satisfies the additional condition

(iv) $\lim_{\lambda \to \infty} \lambda \int_0^\infty \exp(-\lambda\sigma) T(\sigma)x d\sigma = x$ for each $x \in \mathfrak{X}$,

then $T(s)$ is said to be *of class* $(0, A)$. If, instead of (iv), $T(s)$ satisfies the stronger condition

(v) $\lim_{\tau \to 0} \tau^{-1} \int_0^\tau T(\sigma)x d\sigma = x$ for each $x \in \mathfrak{X}$,

then $T(s)$ is said to be *of class* $(0, C)$. Finally if $T(s)$ satisfies (i), (ii), (iii), and the still stronger continuity condition

(vi) $\lim_{s \to 0} T(s)x = x$ for each $x \in \mathfrak{X}$,

then $T(s)$ is said to be *of class* C.

The domain $\mathfrak{D}(A)$ of the infinitesimal generator A is the set of elements x for which

$$\lim_{\tau \to 0} \tau^{-1}[T(\tau) - I]x$$

exists, and this limit is defined to be Ax. It follows from (iv) (and hence (v) or (vi)) that $\mathfrak{D}(A)$ is dense in $\mathfrak{X}$ (cf. [5, Theorem 9.3.1]). We have previously shown [6] that A is closed if and only if $T(s)$ is of class $(0, C)$. However, even when $T(s)$ is of class $(0, A)$, the infinitesimal generator has a smallest closed extension, called the complete infinitesimal generator (c.i.g.) and denoted by $\bar{A}$. For each $x_0 \in \mathfrak{D}(\bar{A})$ there is a sequence $\{x_n\} \subset \mathfrak{D}(A)$ such that $x_n \longrightarrow x_0$ and $Ax_n \longrightarrow \bar{A}x_0$. It follows that $R(\lambda; \bar{A})$ is an extension of $R(\lambda; A)$, that $\rho(A) = \rho(\bar{A})$, that $A^* = (\bar{A})^*$, and that

$$[R(\lambda; A)]^* = [R(\lambda; \bar{A})]^*.$$ [3]

It can be shown that

$$\omega_0 = \inf_{s>0} \log \|T(s)\|/s = \lim_{s\to\infty} \log \|T(s)\|/s. \tag{2.1}$$

Each $\lambda > \omega_0$ belongs to the resolvent set for $\bar{A}$, and the resolvent is given by

$$R(\lambda; \bar{A})x = \int_0^\infty \exp(-\lambda\sigma) T(\sigma) x d\sigma; \tag{2.2}$$

see [6].

DEFINITION 2.1. The semi-group $T(s)$ is said to be *of class* $(0, A)^*$, $(0, C)^*$, or C^* if it is of class $(0, A)$, $(0, C)$, or C, respectively, and if in addition $\|T^*(s)x^*\|$, $0 \leq s \leq 1$, is majorized by integrable function for each $x^* \in \mathfrak{X}^*$.[4]

DEFINITION 2.2. Let $T(s)$ be a semi-group of class $(0, A)$ with infinitesimal generator A. We define the *adjoint semi-group* to be the restriction of $T^*(s)$ to $\mathfrak{X}^+ = \overline{\mathfrak{D}(A^*)}$ and denote it by $T^+(s)$. We denote the infinitesimal generator of $T^+(s)$ by A^+.

---

[3] For $\lambda \in \rho(A)$, the resolvent $R(\lambda; A)$ has a unique bounded linear extension $R(\lambda; A)_1$ on $\mathfrak{X}$. If $\{x_n\} \subset \mathfrak{D}(A)$, $x_n \longrightarrow x_0 \in \mathfrak{D}(\bar{A})$, and $Ax_n \longrightarrow \bar{A}x_0$, then $R(\lambda; A)(\lambda I - A)x_n = x_n$ implies that $R(\lambda; A)_1(\lambda I - \bar{A})x_0 = x_0$. Likewise for $\{y_n\} \subset \mathfrak{R}(\lambda I - A)$ and $y_n \longrightarrow y_0$, the relation $(\lambda I - A)R(\lambda; A)y_n = y_n$ implies that $(\lambda I - \bar{A})R(\lambda; A)_1 y_0 = y_0$. It follows that $R(\lambda; \bar{A})$ exists and is identical with $R(\lambda; A)_1$. This shows that $\rho(A) \subset \rho(\bar{A})$. A similar argument can be used to prove $A^* = \bar{A}^*$, and the last relation is obvious.

[4] This condition is automatically satisfied if $\int_0^1 \|T(\sigma)\| d\sigma < \infty$ or if $T(s)$ if of class C.

THEOREM 2.1. *If $T(s)$ is a semi-group of class $(0, A)^*$, $(0, C)^*$, or C^*, then the adjoint semi-group is of class $(0, A)$, $(0, C)$ or C, respectively. The* c.i.g. $\overline{A^+}$ *is the largest restriction of A^* with domain and range in $\mathfrak{X}^+$.*

Proof. According to Theorem 1.5,

$$R(\lambda; A^*) = R(\lambda; \overline{A^*}) = R^*(\lambda; A)$$

and hence $\mathfrak{D}(A^*)$ is simply the range of $R^*(\lambda; A)$. For $\lambda > \omega_0$, $R^*(\lambda; A)$ can be expressed by means of a Dunford integral [2] as

$$R^*(\lambda; A)x^* = \int_0^\infty \exp(-\lambda\sigma)\, T^*(\sigma)x^*\, d\sigma\,. \tag{2.3}$$

It is clear from this that

$$T^*(s)R^*(\lambda; A) = R^*(\lambda; A)T^*(s),$$

so that $T^*(s)$ takes $\mathfrak{D}(A^*)$ into $\mathfrak{D}(A^*)$. Since $T^*(s)$ is bounded, it follows that $T^*(s)(\mathfrak{X}^+) \subset \mathfrak{X}^+$; that is, $T^+(s) \in \mathfrak{E}(\mathfrak{X}^+)$. It is obvious that $T^*(s)$ and hence $T^+(s)$ satisfies (i).

In order to establish continuity we first note that

$$[T^*(\tau) - I^*]R^*(\lambda; A)x^* = [\exp(\lambda\tau) - 1] \int_0^\infty \exp(-\lambda\sigma)\, T^*(\sigma)x^*\, d\sigma - \exp(\lambda\tau) \int_0^\tau \exp(-\lambda\sigma)\, T^*(\sigma)x^*\, d\sigma. \tag{2.4}$$

The first term in the right member is simply $[\exp(\lambda\tau) - 1]\, R^*(\lambda; A)x^*$, and it clearly converges to zero with τ; further the assumption that $\| T^*(\sigma)x^* \|$ is majorized by a function in $L_1(0, 1)$ implies that the second term also goes to zero with τ. Thus

$$\lim_{s \to 0} T^*(s)y^* = y^*$$

for all $y^* \in \mathfrak{D}(A^*)$. It follows from this (cf. [5, Theorem 9.4.1]) that $T^*(s)y^*$ is strongly continuous for $s \geq 0$, $y^* \in \mathfrak{D}(A^*)$. Further since $\| T^*(s) \| = \| T(s) \|$ is uniformly bounded in each interval of the form $(\delta, 1/\delta)$, we see that $T^*(s)x^*$ is strongly continuous for $s > 0$ and all $x^* \in \mathfrak{X}^+$. Thus $T^+(s)$ satisfies (i), (ii), and (iii). Again, for each $x^* \in \mathfrak{D}(A^*)$,

$$T^{+}(s)x^{*} \longrightarrow x^{*} \text{ as } s \longrightarrow 0$$

and *a fortiori*

$$\tau^{-1}\int_{0}^{\tau} T^{*}(\sigma)x^{*}\,d\sigma \longrightarrow x^{*} \text{ as } \tau \longrightarrow 0$$

and

$$\lambda R^{*}(\lambda; A)x^{*} \longrightarrow x^{*} \text{ as } \lambda \longrightarrow \infty.$$

Now if $T(s)$ is of class C, then $\|T^{*}(s)\| = O(1)$; if $T(s)$ is of class $(0, C)$ then

$$\|[\tau^{-1}\int_{0}^{\tau} T(\sigma)\,d\sigma]^{*}\| = O(1);$$

and if $T(s)$ is of class $(0, A)$ then $\|\lambda R^{*}(\lambda; A)\| = O(1)$. It now follows from the Banach-Steinhaus theorem that $T^{+}(s)$ will satisfy (vi), (v), or (iv) with $T(s)$.

Finally, the c.i.g. $\overline{A^{+}}$ of $T^{+}(s)$ is determined by its resolvent (cf. [**6**]), which for $\lambda > \omega_0$ can be expressed by the Bochner integral

$$R(\lambda; \overline{A^{+}})x^{*} = \int_{0}^{\infty} \exp(-\lambda\sigma)\, T^{+}(\sigma)x^{*}\,d\sigma \qquad (x^{*} \in \mathfrak{X}^{+}).$$

According to formula (2.3) this is simply the restriction of $R(\lambda; A^{*})$ to $\mathfrak{X}^{+}$; thus $\overline{A^{+}}$ is a restriction of A^{*}. Now if $x^{*} \in \mathfrak{D}(A^{*})$ and $A^{*}(x^{*}) \in \mathfrak{X}^{+}$, then $(\lambda I^{*} - A^{*})x^{*} \in X^{+}$ and hence

$$R(\lambda; A^{*})\,(\lambda I^{*} - A^{*})x^{*} = x^{*} \in \mathfrak{D}(\overline{A^{+}}).$$

Conversely if $x^{*} \in \mathfrak{D}(\overline{A^{+}})$, then $x^{*} \in \mathfrak{D}(A^{*})$ and $A^{*}x^{*} = \overline{A^{+}}x^{*} \in \mathfrak{X}^{+}$. In other words, $\overline{A^{+}}$ is the maximal restriction of A^{*} which maps $\mathfrak{X}^{+}$ into $\mathfrak{X}^{+}$. This concludes the proof.

COROLLARY. *If $\lambda \in \rho(\overline{A})$, then $\lambda \in \rho(\overline{A^{+}})$ and $R(\lambda; \overline{A^{+}})$ equals the restriction of $R(\lambda; A^{*})$ to $\mathfrak{X}^{+}$.*

Proof. If $\lambda \in \rho(A)$, then $R(\lambda; A^{*})$ exists. Let $R(\lambda; A^{*})_0$ be the restriction of $R(\lambda; A^{*})$ to $\mathfrak{X}^{+}$. For $x^{*} \in \mathfrak{D}(\overline{A^{+}})$, we have

$$(\lambda I^{+} - \overline{A^{+}})x^{*} = (\lambda I^{*} - A^{*})x^{*}$$

and hence $R(\lambda; A^*)_0$ is a left inverse for $\lambda I^+ - \overline{A^+}$. On the other hand if $x^* \in \mathfrak{X}^+$, then

$$(\lambda I^* - A^*) R(\lambda; A^*)_0 x^* = x^*.$$

Since $R(\lambda; A^*)_0 x^* \in \mathfrak{D}(A^*) \subset \mathfrak{X}^+$ we also have $A^* R(\lambda; A^*)_0 x^* \in \mathfrak{X}^+$ and hence by the above theorem $R(\lambda; A^*)_0 x^* \in \mathfrak{D}(\overline{A^+})$. It follows that $R(\lambda; A^*)_0$ is also the right inverse for $\lambda I^+ - \overline{A^+}$ so that $\lambda \in \rho(\overline{A^+})$.

A converse to the above corollary is obtained in Theorem 3.2 where it is shown that $\rho(\overline{A}) = \rho(\overline{A^+})$.

COROLLARY. *If $\mathfrak{X}$ is reflexive, then $\mathfrak{X}^+ = \mathfrak{X}^*$.*

Proof. If $\mathfrak{X}$ is reflexive, then, according to Theorem 1.1, $\mathfrak{D}(A^*)$ is dense in $\mathfrak{X}^*$. Hence $\mathfrak{X}^+ = \overline{\mathfrak{D}(A^*)} = \mathfrak{X}^*$.

We conclude this section with two other characterizations of $\mathfrak{X}^+$.

THEOREM 2.2. *For a semi-group $T(s)$ of class $(0, A)^*$, let*

$$\Gamma = [x^*;\ T^*(s) x^* \longrightarrow x^* \text{ as } s \longrightarrow 0].$$

Then $\mathfrak{X}^+ = \overline{\Gamma}$.

Proof. It is clear that $\mathfrak{D}(A^*) \subset \Gamma$; and since $\mathfrak{D}(A^*)$ is dense in $\mathfrak{X}^+$, we have $\mathfrak{X}^+ \subset \overline{\Gamma}$. On the other hand if $x^* \in \Gamma$, then a direct calculation shows that

$$\lambda R(\lambda; A^*) x^* = \lambda \int_0^\infty \exp(-\lambda\sigma) T^*(\sigma) x^* d\sigma \longrightarrow x^* \qquad \text{as } \lambda \longrightarrow \infty.$$

Consequently $x^* \in \overline{\mathfrak{D}(A^*)} = \mathfrak{X}^+$.

THEOREM 2.3. *For a semi-group $T(s)$ of class $(0, A)^*$ let*

$$\Gamma_0 = [y^*_{\alpha\beta};\ y^*_{\alpha\beta} = \int_\alpha^\beta T^*(\sigma) x^* d\sigma,\ x^* \in \mathfrak{X}^*,\ 0 \le \alpha < \beta].$$

Then $\mathfrak{X}^+ = \overline{\Gamma}_0$.

Proof. An easy calculation shows that $\Gamma_0 \subset \Gamma$. On the other hand if $x^* \in \Gamma$ then

$$\tau^{-1} \int_0^\tau T^*(\sigma) x^* d\sigma \longrightarrow x^* \qquad \text{as } \tau \longrightarrow 0$$

and belongs to Γ_0; thus $\overline{\Gamma}_0 \supset \Gamma$ and therefore $\overline{\Gamma}_0 = \overline{\Gamma} = \mathfrak{X}^+$.

3. The adjoint space. We shall call $\mathfrak{X}^+$ the *adjoint space to* $\mathfrak{X}$ *relative to the semi-group* $[T(s)]$, or simply, the *adjoint space*; and we shall denote the generic element of $\mathfrak{X}^+$ by x^+. To avoid confusion we shall hereafter refer to $\mathfrak{X}^*$ as the *full adjoint space*. This section is devoted to a study of the hierarchy of adjoint spaces which arise from a given semi-group of operators of class $(0, A)^*$.

It will be observed that whereas

$$||x^*|| = \sup[\,|x^+(x)|;\ ||x|| \leq 1,\ x \in \mathfrak{X}\,],$$

it is not in general true that $||x||$ can be obtained in like manner as

$$||x||' = \sup[\,|x^+(x)|;\ ||x^+|| \leq 1,\ x^+ \in \mathfrak{X}^+\,]. \tag{3.1}$$

All that can be asserted here is that $||x||' \leq ||x||$. If $\mathfrak{X}^+$ is equal to the full adjoint space, then it is clear that $||x||' = ||x||$. This occurs when $\mathfrak{X}$ is reflexive or when A is bounded. In any case we see that the function $||x||'$ satisfies the postulates of a pseudo-norm. However, more is true:

THEOREM 3.1. *The norm* $||x||'$ *defines an equivalent topology for* $\mathfrak{X}$; *in fact, there exists an* $m > 0$ *such that*

$$||x|| \geq ||x||' \geq m\,||x||$$

for all $x \in \mathfrak{X}$. *In particular if*

$$\liminf_{\lambda \to \infty} ||\lambda R(\lambda; \overline{A})|| = 1,$$

then $||x|| \equiv ||x||'$.

Proof. For a fixed $x \in \mathfrak{X}$ there exists an $x^* \in \mathfrak{X}^*$, $||x^*|| = 1$, such that $x^*(x) = ||x||$. It follows from (iv) that

$$[\lambda R^*(\lambda; \overline{A})x^*](x) = x^*[\lambda R(\lambda; \overline{A})x] \longrightarrow x^*(x) \qquad \text{as } \lambda \longrightarrow \infty,$$

and from (iv) together with the uniform boundedness theorem that

$$\overline{\lim_{\lambda \to \infty}}\ ||\lambda R(\lambda; \overline{A})|| = M < \infty.$$

Consequently, given $\epsilon > 0$, there is a λ_ϵ with

$$||\lambda_\epsilon R^*(\lambda_\epsilon;\overline{A})|| \le M+\epsilon \quad \text{and} \quad |[\lambda_\epsilon R^*(\lambda_\epsilon;\overline{A})x^*](x) - ||x||| < \epsilon.$$

Now

$$y_\epsilon^* \equiv \lambda_\epsilon R^*(\lambda_\epsilon;A)x^* \in \mathfrak{X}^+ \quad \text{and} \quad ||y_\epsilon^*|| \le M+\epsilon.$$

Hence

$$\frac{|y_\epsilon^*(x)|}{||y_\epsilon^*||} \ge \frac{||x||-\epsilon}{M+\epsilon};$$

and since ϵ is arbitrary this gives the desired result with $m = 1/M$. In particular if $M = 1$, then $||x|| = ||x||'$.

THEOREM 3.2. *If* $[T(s)]$ *is a semi-group of operators of class* $(0,A)^*$, *then* $\rho(\overline{A}) = \rho(\overline{A^+})$.

Proof. We have already shown in the first corollary to Theorem 2.1 that $\rho(\overline{A}) \subset \rho(\overline{A^+})$. If $\lambda \in \rho(\overline{A^+})$, then

$$\mathfrak{R}(\lambda I^* - \overline{A}^*) \supset \mathfrak{R}(\lambda I^+ - \overline{A^+}) = \mathfrak{X}^+.$$

Since, by Theorem 1.1, $\mathfrak{D}(\overline{A}^*) \subset \mathfrak{X}^+$ is weakly* dense in $\mathfrak{X}^*$, the same is true of $\mathfrak{R}(\lambda I^* - \overline{A}^*)$. It now follows from Theorem 1.3 that $\lambda I - \overline{A}$ has an inverse. Further, if

$$(\lambda I^* - \overline{A}^*)x_0^* = 0$$

then $x_0^* \in \mathfrak{D}(\overline{A}^*)$ and $\overline{A}^* x_0^* \in \mathfrak{D}(\overline{A}^*) \subset \mathfrak{X}^+$, so that $x_0^* \in \mathfrak{D}(\overline{A^+})$. Since $\overline{A^+}$ is a restriction of $\overline{A}^*$, this implies that $(\lambda I^+ - \overline{A^+})x_0^* = 0$ and hence that $x_0^* = 0$. Theorem 1.2 now asserts that $\mathfrak{R}(\lambda I - \overline{A})$ is dense in $\mathfrak{X}$. Finally for $x \in \mathfrak{R}(\lambda I - \overline{A})$ we have

$$\begin{aligned} ||(\lambda I - \overline{A})^{-1}x|| &\le m^{-1}\, ||(\lambda I - \overline{A})^{-1}x||' \\ &= m^{-1} \sup\, [\,|x^+[(\lambda I - \overline{A})^{-1}x]\,|;\ ||x^+|| \le 1,\ x^+ \in \mathfrak{X}^+] \\ &\le m^{-1}\, ||R(\lambda;\overline{A^+})||\ ||x||; \end{aligned}$$

and this shows that $(\lambda I - \overline{A})^{-1}$ is bounded. It follows that $\lambda \in \rho(\overline{A})$.

We see from the above theorem that $\overline{A^+}$ has the same resolvent set as $\bar{A}^*$ (and $\bar{A}$) in spite of the fact that it is a restriction of $\bar{A}^*$.

Renorming $\mathfrak{X}$ by $\|x\|'$ has no effect on our determination of $\mathfrak{X}^+$; in fact, even the norm of the elements of $\mathfrak{X}^+$ remains the same. For

$$\|x\|' \leq \|x\| \text{ and } |x^+(x)| \leq \|x^+\| \, \|x\|'$$

imply that

$$\|x^+\| \leq \sup [\,|x^+(x)|;\ \|x\|' \leq 1,\ x \in \mathfrak{X}] \leq \|x^+\|.$$

Nevertheless, when we deal with the second adjoint space relative to a given semi-group $[T(s)]$, a slight advantage is obtained by renorming $\mathfrak{X}$ in this way.

THEOREM 3.3. *Suppose that both* $[T(s)]$ *and* $[T^+(s)]$ *are of class* $(0, A)^*$, *and let the norm of* $\mathfrak{X}$ *be given by* $\|x\|'$. *Then* $\mathfrak{X}$ *can be embedded in* $\mathfrak{X}^{++}$ *by means of the natural mapping.*

Proof. Each $x_0 \in \mathfrak{X}$ defines a unique bounded linear functional $F_0 \in (\mathfrak{X}^+)^*$, namely $F_0(x^+) = x^+(x_0)$. Further,

$$\|F_0\| = \sup [\,|F_0(x^+)| = |x^+(x_0)|;\ \|x^+\| \leq 1,\ x^+ \in \mathfrak{X}^+] = \|x_0\|'.$$

Hence $x_0 \longrightarrow F_0$ is a linear isometric mapping of $\mathfrak{X}$ onto a subspace of $(\mathfrak{X}^+)^*$. It remains to show that $\mathfrak{X} \subset (\mathfrak{X}^+)^+$ in the above sense. This in turn requires that $\mathfrak{X} \subset \mathfrak{D}[(\overline{A^+})^*]$. However, if $x_0 \longrightarrow F_0$ then

$$[R^*(\lambda; \overline{A^+}) F_0](x^+) = F_0[R(\lambda; \overline{A^+}) x^+] = [R(\lambda; \overline{A^+}) x^+](x_0) = x^+[R(\lambda; \bar{A}) x_0].$$

Hence

$$R(\lambda; \bar{A}) x_0 \longrightarrow R^*(\lambda; \overline{A^+}) F_0 .$$

Now

$$\lim_{\lambda \to \infty} \lambda R(\lambda; \bar{A}) x_0 = x_0$$

implies that

$$\lim_{\lambda \to \infty} \lambda R^*(\lambda; \overline{A^+}) F_0 = F_0 ;$$

and since

$$R^*(\lambda;\overline{A^+})F_0 \in \mathfrak{D}[(\overline{A^+})^*],$$

it follows that $x_0 \in \mathfrak{D}[(\overline{A^+})^*]$.

The space $\mathfrak{X}^{++}$ depends only on $T^+(s)$ and $\mathfrak{X}^+$. Further, the norm in $\mathfrak{X}^+$ is not effected by renorming $\mathfrak{X}$ with the norm $||x||'$; in fact

$$||x^+|| = \sup[|x^+(x)|;\ ||x||' \leq 1,\ x \in \mathfrak{X}].$$

Since $\mathfrak{X}$ with the norm $||x||'$ is a subset of $\mathfrak{X}^{++}$, it follows that

$$||x^+||' \equiv \sup[|x^{++}(x^+)|;\ ||x^{++}|| \leq 1,\ x^{++} \in \mathfrak{X}^{++}] = ||x^+||.$$

Thus it is only in the case of $\mathfrak{X}$ and $\mathfrak{X}^+$ that a nonsymmetric condition between norms may arise; for all other pairs of successive adjoint spaces the norms are symmetric. Even if $\mathfrak{X}$ is not renormed, $\mathfrak{X}$ will be isomorphic with its image in $\mathfrak{X}^{++}$ under the natural mapping.

DEFINITION 3.1. We define the (Γ)-*weak topology* in $\mathfrak{X}$ in the usual way be means of the generic neighborhood

$$N(x_0; x_1^*, \cdots, x_n^*; \epsilon) \equiv [x; |x_k^*(x - x_0)| < \epsilon, k = 1, \cdots, n],$$

where the $(x_1^*, \cdots, x_n^*)$ can be any finite subset of Γ and ϵ is an arbitrary positive number.

It is of interest to determine when, under the natural mapping, $\mathfrak{X} = \mathfrak{X}^{++}$; that is, under what conditions $\mathfrak{X}$ is reflexive relative to a given semi-group of operators $[T(s)]$. Here we assume that $\mathfrak{X}$ has been renormed with norm $||x||'$. If $\mathfrak{X}$ is a reflexive in the usual sense, then the second corollary to Theorem 2.1 asserts that $\mathfrak{X}^+ = \mathfrak{X}^*$, and likewise that

$$\mathfrak{X}^{++} = (\mathfrak{X}^+)^* = \mathfrak{X}^{**} = \mathfrak{X}.$$

More generally, we have:

THEOREM 3.4. *Suppose that both* $[T(s)]$ *and* $[T^+(s)]$ *are of class* $(0, A)^*$, *and let the norm of* $\mathfrak{X}$ *be given by* $||x||'$. *A necessary and sufficient condition for* $\mathfrak{X} = \mathfrak{X}^{++}$ *is that* $R(\lambda; \overline{A})$ *be* $(\mathfrak{X}^+)$-*weakly compact.*

Proof. Suppose first that $R(\lambda; \overline{A})$ is $(\mathfrak{X}^+)$-weakly compact; that is, the image of each bounded set is contained in an $(\mathfrak{X}^+)$-weakly compact subset of $\mathfrak{X}$. Let F_0 be an arbitrary element of $(\mathfrak{X}^+)^*$. Then by Helly's theorem, given a finite subset $\pi \subset \mathfrak{X}^+$, there exists an

$$x_\pi \in \mathfrak{X}, \quad ||x_\pi|| \le 2\, ||F_0||,$$

such that $F_0(x^+) = x^+(x_\pi)$ for all $x^+ \in \pi$. Ordering the π's by inclusion, we easily see that they form a directed set. Consequently,

$$[R^*(\lambda; \overline{A^+})F_0](x^+) = F_0[R(\lambda; \overline{A^+})x^+] = \lim_\pi [R(\lambda; \overline{A^+})x^+](x_\pi)$$

$$= \lim_\pi x^+[R(\lambda; \overline{A})x_\pi].$$

Since the $R(\lambda; \overline{A})$ image of any bounded set is contained in an $(\mathfrak{X}^+)$-weakly compact subset of $\mathfrak{X}$, it is easily shown that there exists an $x_0 \in \mathfrak{X}$ such that

$$\lim_\pi x^+[R(\lambda; A)x_\pi] = x^+(x_0)$$

for all $x^+ \in \mathfrak{X}^+$. Thus $R^*(\lambda; \overline{A^+})F_0$ is the image of x_0 under the natural mapping; in other words, $\mathfrak{X} \supset \mathfrak{D}[(\overline{A^+})^*]$. This together with Theorem 3.3 shows that $\mathfrak{X} = \mathfrak{X}^{++}$.

Conversely, suppose that $\mathfrak{X} = \mathfrak{X}^{++}$. Then $R^*(\lambda; \overline{A^+})[(\mathfrak{X}^+)^*]$ is contained in the images of $\mathfrak{X}$. Now $R^*(\lambda; \overline{A^+})$ is continuous in the usual weak* topology of $(\mathfrak{X}^+)^*$; hence the unit sphere, which is weakly* compact, maps onto a weakly* compact subset. Now this image lies in $\mathfrak{X}$ and the weak* topology in $\mathfrak{X} \subset (\mathfrak{X}^+)^*$ is the same as the $(\mathfrak{X}^+)$-weak topology for $\mathfrak{X}$. Hence $R(\lambda; \overline{A})$, which is essentially a restriction of $R^*(\lambda; \overline{A^+})$, takes bounded sets into $(\mathfrak{X}^+)$-weakly compact subsets of $\mathfrak{X}$. This concludes the proof.

COROLLARY *If $R(\lambda; \overline{A})$ is weakly compact relative to the usual weak topology of $\mathfrak{X}$, then $\mathfrak{X} = \mathfrak{X}^{++}$.*

Proof. It is clear that a weakly compact subset of $\mathfrak{X}$ is also weakly compact relative to any weaker topology such as the $(\mathfrak{X}^+)$-weak topology of $\mathfrak{X}$.

REFERENCES

1. Leonidas Alaoglu, *Weak topologies of normed linear spaces*, Ann. of Math. **41** (1940), 252-267.

2. Nelson Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc. **44** (1938), 305-356.

3. William Feller, *The parabolic differential equations and the associated semi-groups of transformations*, Ann. of Math. **55** (1952), 468-519.

4. ———, *Semi-groups of transformations in general weak topologies*, Ann. of Math. **57-58** (1953), 287-308.

5. Einar Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publ. vol. 31, New York, 1948.

6. R. S. Phillips, *An inversion formula for Laplace transforms and semi-groups of linear operators*, Ann. of Math. **59** (1954), 325-356.

UNIVERSITY OF SOUTHERN CALIFORNIA

A LATTICE-THEORETICAL FIXPOINT THEOREM AND ITS APPLICATIONS

ALFRED TARSKI

1. A lattice-theoretical fixpoint theorem. In this section we formulate and prove an elementary fixpoint theorem which holds in arbitrary complete lattices. In the following sections we give various applications (and extensions) of this result in the theories of simply ordered sets, real functions, Boolean algebras, as well as in general set theory and topology.[1]

By a *lattice* we understand as usual a system $\mathfrak{A} = \langle A, \leq \rangle$ formed by a non-empty set A and a binary relation $\leq$; it is assumed that $\leq$ establishes a partial order in A and that for any two elements $a,b \in A$ there is a least upper bound (join) $a \cup b$ and a greatest lower bound (meet) $a \cap b$. The relations $\geq$, $<$, and $>$ are defined in the usual way in terms of $\leq$.

The lattice $\mathfrak{A} = \langle A, \leq \rangle$ is called *complete* if every subset B of A has a least upper bound $\bigcup B$ and a greatest lower bound $\bigcap B$. Such a lattice has in particular two elements 0 and 1 defined by the formulas

$$0 = \bigcap A \quad \text{and} \quad 1 = \bigcup A\,.$$

Given any two elements $a,b \in A$ with $a \leq b$, we denote by $[a,b]$ the *interval* with the endpoints a and b, that is, the set of all elements $x \in A$ for which $a \leq x \leq b$; in symbols,

$$[a,b] = \mathrm{E}_x[x \in A \quad \text{and} \quad a \leq x \leq b].$$

The system $\langle [a,b], \leq \rangle$ is clearly a lattice; it is a complete if $\mathfrak{A}$ is complete.

We shall consider functions on A to A and, more generally, on a subset B of A to another subset C of A. Such a function f is called *increasing* if, for any

[1] For notions and facts concerning lattices, simply ordered systems, and Boolean algebras consult [1].

Received June 29, 1953. Most of the results contained in this paper were obtained in 1939. A summary of the results was given in [6]. The paper was prepared for publication when the author was working on a research project in the foundations of mathematics sponsored by the Office of Ordnance Research, U.S. Army.

Pacific J. Math. 5 (1955), 285-309

elements $x,y \in B$, $x \leq y$ implies $f(x) \leq f(y)$. By a *fixpoint* of a function f we understand, of course, an element x of the domain of f such that $f(x) = x$.

Throughout the discussion the variables $a, b, \cdots, x, y, \cdots$ are assumed to represent arbitrary elements of a lattice (or another algebraic system involved).

THEOREM 1 (LATTICE-THEORETICAL FIXPOINT THEOREM). *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a complete lattice,*

(ii) f *be an increasing function on* A *to* A,

(iii) P *be the set of all fixpoints of* f.

Then the set P *is not empty and the system* $\langle P, \leq \rangle$ *is a complete lattice; in particular we have*

$$\mathsf{U}P = \mathsf{U}\mathsf{E}_x[f(x) \geq x] \in P$$

and

$$\mathsf{\Pi}P = \mathsf{\Pi}\mathsf{E}_x[f(x) \leq x] \in P.$$ [2]

Proof. Let

(1) $$u = \mathsf{U}\mathsf{E}_x[f(x) \geq x].$$

We clearly have $x \leq u$ for every element x with $f(x) \geq x$; hence, the function f being increasing,

$$f(x) \leq f(u) \text{ and } x \leq f(u).$$

By (1) we conclude that

(2) $$u \leq f(u).$$

---

[2] In 1927 Knaster and the author proved a set-theoretical fixpoint theorem by which every function, on and to the family of all subsets of a set, which is increasing under set-theoretical inclusion has at least one fixpoint; see [3], where some applications of this result in set theory (a generalization of the Cantor-Bernstein theorem) and topology are also mentioned. A generalization of this result is the lattice-theoretical fixpoint theorem stated above as Theorem 1. The theorem in its present form and its various applications and extensions were found by the author in 1939 and discussed by him in a few public lectures in 1939-1942. (See, for example, a reference in the American Mathematical Monthly **49** (1942), 402.) An essential part of Theorem 1 was included in [**1**, p. 54]; however, the author was informed by Professor Garrett Birkhoff that a proper historical reference to this result was omitted by mistake.

Therefore

$$f(u) \leq f(f(u)),$$

so that $f(u)$ belongs to the set $\mathsf{E}_x[f(x) \geq x]$; consequently, by (1),

$$f(u) \leq u. \tag{3}$$

Formulas (2) and (3) imply that u is a fixpoint of f; hence we conclude by (1) that u is the join of all fixpoints of f, so that

$$\cup P = \cup \mathsf{E}_x[f(x) \geq x] \in P. \tag{4}$$

Consider the dual lattice $\mathfrak{A}' = \langle A, \geq \rangle$. $\mathfrak{A}'$, like $\mathfrak{A}$, is complete, and f is again an increasing function in $\mathfrak{A}'$. The join of any elements in $\mathfrak{A}'$ obviously coincides with the meet of these elements in $\mathfrak{A}$. Hence, by applying to $\mathfrak{A}'$ the result established for $\mathfrak{A}$ in (4), we conclude that

$$\cap P = \cap \mathsf{E}_x[f(x) \leq x] \in P. \tag{5}$$

Now let Y be any subset of P. The system

$$\mathfrak{B} = \langle [\cup Y, 1], \leq \rangle$$

is a complete lattice. For any $x \in Y$ we have $x \leq \cup Y$ and hence

$$x = f(x) \leq f(\cup Y);$$

therefore $\cup Y \leq f(\cup Y)$. Consequently, $\cup Y \leq z$ implies

$$\cup Y \leq f(\cup Y) \leq f(z).$$

Thus, by restricting the domain of f to the interval $[\cup Y, 1]$, we obtain an increasing function f' on $[\cup Y, 1]$ to $[\cup Y, 1]$. By applying formula (5) established above to the lattice $\mathfrak{B}$ and to the function f', we conclude that the greatest lower bound v of all fixpoints of f' is itself a fixpoint of f'. Obviously, v is a fixpoint of f, and in fact the least fixpoint of f which is an upper bound of all elements of Y; in other words, v is the least upper bound of Y in the system $\langle P, \leq \rangle$. Hence, by passing to the dual lattices $\mathfrak{A}'$ and $\mathfrak{B}'$, we see that there exists also a greatest lower bound of Y in $[P, \leq]$. Since Y is an arbitrary subset of P, we finally conclude that

(6) the system $\langle P, \leq \rangle$ is a complete lattice.

In view of (4)-(6), the proof has been completed.

By the theorem just proved, the existence of a fixpoint for every increasing function is a necessary condition for the completeness of a lattice. The question naturally arises whether this condition is also sufficient. It has been shown that the answer to this question is affirmative.[3]

A set F of functions is called *commutative* if

(i) all the functions of F have a common domain, say B, and the ranges of all functions of F are subsets of B;

(ii) for any $f, g \in F$ we have $fg = gf$, that is,

$$f(g(x)) = g(f(x)) \text{ for every } x \in B.$$

Using this notion we can improve Theorem 1 in the following way:

THEOREM 2 (GENERALIZED LATTICE-THEORETRICAL FIXPOINT THEOREM). *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a complete lattice,*

(ii) F *be any commutative set of increasing functions on* A *to* A,

(iii) P *be the set of all common fixpoints of all the functions* $f \in F$.

Then the set P *is not empty and the system* $\langle P, \leq \rangle$ *is a complete lattice; in particular, we have*

$$\bigcup P = \bigcup E_x [f(x) \geq x \text{ for every } f \in F] \in P$$

and

$$\bigcap P = \bigcap E_x [f(x) \leq x \text{ for every } f \in F] \in P.$$

Proof. Let

(1) $$u = \bigcup E_x [f(x) \geq x \text{ for every } f \in F].$$

As in the proof of Theorem 1 we show that

(2) $$u \leq f(u) \text{ for every } f \in F.$$

Given any function $g \in F$, we have, by (2),

[3] This is a result of Anne C. Davis; see her note [2] immediately following this this paper.

$$g(u) \leq g(f(u))$$

and hence, the set F being commutative,

$$g(u) \leq f(g(u))$$

for every $f \in F$. Thus

$$g(u) \in \mathsf{E}_x[f(x) \geq x \;\; for\ every \;\; f \in F].$$

Therefore, by (1),

$$g(u) \leq u;$$

since g is an arbitrary function of F, we have

(3) $$f(u) \leq u \text{ for every } f \in F.$$

From (1)-(3) we conclude that u is a common fixpoint of all functions $f \in F$, and, in fact, the least upper bound of all such common fixpoints. In other words,

$$\mathsf{U}P = \mathsf{U}\mathsf{E}_x[f(x) \geq x \;\; for\ every \;\; f \in F] \in P.$$

In its remaining part the proof is entirely analogous to that of Theorem 1.

Since every set consisting of a single function is obviously commutative, Theorem 2 comprehends Theorem 1 as a particular case. Theorem 2 will not be involved in our further discussion.

2. Applications and extensions in the theories of simply ordered sets and real functions. A *simply ordered system* $\mathfrak{A} = \langle A, \leq \rangle$, that is, a system formed by a nonempty set A and a binary relation $\leq$ which establishes a simple order in A, is obviously a lattice. If it is a complete lattice, it is called a *continuously* (or *completely*) *ordered system*. The system $\mathfrak{A}$ is said to be a *densely ordered system* if, for all $x, y \in A$ with $x < y$, there is a $z \in A$ with $x < z < y$.

Theorems 1 and 2 obviously apply to every continuously ordered system $\mathfrak{A}$. Under the additional assumption that $\mathfrak{A}$ is densely ordered we can improve Theorem 1 by introducing the notions of quasi-increasing and quasi-decreasing functions.

Given a function f and a subset X of its domain, we denote by $f^*(X)$ the set of all elements $f(x)$ correlated with elements $x \in X$. A function f on B to C, where B and C are any two subsets of A, is called *quasi-increasing* if it satisfies the formulas

$$f(\bigcup X) \geq \bigcap f^*(X) \quad \text{and} \quad f(\bigcap X) \leq \bigcup f^*(X)$$

for every nonempty subset X of B. It is called *quasi-decreasing* if it satisfies the formulas

$$f(\bigcup X) \leq \bigcup f^*(X) \text{ and } f(\bigcap X) \geq \bigcap f^*(X)$$

for every nonempty subset X of A. A function which is both quasi-increasing and quasi-decreasing is called *continuous*.

THEOREM 3. *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a continuously and densely ordered set,*

(ii) f *be a quasi-increasing function and* g *a quasi-decreasing function on* A *to* A *such that*

$$f(0) \geq g(0) \textit{ and } f(1) \leq g(1),$$

(iii) $P = \mathsf{E}_x[f(x) = g(x)]$.

Then P *is not empty and* $\langle P, \leq \rangle$ *is a continuously ordered system; in particular we have*

$$\bigcup P = \bigcup \mathsf{E}_x[f(x) \geq g(x)] \in P$$

and

$$\bigcap P = \bigcap \mathsf{E}_x[f(x) \leq g(x)] \in P.$$

Proof. Let B be any subset of A such that

(1) $$f(x) \geq g(x) \text{ for } x \in B.$$

Assume that

(2) $$f(\bigcup B) < g(\bigcup B).$$

Since, by hypothesis, $f(0) \geq g(0)$, we conclude that

(3) $$\bigcup B \neq 0.$$

The system $\mathfrak{A}$ being densely ordered, we also conclude from (2) that there is an element $a \in A$ for which

(4) $$f(\bigcup B) < a < g(\bigcup B).$$

Let

(5) $$D = \mathsf{E}_x[x \leq \bigcup B \ \textit{and} \ g(x) \leq a],$$

whence

(6) $$\bigcup D \leq \bigcup B$$

and

(7) $$\bigcup g^*(D) \leq a.$$

If $\bigcup D = \bigcup B$, we see from (3) that $\bigcup D \neq 0$ and that consequently the set D is not empty; hence, the function g being by hypothesis quasi-decreasing, we obtain

$$g(\bigcup B) = g(\bigcup D) \leq \bigcup g^*(D),$$

and therefore, by (7),

$$g(\bigcup B) \leq a.$$

Since this formula clearly contradicts (4) we conclude that $\bigcup D \neq \bigcup B$ and thus, by (6),

(8) $$\bigcup D < \bigcup B.$$

Let

(9) $$E = \mathsf{E}_x[\bigcup D < x \ \textit{and} \ x \in B].$$

If the set E were empty, we would have $x \leq \bigcup D$ for every $x \in B$ and consequently $\bigcup B \leq \bigcup D$, in contradiction to (8). Hence E is not empty. We easily conclude by (9) that $\bigcup E = \bigcup B$. Since, by hypothesis, the function f is quasi-increasing, we have

$$f(\bigcup B) = f(\bigcup E) \geq \bigcap f^*(E)$$

and therefore, by (4),

$$a > \bigcap f^*(E).$$

Hence we must have $a > f(z)$ for some $z \in E$, for otherwise

$$a \leq \bigcap f^*(E).$$

Thus, by (1) and (9),

$$\bigcup D < z, \quad z \in B, \text{ and } g(z) \leq a;$$

therefore, by (5), $z \in D$. The formulas

$$\bigcup D < z \text{ and } z \in D$$

clearly contradict each other.

We have thus shown that formula (2) cannot hold for any non-empty set B satisfying (1). In other words, we have

(10) $$f(\bigcup B) \geq g(\bigcup B) \text{ for every non-empty subset } B \text{ of}$$
$$\mathsf{E}_x[f(x) \geq g(x)].$$

By applying the result just obtained to the dual system $\mathfrak{A}' = \langle A, \geq \rangle$, we conclude that

(11) $$f(\bigcap C) \leq g(\bigcap C) \text{ for every subset } C \text{ of}$$
$$\mathsf{E}_x[f(x) \leq g(x)].$$

Now let Y be any subset (whether empty or not) of the set

$$P = \mathsf{E}_x[f(x) = g(x)],$$

and let

(12) $$u = \bigcup \mathsf{E}_x[f(x) \geq g(x) \textit{ and } x \leq \bigcap Y].$$

By (10) and (11) we have

(13) $$f(u) \geq g(u) \text{ and } f(\bigcap Y) \leq g(\bigcap Y).$$

Hence, in case $u = \bigcap Y$, we obtain at once

(14) $$f(u) = g(u), \text{ that is, } u \in P.$$

In case $u \neq \bigcap Y$ we see from (12) that $u < \bigcap Y$. The system $\mathfrak{A}$ being densely ordered, we conclude that

(15) $$u = \bigcap \mathsf{E}_x[u < x \leq \bigcap Y].$$

We also see from (12) that $f(x) < g(x)$ for every element x of the set

$$\mathsf{E}_x[u < x \leq \bigcap Y].$$

Hence, by (11) and (15), we obtain

$$f(u) \leq g(u),$$

and this formula, together with (13), implies (14) again. Thus we have shown that

(16) for every subset Y of P, if $u = \bigcup \mathsf{E}_x[f(x) \geq g(x)$ *and* $x \leq \bigcup Y]$, then $u \in P$.

Dually we have

(17) for every subset Y of P, if $v = \bigcap \mathsf{E}_x[f(x \leq g(x)$ *and* $x \geq \bigcap Y]$, then $v \in P$.

We see immediately that the element u in (16) is the largest element of P which is a lower bound of all elements of Y; in other words, u is the greatest lower bound of Y in the system $\langle P, \leq \rangle$. Similarly, the element v in (17) is the least upper bound of Y in $\langle P, \leq \rangle$. Consequently,

(18) $\langle P, \leq \rangle$ is a continuously ordered system.

Finally, let us take in (16) and (17) the empty set for Y, so that $\bigcap Y = 1$ and $\bigcup Y = 0$. We then easily arrive at formulas

(19) $$\bigcup P = \bigcup \mathsf{E}_x[f(x) \geq g(x)] \in P$$

and

(20) $$\bigcap P = \bigcap \mathsf{E}_x[f(x) \leq g(x)] \in P.$$

By (18)-(20) the proof is complete.

Every increasing function is clearly quasi-increasing. The identity function, $g(x) = x$ for every $x \in A$, is continuous, that is, both quasi-increasing and quasi-decreasing, and the same applies to every constant function, $g(x) = c \in A$ for every $x \in A$. Hence we can take in Theorem 3 an arbitrary increasing function for f and the identity function for g; we thus obtain Theorem 1 in its application to continuously and densely ordered systems. On the other hand, by taking for g a constant function, we arrive at:

THEOREM 4 (GENERALIZED WEIERSTRASS THEOREM). *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a continuously and densely ordered system,*

(ii) f *be a quasi-increasing function on* A *to* A *and* c *be an element of* A *such that*

$$f(0) \geq c \geq f(1),$$

(iii) $P = \mathsf{E}_x[f(x) = c]$.

Then P *is not empty and* $\langle P, \leq \rangle$ *is a continuously ordered system; in particular, we have*

$$\bigcup P = \bigcup \mathsf{E}_x[f(x) \geq c] \in P$$

and

$$\bigcap P = \bigcap \mathsf{E}_x[f(x) \leq c] \in P.$$

An analogous theorem for pseudo-decreasing functions can be derived from Theorem 3 by taking an arbitrary constant function for f.

It can be shown by means of simple examples that Theorems 3 and 4 do not extend either to arbitrary continuously ordered systems or to arbitrary complete lattices which satisfy the density condition (that is, in which, for any elements x and y, $x < y$ implies the existence of an element z with $x < z < y$).

We can generalize Theorem 3 by considering two simply ordered systems,

$$\mathfrak{A} = \langle A, \leq \rangle \text{ and } \mathfrak{B} = \langle B, \leq \rangle,$$

as well as two functions on A to B, a quasi-increasing function f and a quasi-decreasing function g. The system $\mathfrak{A}$ is assumed to be continuously and densely ordered. No such assumptions regarding B are needed. Instead, the definitions of quasi-increasing and quasi-decreasing functions must be slightly modified. For example, a function f on A to B will be called quasi-increasing if, for every non-empty subset X of A and for every $b \in B$ we have

$$f(\bigcup X) \geq b \text{ whenever } f(x) \geq b \text{ for every } x \in X$$

and

$$f(\cap X) \leq b \text{ whenever } f(x) \leq b \text{ for every } x \in X.$$

By repeating with small changes the proof of Theorem 3, we see that under these assumptions the conclusions of the theorem remain valid. (The only change which is not obvious is connected with the fact that the system $\mathfrak{B}$ is not assumed to be densely ordered; therefore we cannot claim the existence of an element $a \in B$ which satisfies (4), and we have to distinguish two cases, dependent on whether an element a with this property exists or not.) Theorem 4 can of course be generalized in the same way.

Theorems 3 and 4 thus generalized can be applied in particular to real functions defined on a closed interval $[a,b]$ of real numbers. In application to real functions Theorem 3 can easily be derived from Theorem 4. In fact, if f is a quasi-increasing real function and g a quasi-decreasing real function on the interval $[a,b]$, then the function f' defined by the formula

$$f'(x) = f(x) - g(x)$$

is clearly quasi-increasing; by applying Theorem 4 to this function, we obtain the conclusions of Theorem 3 for f and g. Hence the fixpoint theorem (Theorem 1) for increasing real functions is also a simple consequence of Theorem 4. Finally, since every continuous function is quasi-increasing, and since, in the real domain, continuous functions in our terminology coincide with continuous functions in the usual sense, Theorem 4 is a generalization of the well-known Weierstrass theorem on continuous real functions.[4]

Returning to Theorem 3 for simply ordered systems, if we assume that both functions f and g are continuous, we can strengthen the conclusion of the theorem; in fact we can show, not only that the system $\langle P, \leq \rangle$ is continuously

[4] Theorem 3 (for both simply ordered systems and real functions) was originally stated under the assumption that the function f is increasing and the function g is continuous; see [3]. In 1949 A. P. Morse noticed that this result in the real domain could be improved; in fact, he obtained Theorem 4 for real functions—under a different, though equivalent, definition of a quasi-increasing function. By his definition, a real function f on an interval $[a,b]$ is quasi-increasing if it is upper semicontinuous on the left and lower semicontinuous on the right, that is, if

$$\text{(i)} \qquad \overline{\lim_{x \to d-}} f(x) \leq f(d) \leq \underline{\lim_{x \to d+}} f(x) \text{ for every } d \in [a,b].$$

By generalizing this observation, the author arrived at the present abstract formulations of Theorems 3 and 4. According to a recent remark of Morse, the first part of the conclusion of Theorem 4, that is, the statement that the set P is not empty, holds in the real domain for a still more comprehensive class of functions; in fact, for all real functions which satisfy the condition obtained from (i) by replacing $\underline{\lim}$ by $\overline{\lim}$ on the right side of the double inequality (or else by replacing $\overline{\lim}$ by $\underline{\lim}$ on the left side).

ordered, but also that, for every nonempty subset X of P, the least upper bound of X in $\langle P, \leq \rangle$ coincides with the least upper bound of X in $\langle A, \leq \rangle$, and similarly for the greatest lower bound. In application to real functions this means that the set P of real numbers is, not only continuously ordered, but also closed in the topological sense. Analogous remarks apply to Theorem 4.

3. Applications to Boolean algebras and the theory of set-theoretical equivalence. As is known, a *Boolean algebra* can be defined as a lattice $\mathfrak{A} = \langle A, \leq \rangle$, with 0 and 1, in which for every element $b \in A$ there is a uniquely determined element $\overline{b} \in A$ (called the *complement* of b), such that

$$b \cup \overline{b} = 1 \text{ and } b \cap \overline{b} = 0.$$

Given any two elements $a,b \in A$, we shall denote by $a - b$ their difference, that is, the element $a \cap \overline{b}$. If $\mathfrak{A} = \langle A, \leq \rangle$ is a Boolean algebra and $a \in A$, then $\mathfrak{A}' = \langle [0,a], \leq \rangle$ is also a Boolean algebra, though the complement of an element b in $\mathfrak{A}'$ does not coincide with the complement of b in $\mathfrak{A}$.

By applying the lattice-theoretical fixpoint theorem we obtain:

THEOREM 5. *Let*

(i) $\mathfrak{A} = \langle A, \leq \rangle$ *be a complete Boolean algebra,*

(ii) a,b *be any elements of* A, f *be an increasing function on* $[0,a]$ *to* A, *and* g *an increasing function on* $[0,b]$ *to* A.

Then there are elements $a',b' \in A$ *such that*

$$f(a - a') = b' \text{ and } g(b - b') = a'.$$

Proof. Consider the function h defined by the formula

$$h(x) = f(a - g(b - x)) \text{ for every } x \in A. \tag{1}$$

Let x and y be any elements in A such that

$$x \leq y.$$

We have then

$$b - x \geq b - y;$$

and since $b - x$ and $b - y$ are in $[0,b]$, and g is an increasing function on $[0,b]$ to A, we conclude that

$$g(b-x) \geq g(b-y)$$

and

$$a-g(b-x) \leq a-g(b-y).$$

Hence, the elements $a-g(b-x)$ and $a-g(b-y)$ being in $[0,a]$, and f being an increasing function on $[0,a]$ to A, we obtain

$$f(a-g(b-x)) \leq f(a-g(b-y)),$$

that is, by (1),

$$h(x) \leq h(y).$$

Thus h is an increasing function on A to A, and consequently, by Theorem 1, it has a fixpoint b'. Hence, by (1),

$$(2) \qquad f(a-g(b-b')) = b'.$$

We put

$$(3) \qquad g(b-b') = a'.$$

From (2) and (3) we see at once that the elements a' and b' satisfy the conclusion of our theorem.

If in the hypothesis of Theorem 5 we assume in addition that $f(a) \leq b$ and $g(b) \leq a$, we can obviously improve the conclusion by stating that there are elements $a', a'', b', b'' \in A$ for which

$$a = a' \cup a'', \quad b = b' \cup b'', \quad a' \cap a'' = b' \cap b'' = 0,$$

$$f(a'') = b' \text{ and } g(b'') = a'.$$ [5]

Theorem 5 has interesting applications in the discussion of homogeneous elements. Given a Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$, two elements $a, b \in A$ are called *homogeneous*, in symbols $a \approx b$, if the Boolean algebras $\langle [0,a], \leq \rangle$ and $\langle [0,b], \leq \rangle$ are isomorphic. In other words, $a \approx b$ if and only if there is a function f satisfying the following conditions: the domain of f is $[0,a]$; the range of f is $[0,b]$; the formulas $x \leq y$ and $f(x) \leq f(y)$ are equivalent for any

[5] In this more special form Theorem 5 is a generalization of a set-theoretical theorem obtained by Knaster and the author; see [3].

$x, y \in [0, a]$. Various fundamental properties of the homogeneity relation easily follow from this definition; for example, we have:

THEOREM 6. *$\mathfrak{A} = \langle A, \leq \rangle$ being an arbitrary Boolean algebra,*

(i) *$a \approx a$ for every $a \in A$;*

(ii) *if $a, b \in A$ and $a \approx b$, then $b \approx a$;*

(iii) *if $a, b, c \in A$, $a \approx b$, and $b \approx c$, then $a \approx c$;*

(iv) *if $a_1, a_2, b_1, b_2 \in A$, $a_1 \cap a_2 = 0 = b_1 \cap b_2$,*
$a_1 \approx b_1$, and $a_2 \approx b_2$, then $a_1 \cup a_2 \approx b_1 \cup b_2$;

(v) *if $a, b_1, b_2 \in A$, $b_1 \cap b_2 = 0$, and $a \approx b_1 \cup b_2$,*
then there are elements $a_1, a_2 \in A$ such that $a_1 \cup a_2 = a$, $a_1 \cap a_2 = 0$, $a_1 \approx b_1$, and $a_2 \approx b_2$.

In what follows we shall use parts (i)-(iii) of Theorem 6 without referring to them explicitly. If now we restrict our attention to complete Boolean algebras, we can establish various deeper properties of the homogeneity relation by applying Theorem 5. We start with the following:

THEOREM 7. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if*

$$a, b_1, b_2, c, d \in A, \quad b_1 \cap b_2 = 0, \quad c \approx d, \quad \text{and } a \cup c \approx b_1 \cup b_2 \cup d,$$

then there are elements $a_1, a_2 \in A$ such that

$$a_1 \cup a_2 = a, \quad a_1 \cap a_2 = 0, \quad a_1 \cup c \approx b_1 \cup d, \quad \text{and} \quad a_2 \cup c \approx b_2 \cup d.$$

Proof. By the definition of homogeneity, the formula $c \approx d$ implies the existence of a function f which maps isomorphically the Boolean algebra $\langle [0, c], \leq \rangle$ onto the Boolean algebra $\langle [0, d], \leq \rangle$; we have in particular

$$(1) \qquad f(c) = d.$$

Similarly, the formula $a \cup c \approx b_1 \cup b_2 \cup d$ implies the existence of a function g which maps isomorphically $\langle [0, b_1 \cup b_2 \cup d], \leq \rangle$ onto $\langle [0, a \cup c], \leq \rangle$, and we have

$$(2) \qquad g(b_1 \cup b_2 \cup d) = a \cup c.$$

We can assume for a while that the domain of g has been restricted to the interval $[0, b_1 \cup d]$. Thus, f is an increasing function on $[0,c]$ to A, g is an increasing function on $[0, b_1 \cup d]$ to A, and by applying Theorem 5 we obtain two elements c', d' such that

(3) $$f(c - c') = d' \text{ and } g((b_1 \cup d) - d') = c'.$$

The functions f and g being increasing, formulas (1)-(3) imply

(4) $$d' \leq d \text{ and } c' \leq a \cup c.$$

We now let

(5) $$a_1 = c' - c \text{ and } a_2 = a - a_1.$$

By (4) we have $c' - c \leq a$, and hence, by (5),

(6) $$a_1 \cup a_2 = a \text{ and } a_1 \cap a_2 = 0.$$

From (4) and (5) we also obtain

(7) $$(c - c') \cup c' = a_1 \cup c \text{ and } (c - c') \cap c' = 0,$$

(8) $$d' \cup [(b_1 \cup d) - d'] = b_1 \cup d \text{ and } d' \cap [(b_1 \cup d) - d'] = 0.$$

Since f maps isomorphically $\langle [0,c], \leq \rangle$ onto $\langle [0,d], \leq \rangle$, we conclude from (3) that it also maps isomorphically $\langle [0,c - c'], \leq \rangle$ onto $\langle [0,d'], \leq \rangle$ and that consequently

(9) $$c - c' \approx d'.$$

Analogously, by (3),

(10) $$c' \approx (b_1 \cup d) - d'.$$

By Theorem 6 (iv), formulas (7)-(10) imply

(11) $$a_1 \cup c \approx b_1 \cup d.$$

Furthermore, from (4) and (5) we derive

(12) $$(c \cap c') \cup [(a \cup c) - c'] = a_2 \cup c \text{ and } (c \cap c') \cap [(a \cup c) - c'] = 0,$$

$$(13) \quad (d-d')\cup[(b_2-d)\cup d'] = b_2\cup d \text{ and } (d-d')\cap[(b_2-d)\cup d'] = 0.$$

The function f being an isomorphic transformation, we obtain, with the help of (1) and (3),

$$f(c\cap c') = f(c-(c-c')) = f(c)-f(c-c') = d-d',$$

and hence, by arguing as above in the proof of (9),

$$(14) \qquad c\cap c' \approx d-d'.$$

Since, by (4) and the hypothesis,

$$(b_2-d)\cup d' = (b_1\cup b_2\cup d)-[(b_1\cup d)-d'],$$

we conclude analogously, with the help of (2) and (3), that

$$g(b_2-d)\cup d') = g(b_1\cup b_2\cup d)-g((b_1\cup d)-d') = (a\cup c)-c'$$

and therefore

$$(15) \qquad (a\cup c)-c' \approx (b_2-d)\cup d'.$$

From (12)-(15), by applying Theorem 6 (iv) again, we get

$$(16) \qquad a_2\cup c \approx b_2\cup d.$$

By (6), (11), and (16), the proof is complete.

In deriving the remaining theorems of this section we shall apply exclusively those properties of the homogeneity relation which have been established in Theorems 6 and 7; thus the results obtained will apply to every binary relation (between elements of a complete Boolean algebra) for which these two theorems hold. It may be noticed in this connection that Theorem 6 (v) restricted to complete Boolean algebras is a simple consequence of Theorems 6 (i) and 7.

THEOREM 8 (MEAN-VALUE THEOREM). *$\mathfrak{A} = \langle A, \leq\rangle$ being a complete Boolean algebra, if $a,b,c,a',c' \in A$, $a\leq b\leq c$, $a'\leq c'$, $a\approx a'$, and $c\approx c'$, then there is an element $b'\in A$ such that $a'\leq b'\leq c'$ and $b\approx b'$.*

Proof. We apply Theorem 7, with a,b_1,b_2,c,d respectively replaced by $c'-a'$, $b-a$, $c-b$, a', a, and we conclude that there are elements $a_1,a_2\in A$

such that

$$c' - a' = a_1 \cup a_2 \text{ and } (b - a) \cup a \approx a_1 \cup a'.$$

The element $b' = a_1 \cup a'$ clearly satisfies the conclusion of our theorem.

THEOREM 9. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, for any elements $a,b \in A$ the following two conditions are equivalent:*

(i) *there is an element $a_1 \in A$ such that $a \approx a_1 \leq b$;*

(ii) *there is an element $b_1 \in A$ such that $a \leq b_1 \approx b$.*

Proof. To derive (ii) from (i), we consider an arbitrary element a_1 satisfying (i), and we apply Theorem 8 with a,c,a',c' respectively replaced by a_1, 1, a, 1. The implication in the opposite direction follows immediately from Theorem 6 (v) (and hence holds in an arbitrary Boolean algebra).

THEOREM 10 (EQUIVALENCE THEOREM). *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a,b,c \in A$, $a \leq b \leq c$, and $a \approx c$, then $a \approx b \approx c$.*

Proof. This follows immediately from Theorem 8 with $a' = c' = c$.

THEOREM 11. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, for any elements $a_1,a_2,b \in A$ the formulas*

(i) $a_1 \cup b \approx a_2 \cup b \approx b$ *and*

and

(ii) $a_1 \cup a_2 \cup b \approx b$

are equivalent.

Proof. Obviously,

$$b \leq a_1 \cup b \leq a_1 \cup a_2 \cup b \text{ and } b \leq a_2 \cup b \leq a_1 \cup a_2 \cup b.$$

Hence (ii) implies (i) by Theorem 10.

Assume now, conversely, that (i) holds. We clearly have

$$[a_2 - (a_1 \cup b)] \cap (a_1 \cup b) = [a_2 - (a_1 \cup b)] \cap b = 0$$

and

$$a_2 - (a_1 \cup b) \approx a_2 - (a_1 \cup b).$$

By Theorem 6 (iv), these two formulas together with (i) imply

(1) $$a_1 \cup a_2 \cup b = [a_2 - (a_1 \cup b)] \cup (a_1 \cup b) \approx [a_2 - (a_1 \cup b)] \cup b.$$

Since

$$[a_2 - (a_1 \cup b)] \cup b \leq a_2 \cup b \leq a_1 \cup a_2 \cup b,$$

we derive from (1), by applying Theorem 10,

(2) $$a_2 \cup b \approx a_1 \cup a_2 \cup b.$$

Formulas (i) and (2) obviously imply (ii), and the proof is complete.

Various properties of the relation of homogeneity can conveniently be expressed in terms of another, related relation which is denoted by $\precsim$. Thus $\mathfrak{A} = \langle A, \leq \rangle$ being a Boolean algebra, and a,b being any elements of A, we write $a \precsim b$ if there is an element $a_1 \in A$ such that $a \approx a_1 \leq b$; in case the algebra $\mathfrak{A}$ is complete, an equivalent formulation of this condition is given in Theorem 9(ii). Theorems 8 and 10 can now be put in a somewhat simpler, though essentially equivalent, form:

MEAN-VALUE THEOREM. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a,b,c \in A$, $a \leq c$, and $a \precsim b \precsim c$, then there is an element $b' \in A$ such that $a \leq b' \leq c$ and $b \approx b'$.*

EQUIVALENCE THEOREM. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a,b \in A$, $a \precsim b$, and $b \precsim a$, then $a \approx b$.*

We shall give two further results formulated in terms of $\precsim$.

THEOREM 12. *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if*

$$a_1,a_2,c_1,c_2 \in A,\ a_1 \leq c_1,\ a_1 \precsim c_2,\ a_2 \precsim c_1,\ \textit{and}\ a_2 \leq c_2,$$

then there are elements $b_1,b_2 \in A$ such that $a_1 \leq b_1 \leq c_1$, $a_2 \leq b_2 \leq c_2$, and $b_1 \approx b_2$.

Proof. The hypothesis implies the existence of two elements a_1',a_2' such that

(1) $$a_1 \approx a_1' \leq c_2 \text{ and } a_2 \approx a_2' \leq c_1.$$

Since, by (1),

$$a_1 \cap a_2' \leq a_1 \approx a_1',$$

we conclude from Theorem 9 that there is an element d for which

(2) $$a_1 \cap a_2' \approx d \leq a_1'.$$

We have, by (1),

$$a_2 \approx (a_1 \cap a_2') \cup (a_2' - a_1) \text{ and } (a_1 \cap a_2') \cap (a_2' - a_1) = 0;$$

hence, by Theorem 6(v), there are elements e_1, e_2 such that

(3) $$a_2 = e_1 \cup e_2 \text{ and } e_1 \cap e_2 = 0,$$

(4) $$e_1 \approx a_1 \cap a_2' \text{ and } e_2 \approx a_2' - a_1.$$

By (1)-(4) and the hypothesis,

$$d \leq a_1' \leq c_2, \quad e_1 < c_2, \quad d \approx e_1, \text{ and } c_2 \approx c_2;$$

hence, by Theorem 8, there is an element f for which

(5) $$e_1 \leq f \leq c_2 \text{ and } a_1' \approx f.$$

Since, by (4),

$$e_2 - f \leq e_2 \approx a_2' - a_1,$$

Theorem 9 implies the existence of an element g with

(6) $$e_2 - f \approx g \leq a_2' - a_1.$$

We now put

(7) $$b_1 = a_1 \cup g \text{ and } b_2 = f \cup (e_2 - f) = f \cup e_2.$$

By (1), (3), (5), (6), (7), and the hypothesis, we obtain

(8) $$a_1 \leq b_1 \leq c_1 \text{ and } a_2 \leq b_2 \leq c_2.$$

By (5) and (6) we have

$$a_1 \cap g = f \cap (e_2 - f) = 0, \quad a_1 \approx f, \quad g \approx e_2 - f;$$

hence, by (7) and Theorem 6(iv), we get

(9) $$b_1 \approx b_2.$$

From (8) and (9) we see that the elements b_1 and b_2 satisfy the conclusion of our theorem.

From the theorem just proved, by letting $a_1 = c_1$, we derive as an immediate consequence the mean-value theorem; if we put $a_1 = c_1$ and $a_2 = c_2$, we obtain the equivalence theorem. A further consequence of Theorem 12 is:

THEOREM 13 (INTERPOLATION THEOREM). *$\mathfrak{A} = \langle A, \leq \rangle$ being a complete Boolean algebra, if $a_1, a_2, c_1, c_2 \in A$ and $a_i \preceq c_j$ for $i,j = 1,2$, then there is an element $b \in A$ such that $a_i \preceq b \preceq c_j$ for $i,j = 1,2$.*

Proof. The hypothesis implies the existence of two elements a_1' and a_2' for which

$$(1) \qquad a_1 \approx a_1' \leq c_1 \text{ and } a_2 \approx a_2' \leq c_2 .$$

Hence, as is easily seen,

$$a_1' \leq c_1,\ a_1' \preceq c_2,\ a_2' \preceq c_1,\ a_2' \leq c_2 .$$

Consequently, by Theorem 12, there are elements b_1, b_2 such that

$$(2) \qquad a_1' \leq b_1 \leq c_1,\ a_2' \leq b_2 \leq c_2, \text{ and } b_1 \approx b_2 .$$

From (1) and (2), with the help of Theorem 9, we obtain

$$a_i \preceq b_1 \preceq c_j \qquad \text{for} \qquad i,j = 1,2 .$$

Thus the element $b = b_1$ satisfies the conclusion of our theorem.

From Theorems 7 and 11-13 we obtain by induction more general results in which the couples $\langle a_1, a_2 \rangle$, $\langle b_1, b_2 \rangle$ $\langle c_1, c_2 \rangle$ are replaced by finite sequences

$$\langle a_1, \cdots, a_n \rangle,\ \langle b_1, \cdots, b_n \rangle,\ \langle c_1, \cdots, c_n \rangle$$

with an arbitrary number n of terms; in Theorem 13 the couples $\langle a_1, a_2 \rangle$ and $\langle c_1, c_2 \rangle$ can be replaced by two finite sequences with different numbers of terms. The results discussed can be further extended to infinite sequences; however, these extensions seem to require a different method of proof, and we see no way of deriving them by means of elementary arguments from the fixpoint theorem of § 1.[6]

[6] Theorems 6-13 concerning the relation of homogeneity and their applications to cardinal products of Boolean algebras and to the theory of set-theoretical equivalence are not essentially new. (Theorem 12 is new, but it can be regarded simply as a new formulation of the interpolation theorem 13.) All these results are stated explicitly or implicitly in [7, §§ 11, 12, 15-17], where historical references to earlier publications can also be found. However, the method applied in [7] is different from that in the present paper and is not directly related to any fixpoint theorem. Also, the axiom of choice is not involved at all in the present discussion, while the situation in [7] is in this respect more complicated (compare, for instance, the remarks starting on page 239).

All the results of this section, except Theorem 5, remain valid if the Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$ is assumed to be not necessarily complete, but only *countably-complete* (*σ-complete*). This can be seen in the following way. To prove Theorem 5 we have constructed, in terms of two given increasing functions f and g, a new function h, and we have shown that this function h is increasing and hence has a fixpoint. In the subsequent discussion, Theorem 5 has been applied only once, namely in the proof of Theorem 7. The functions f and g involved in this application not only are increasing, but have much stronger properties, in fact, the *distributive properties* under countable joins and meets; that is, for every infinite sequence $\langle a_1, \cdots, a_n, \cdots \rangle$ we have

$$f(a_1 \cup \cdots \cup a_n \cup \cdots) = f(a_1) \cup \cdots \cup f(a_n) \cup \cdots,$$

$$f(a_1 \cap \cdots \cap a_n \cap \cdots) = f(a_1) \cap \cdots \cap f(a_n) \cap \cdots,$$

and similarly for g. It can be shown that the function h constructed from f and g in the way indicated in the proof of Theorem 5 also has these distributive properties. It is also easily seen that, in any countably-complete Boolean algebra (and, more generally, in any countably-complete lattice with 0), every function h which is distributive under countable joins has at least one fixpoint a; in fact,

$$a = 0 \cup h(0) \cup h(h(0)) \cup \cdots.$$

The results obtained in this section have interesting consequences concerning isomorphism of cardinal (direct) products of Boolean algebras. To obtain these consequences it suffices to notice that every system of Boolean algebras $\langle \mathfrak{A}_i \rangle$ can be represented by means of a system of disjoint elements $\langle a_i \rangle$ of a single Boolean algebra $\mathfrak{A}$ (in fact, of the cardinal product of all algebras $\mathfrak{A}_i$) in such a way that (i) each algebra $\mathfrak{A}_i$ is isomorphic to the subalgebra $\langle [0, a_i], \leq \rangle$ of $\mathfrak{A}$; hence (ii) two algebras $\mathfrak{A}_i$ and $\mathfrak{A}_j$ are isomorphic ($\mathfrak{A}_i \cong \mathfrak{A}_j$) if and only if the elements a_i and a_j are homogeneous ($a_i \approx a_j$); (iii) for $i \neq j$, we have $\mathfrak{A}_i \times \mathfrak{A}_j = \mathfrak{A}_k$ if and only if $a_i \cup a_j \approx a_k$; (iv) $\mathfrak{A}_i$ is isomorphic to a factor of $\mathfrak{A}_k$ if and only if $a_i \preceq a_k$. Keeping this in mind, we derive, for example, the following corollary from Theorem 11:

$\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}$ *being three complete Boolean algebras, we have*

$$\mathfrak{A}_1 \times \mathfrak{B} \cong \mathfrak{A}_2 \times \mathfrak{B} \cong \mathfrak{B}$$

if and only if

$$\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{B} \cong \mathfrak{B}.$$

Results of this type can again be extended to countably-complete Boolean algebras.

Any given sets A, B, C, $\cdots$ can be regarded as elements of a complete Boolean algebra; in fact, of the algebra formed by all subsets of the union $A \cup B \cup C \cup \cdots$, with set-theoretical inclusion as the fundamental relation. As is easily seen, two sets A and B treated this way are homogeneous in the Boolean-algebraic sense if and only if they are set-theoretically equivalent, that is, have the same power. Hence, as particular cases of theorems on homogeneous elements, we obtain various results concerning set-theoretical equivalence; for instance, Theorem 10 yields the well-known Cantor-Bernstein theorem.[7]

4. **Applications to topology.**[8] By a *derivative algebra* we understand a system $\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$ in which $\langle A, \leq \rangle$ is a Boolean algebra and D is a unary operation (function) on A to A assumed to satisfy certain simple postulates; the main consequence of these postulates which is involved in our further discussion is the fact that D is increasing. The element $\mathrm{D}x$ (for any given $x \in A$) is referred to as the *derivative* of x. The derivative algebra $\mathfrak{A}$ is called *complete* if the Boolean algebra $\langle A, \leq \rangle$ is complete.

In topology the notion of the derivative of a set is either treated as a fundamental notion in terms of which the notion of a topological space is characterized, or else it is defined in terms of other fundamental notions (for example, the derivative of a point set X is defined as the set of all limit points of X). At any rate, all point sets of a topological space form a complete derivative algebra under the set-theoretical relation of inclusion and the topological operation of derivative. Hence the theorems on complete derivative algebras can be applied to arbitrary topological spaces.

$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$ being a derivative algebra, an element $a \in A$ is called *closed* if $\mathrm{D}a \leq a$; it is called *dense-in-itself* if $\mathrm{D}a \geq a$, and *perfect* if $\mathrm{D}a = a$; it is called *scattered* if there is no element $x \leq a$ different from 0 which is dense-in-itself.

As a consequence of the fixpoint theorem we obtain:

THEOREM 14 (GENERALIZED CANTOR-BENDIXON THEOREM).

[7] These extensions can be found in [7]. The proof of Theorems 12 and 13 extended to infinite sequences requires an application of the axiom of choice (to denumerable families of sets). Compare the preceding footnote.

[8] In connection with this section see [4, pp. 182 f.]; compare also [5], in particular pp. 38 f. and 44.

$$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$$

being a complete derivative algebra, every closed element $a \in A$ has a decomposition

$$a = b \cup c, \quad b \cap c = 0,$$

where the element $b \in A$ is perfect and the element $c \in A$ is scattered.

Proof. We put

(1) $$b = \mathsf{U}\mathrm{E}_x [a \cap \mathrm{D}x \geq x] \text{ and } c = a - b.$$

Hence obviously

(2) $$a = b \cup c \text{ and } b \cap c = 0.$$

D being an increasing function on A to A, the same clearly applies to the function D_a defined by the formula

$$\mathrm{D}_a x = a \cap \mathrm{D}x \text{ for every } x \in A.$$

Hence, by Theorem 1, we conclude from (1) that b is a fixpoint of D_a; that is,

(3) $$b = \mathrm{D}_a b = a \cap \mathrm{D}b.$$

Consequently $b \leq a$ and $\mathrm{D}b \leq \mathrm{D}a$; since the element a is closed, we have $\mathrm{D}a \leq a$, $\mathrm{D}b \leq a$ and therefore, by (3), $b = \mathrm{D}b$; that is, the element b is perfect. If an element $x \leq c$ is dense-in-itself, that is, $\mathrm{D}x \geq x$, we have, by (1),

$$a \cap \mathrm{D}x \geq x,$$

and hence $x \leq b$; therefore, by (2), $x = 0$. Thus the element c is scattered. This completes the proof.

It should be mentioned that the operation D in a derivative algebra

$$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$$

is assumed to be not only increasing, but distributive under finite joins, that is,

$$\mathrm{D}(x \cup y) = \mathrm{D}x \cup \mathrm{D}y \text{ for any } x, y \in A.$$

Under this assumption we can improve Theorem 14 by showing that every closed element $a \in A$ has a unique decomposition

$$a = b \cup c, \quad b \cap c = 0,$$

where b is perfect and c is scattered. In fact, let

$$a = b' \cup c', \quad b' \cap c' = 0$$

be another decomposition of this kind. We then have

$$b = \mathrm{D}b \leq \mathrm{D}(b \cup b') = \mathrm{D}((b - b') \cup b') = \mathrm{D}(b - b') \cup \mathrm{D}b' = \mathrm{D}(b - b') \cup b'.$$

Hence

$$b - b' \leq \mathrm{D}(b - b');$$

that is, $b - b'$ is dense-in-itself. Since, moreover, $b - b' \leq c'$, and c' is scattered, we conclude that $b - b' = 0$. Similarly we get $b' - b = 0$. Consequently $b = b'$, and hence also $c = c'$.

If, instead of Theorem 1, we apply Theorem 5, we obtain the following result (of which, however, no interesting topological consequences are known):

THEOREM 15. *$\mathfrak{A} = \langle A, \leq, \mathrm{D} \rangle$ being a complete derivative algebra, every closed element $a \in A$ has two decompositions*

$$a = b \cup c = b' \cup c', \quad b \cap c = b' \cap c' = 0,$$

where b, c, b', c' are elements of A such that

$$\mathrm{D}b' = b \text{ and } \mathrm{D}c' = c.$$

Proof. From Theorem 5 (with $a = b$) we conclude that there are two elements $c, b' \in A$ such that

(1) $$\mathrm{D}(a - c) = b' \text{ and } \mathrm{D}(a - b') = c.$$

By putting

(2) $$b = a - c \text{ and } c' = a - b'$$

we obtain, from (1),

(3) $$\mathrm{D}b = b' \text{ and } \mathrm{D}c' = c.$$

Since the function D is increasing and the element a is closed, (1) implies

$$c \leq \mathrm{D}a \leq a \text{ and } b' \leq \mathrm{D}a \leq a;$$

hence, by (2),

$$a = b \cup c = b' \cup c' \text{ and } b \cap c = b' \cap c' = 0. \tag{4}$$

By (3) and (4) the proof has been completed.

Theorems 1 and 5 can be applied not only to the operation D, but also to other topological operations which are defined in terms of D and, like the latter, are increasing; for instance, to the operation I defined by the formula

$$\mathrm{I}x = x - \mathrm{D}\overline{x};$$

$\mathrm{I}x$ referred to as the *interior* of the element x. Theorem 5 can of course be applied to two different topological operations, provided both are increasing.

REFERENCES

1. Garrett Birkhoff, *Lattice theory*, revised edition, Amer. Math. Soc. Colloq. Publ. vol. 25, New York, 1948.

2. Anne C. Davis, *A characterization of complete lattices*, Pacific J. Math. **5** (1955), 311-319.

3. B. Knaster, *Un théorème sur les fonctions d'ensembles*, Ann. Soc. Polon. Math. **6** (1928), 133-134.

4. J. C. C. McKinsey and Alfred Tarski, *The algebra of topology*, Ann. of Math. **45** (1944), 141-191.

5. Wacław, Sierpiński, *Introduction to general topology*, Toronto, 1934.

6. Alfred Tarski, *A fixpoint theorem for lattices and its applications* (preliminary report), Bull. Amer. Math. Soc. **55** (1949), 1051-1052 and 1192.

7. ———, *Cardinal algebras*, New York, 1949.

UNIVERSITY OF CALIFORNIA, BERKELEY

A CHARACTERIZATION OF COMPLETE LATTICES

ANNE C. DAVIS

1. Introduction. A complete lattice $\mathfrak{A} = \langle A, \leq \rangle$ has the property that every increasing function on A to A has a fixpoint.[1] Tarski raised the question whether the converse of this result also holds. In this note we shall show that the answer to this question is affirmative, thus establishing a criterion for completeness of a lattice in terms of fixpoints.[2]

We shall use the notation of [**6**]. In addition, the formula $a \nleq b$ will be used to express the fact that $a \leq b$ does not hold. By $\langle a_\xi ; \xi < \alpha \rangle$, where α is any (finite or transfinite) ordinal we shall denote the sequence whose consecutive terms are $a_0, a_1, \cdots, a_\xi, \cdots$ (with $\xi < \alpha$); the set of all terms of this sequence will be denoted by $\{a_\xi ; \xi < \alpha\}$. The sequence $\langle a_\xi ; \xi < \alpha \rangle$ is, of course, called *increasing*, or *strictly increasing*, if $a_\xi \leq a_{\xi'}$, or $a_\xi < a_{\xi'}$, for any $\xi < \xi' < \alpha$; analogously we define *decreasing* and *strictly decreasing* sequences.

2. A lemma. We start with the following:

LEMMA 1. *If the lattice* $\mathfrak{A} = \langle A, \leq \rangle$ *is incomplete, then there exist two sequences* $\langle b_\xi ; \xi < \beta \rangle$ *and* $\langle c_\eta ; \eta < \gamma \rangle$ *such that*

(i) $b_\xi < c_\eta$ *for every* $\xi < \beta$ *and every* $\eta < \gamma$,

(ii) $\langle b_\xi ; \xi < \beta \rangle$ *is strictly increasing and* $\langle c_\eta ; \eta < \gamma \rangle$ *is strictly decreasing,*

(iii) *there is no element* $a \in A$ *which is both an upper bound of* $\{b_\xi ; \xi < \beta\}$ *and a lower bound of* $\{c_\eta ; \eta < \gamma\}$.[3]

[1] See [**6**] (where further historical references can also be found).

[2] This result was found in 1950 and outlined in [**2**].

[3] A related, though weaker, property of incomplete lattices is mentioned implicitly in [**1**, p. 53, Exercise 4].

Received June 29, 1953. The present note was prepared while the author was working on a research project in the foundations of mathematics sponsored by the Office of Ordnance Research, U.S. Army.

Pacific J. Math. 5 (1955), 311-319

Proof. We first notice that there exists at least one subset of A without a least upper bound (for otherwise the lattice would be complete).[4] Hence we can find a subset B of A with the following properties:

(1) $\qquad \bigcup B$ does not exist;

(2) $\qquad$ if X is any subset of A with smaller power than B, then $\bigcup X$ exists.

Let β' be the initial ordinal of the same power as B (that is, the smallest ordinal such that the set of all preceding ordinals has the same power as B). The ordinal β' may be equal to 0; if not, β' is certainly infinite and, since it is initial, it has no predecessor; that is, $\xi < \beta'$ implies $\xi + 1 < \beta'$ for every ordinal ξ. Thus all the elements of B can be arranged in a sequence $\langle b'_\xi;\ \xi < \beta' \rangle$. For every $\xi < \beta'$, the set $\{ b'_\zeta;\ \zeta < \xi + 1\}$ is of smaller power than β' and therefore, by (2), its least upper bound

$$u_\xi = \bigcup \{ b'_\zeta;\ \zeta < \xi + 1\}$$

exists. The sequence $\langle u_\xi;\ \xi < \beta' \rangle$ is clearly increasing but not necessarily strictly increasing. However, by omitting repeating terms in this sequence, we obtain a strictly increasing sequence $\langle b_\xi;\ \xi < \beta \rangle$, where β is an ordinal $\leq \beta'$, such that

$$\{ b_\xi;\ \xi < \beta\} = \{ u_\xi;\ \xi < \beta' \}.$$

(Actually, one can easily prove that $\beta = \beta'$.) Obviously,

(3) $\qquad$ for every $b \in B$ there is a $\xi < \beta$ such that $b \leq b_\xi$;

(4) $\qquad$ for every $\xi < \beta'$ there is a subset X of B such that $b_\xi = \bigcup X$.

By (3) and (4), if the least upper bound $\bigcup \{ b_\xi;\ \xi < \beta\}$ existed, it would coincide with $\bigcup B$; hence, by (1),

(5) $\qquad \bigcup \{ b_\xi;\ \xi < \beta\}$ does not exist.

Let C be the set of all upper bounds of $\{ b_\xi;\ \xi < \beta\}$. Clearly $\bigcap C$ does not exist, for if it did, it would coincide with $\bigcup \{ b_\xi;\ \xi < \beta\}$; this result would contradict (5). Now C, like B, is either empty or infinite. Since C is partly ordered by the relation $\leq$, there is a strictly decreasing sequence $\langle c_\eta;\ \eta < \gamma \rangle$ such

[4] See [1, p. 49].

that $\{c_\eta;\ \eta < \gamma\}$ is a subset of C with which C is coinitial (that is, there is no element of C which is a lower bound of $\{c_\eta;\ \eta < \gamma\}$ without belonging to $\{c_\eta;\ \eta < \gamma\}$).[5] If the greatest lower bound $\bigcap\{c_\eta;\ \eta < \gamma\}$ existed, it would be an upper bound of $\{b_\xi;\ \xi < \beta\}$; but since $\bigcup\{b_\xi;\ \xi < \beta\}$ does not exist, there would be an element $c \in C$ such that $\bigcap\{c_\eta;\ \eta < \gamma\} \not\leq c$. Hence we would have

$$c \cap \bigcap\{c_\eta;\ \eta < \gamma\} \in C \quad \text{and} \quad c \cap \bigcap\{c_\eta;\ \eta < \gamma\} < \bigcap\{c_\eta;\ \eta < \gamma\},$$

in contradiction to the assumption that C is coinitial with $\{c_\eta;\ \eta < \gamma\}$. Consequently,

$$\bigcap\{c_\eta;\ \eta < \gamma\} \text{ does not exist.} \tag{6}$$

The sequences $\langle b_\xi;\ \xi < \beta\rangle$ and $\langle c_\eta;\ \eta < \gamma\rangle$ obviously satisfy conditions (i) and (ii) of our lemma. To show that (iii) is also satisfied, assume that an element $a \in A$ is both an upper bound of $\{b_\xi;\ \xi < \beta\}$ and a lower bound of $\{c_\eta;\ \eta < \gamma\}$. We have then, by definition, $a \in C$. Hence, C being coinitial with $\{c_\eta;\ \eta < \gamma\}$, we must have

$$a \in \{c_\eta;\ \eta < \gamma\},$$

and therefore

$$a = \bigcap\{c_\eta;\ \eta < \gamma\},$$

in contradiction to (6). This completes the proof.

3. The main result. With the help of Lemma 1 we now obtain the main result of this note:

THEOREM 2. *For a lattice $\mathfrak{A} = \langle A, \leq\rangle$ to be complete it is necessary and sufficient that every increasing function on A to A have a fixpoint.*

Proof. Since the condition of the theorem is known to be necessary for the completeness of a lattice, we have only to show that it is sufficient. In other words, we have to show that, under the assumption that the lattice $\mathfrak{A} = \langle A, \leq\rangle$ is incomplete, there exists an increasing function f on A to A without fixpoints.

In fact, let $\langle b_\xi;\ \xi < \beta\rangle$ and $\langle c_\eta;\ \eta < \gamma\rangle$ be any two sequences satisfying conclusions (i)-(iii) of Lemma 1. To define f for any element $x \in A$, we distinguish two cases dependent upon whether x is a lower bound of $\{c_\eta;\ \eta < \gamma\}$ or not.

[5] Cf. [3, p. 141].

In the first case, by conclusion (iii), x is not an upper bound of $\{ b_\xi ; \xi < \beta \}$; that is, the set of ordinals

$$(1) \qquad \Phi(x) = \mathsf{E}_\xi [\xi < \beta \text{ and } b_\xi \nleq x]$$

is non-empty. We put

$$(2) \qquad \phi(x) = \min \Phi(x) \text{ and } f(x) = b_{\phi(x)}.$$

(Δ being any non-empty set of ordinals, min Δ is of course the smallest ordinal belonging to Δ.) In the second case, the set

$$(3) \qquad \Psi(x) = \mathsf{E}_\eta [\eta < \gamma \text{ and } x \nleq c_\eta]$$

is nonempty. We let

$$(4) \qquad \psi(x) = \min \Psi(x) \text{ and } f(x) = c_{\psi(x)}.$$

We have thus defined a function f on A to A. From (1)-(4) it follows clearly that either $f(x) \nleq x$ or $x \nleq f(x)$ for every $x \in A$; thus f has no fixpoints.

Let x and y be any elements of A with $x \leq y$. If x is a lower bound of $\{c_\eta ; \eta < \beta\}$ but y is not, then, by (1)-(4) and conclusion (i) of Lemma 1, $f(x) \leq f(y)$. If both x and y are lower bounds of $\{c_\eta, \eta < \gamma\}$, we see from (1) that $\Phi(y)$ is a subset of $\Phi(x)$; hence, by (2) and conclusion (ii) of Lemma 1, it follows at once that $f(x) \leq f(y)$. Finally, if x is not a lower bound of $\{c_\eta ; \eta < \gamma\}$, then y is not either, and by an argument analogous to that just outlined (using (3) and (4) instead of (1) and (2)) we again obtain $f(x) \leq f(y)$. Thus the function f is increasing, and the proof of the theorem is complete.

4. Extensions. More difficult problems seem to arise if we try to improve Theorem 2 by considering, instead of arbitrary increasing functions, more special classes of functions. In particular, we have in mind *join-distributive* (or *meet-distributive*) functions, that is, functions f on A to A which satisfy the formula

$$f(x \cup y) = f(x) \cup f(y) \quad (\text{or } f(x \cap y) = f(x) \cap f(y))$$

for all $x, y \in A$. The problem is open whether Theorem 2 remains valid if the term "increasing" is replaced by "join-distributive" or by "meet-distributive". We are going to give (in Theorem 4 below) a partial positive result concerning this problem.

The lattice $\mathfrak{A} = \langle A, \leq \rangle$ is called *α-join-complete* (or *α-meet-complete*) if $\cup X$ (or $\cap X$) exists for every nonempty subset X of A with power at most equal to $\aleph_\alpha$.

LEMMA 3. *Let $\mathfrak{A} = \langle A, \leq \rangle$ be an incomplete lattice with the set A of power $\aleph_\alpha$. If $\mathfrak{A}$ is δ-join-complete for every $\delta < \alpha$, then there exist two sequences $\langle b_\xi;\ \xi < \beta \rangle$ and $\langle c_\eta;\ \eta < \gamma \rangle$ which satisfy conclusions* (i)-(iii) *of Lemma* 1 *as well as the following condition*:

(iv) *if an element $x \in A$ is a lower bound of $\{c_\eta;\ \eta < \gamma\}$, then there exists an ordinal ξ such that $\xi < \beta$ and $x \leq b_\xi$.*

Proof. From Lemma 1 we easily conclude that there exists a strictly decreasing sequence $\langle c_\eta;\ \eta < \gamma \rangle$ of elements of A such that $\cap\{c_\eta;\ \eta < \gamma\}$ does not exist. Let B' be the set of all lower bounds of $\{c_\eta;\ \eta < \gamma\}$. Then clearly $\cup B'$ does not exist. Hence, by hypothesis, B' must be either empty or of power $\aleph_\alpha$; since $\mathfrak{A}$ is δ-join-complete for every $\delta < \alpha$, it follows that B' satisfies conditions (1) and (2) in the proof of Lemma 1 (with B replaced by B'). Therefore, by literally repeating the corresponding part of the proof of that lemma, we obtain a strictly increasing sequence $\langle b_\xi;\ \xi < \beta \rangle$ of elements of B' for which conditions (3)-(5) (with $B = B'$) hold. Obviously the sequences $\langle b_\xi;\ \xi < \beta \rangle$ and $\langle c_\eta;\ \eta < \gamma \rangle$ satisfy conclusions (i) and (ii) of Lemma 1. To show that conclusion (iii) is satisfied, assume, to the contrary, that a is both a lower bound of $\{c_\eta;\ \eta < \gamma\}$ and an upper bound of $\{b_\xi;\ \xi < \beta\}$. Therefore, by the definition of B', we have $a \in B'$; using (3) of the proof of Lemma 1 we see that $a \leq b_\xi$ for some $\xi < \beta$, and hence, a being an upper bound of $\{b_\xi;\ \xi < \beta\}$, we conclude that

$$a = \cup\{b_\xi;\ \xi < \beta\},$$

which contradicts (5). Finally, in view of the definition of B', conclusion (iv) of our present lemma simply coincides with condition (3) in the proof of Lemma 1 (again with $B = B'$).

With the help of Lemma 3 we now obtain:

THEOREM 4. *Let $\mathfrak{A} = \langle A, \leq \rangle$ be a lattice with the set A of power $\aleph_\alpha$. For A to be complete it is necessary and sufficient that*

(i) *$\mathfrak{A}$ be δ-join-complete for every $\delta < \alpha$ and*

(ii) *every join-distributive function on A to A have a fixpoint.*

Proof. If $\mathfrak{A}$ is complete, then obviously (i) holds. To show that the completeness of $\mathfrak{A}$ implies (ii) we need only note that every join-distributive function is increasing, and then apply Theorem 2. Thus (i) and (ii) are necessary conditions for the completeness of $\mathfrak{A}$.

In order to show that these conditions (jointly) are also sufficient, we assume that $\mathfrak{A}$ is an incomplete lattice which is δ-join-complete for every $\delta < \alpha$, and we show that there exists a join-distributive function f on A to A without fixpoints.

Let $\langle b_\xi;\ \xi < \beta\rangle$ and $\langle c_\eta;\ \eta < \gamma\rangle$ be any two sequences satisfying conclusions (i)-(iii) of Lemma 1 and the additional conclusion (iv) of Lemma 3. In order to define f for every $x \in A$ we distinguish two cases dependent upon whether x is a lower bound of $\{c_\eta;\ \eta < \gamma\}$ or not.

In the first case, by (iv) of Lemma 3, the set

$$\theta(x) = \mathrm{E}_\xi\,[\xi < \beta \text{ and } x \leq b_\xi] \tag{1}$$

is non-empty. We notice that, by conclusions (ii) and (iii), the sequence $\langle b_\xi;\ \xi < \beta\rangle$ cannot have a last term; that is $\xi < \beta$ always implies $\xi + 1 < \beta$. Hence we may put

$$\vartheta(x) = \min \theta(x) \text{ and } f(x) = b_{\vartheta(x)+1}. \tag{2}$$

In the second case, the set

$$\Psi(x) = \mathrm{E}_\eta\,[\eta < \gamma \text{ and } x \nleq c_\eta] \tag{3}$$

is nonempty. We let

$$\psi(x) = \min \Psi(x) \text{ and } f(x) = c_{\psi(x)}. \tag{4}$$

We have thus defined a function f on A to A. If $x \in A$, and x is a lower bound of $\{c_\eta;\ \eta < \gamma\}$, it follows from (1), (2), and conclusion (ii) of Lemma 1, that

$$x \leq b_{\vartheta(x)} < b_{\vartheta(x)+1} = f(x),$$

while if x is not a lower bound of $\{c_\eta;\ \eta < \gamma\}$, we see from (3) and (4) that $x \nleq f(x)$; thus f has no fixpoints.

Now let x and y be any elements of A. Assume first that both x and y are lower bounds of $\{c_\eta;\ \eta < \gamma\}$. Let, in addition, $\vartheta(x) \leq \vartheta(y)$. Then, obviously,

$$\vartheta(x) + 1 \leq \vartheta(y) + 1$$

and, by (2) and conclusion (ii) of Lemma 1, we obtain

(5) $$f(x) \cup f(y) = b_{\vartheta(x)+1} \cup b_{\vartheta(y)+1} = b_{\vartheta(y)+1}.$$

Clearly, $x \cup y$ is a lower bound of $\{c_\eta;\ \eta < \gamma\}$ and we see from (1) that $\theta(x \cup y)$ is a subset of $\theta(y)$; therefore it follows from (2) that

(6) $$\vartheta(y) \leq \vartheta(x \cup y).$$

On the other hand, by (1), (2), and conclusion (ii) of Lemma 1, we have

$$x \leq b_{\vartheta(x)} \leq b_{\vartheta(y)} \text{ and } y \leq b_{\vartheta(y)};$$

hence $x \cup y \leq b_{\vartheta(y)}$ and, by (1), $\vartheta(y) \in \theta(x \cup y)$. Then, using (2), we obtain

$$\vartheta(x \cup y) \leq \vartheta(y);$$

hence, with the help of (2), (5), and (6), we conclude that

(7) $$f(x \cup y) = f(x) \cup f(y).$$

Assume next that x is a lower bound of $\{c_\eta;\ \eta < \gamma\}$ while y is not. Then, by (2), (4), and conclusion (i) of Lemma 1, we have

(8) $$f(x) \cup f(y) = b_{\vartheta(x)+1} \cup c_{\psi(y)} = c_{\psi(y)}.$$

Since y is not a lower bound of $\{c_\eta;\ \eta < \gamma\}$, $x \cup y$ is not either, and by (3) we see that $\Psi(y)$ is a subset of $\Psi(x \cup y)$; therefore, by (4),

(9) $$\psi(x \cup y) \leq \psi(y).$$

From (3) and (4) it is obvious that

$$x \cup y \not\leq c_{\psi(x \cup y)},$$

and hence either

$$x \not\leq c_{\psi(x \cup y)} \text{ or } y \not\leq c_{\psi(x \cup y)}.$$

But since x is assumed to be a lower bound of $\{c_\eta;\ \eta < \gamma\}$, it follows that

$$y \not\leq c_{\psi(x \cup y)};$$

therefore, by (3), $\psi(x \cup y) \in \Psi(y)$; and, by (4),

$$\psi(y) \leq \psi(x \cup y). \tag{10}$$

Applying (4), (8), (9), and (10), we conclude that (7) holds.

Finally, assume that neither x nor y is a lower bound $\{c_\eta;\ \eta < \gamma\}$, and let $\psi(x) \leq \psi(y)$. From (4) and conclusion (ii) of Lemma 1, we obtain

$$f(x) \cup f(y) = c_{\psi(x)} \cup c_{\psi(y)} = c_{\psi(x)}. \tag{11}$$

Since, by (3), $\Psi(x)$ is a subset of $\Psi(x \cup y)$, it follows from (4) that

$$\psi(x \cup y) \leq \psi(x). \tag{12}$$

Using (3) and (4) again, we see that

$$x \cup y \nleq c_{\psi(x \cup y)},$$

and hence either

$$x \nleq c_{\psi(x \cup y)} \text{ or } y \nleq c_{\psi(x \cup y)}.$$

Therefore,

$$\psi(x) \leq \psi(x \cup y) \text{ or } \psi(y) \leq \psi(x \cup y).$$

But if $\psi(y) \leq \psi(x \cup y)$, then, since $\psi(x) \leq \psi(y)$, it is also the case that

$$\psi(x) \leq \psi(x \cup y). \tag{13}$$

Using (4), (11), (12), and (13), we again obtain (7). Thus the function f is join-distributive, and the proof of the theorem is complete.

As an immediate consequence of Theorem 4 we obtain:

COROLLARY 5. *Let $\mathfrak{A} = \langle A, \leq \rangle$ be a lattice in which the set A is denumerable. For $\mathfrak{A}$ to be complete it is necessary and sufficient that every join-distributive function on A to A have a fixpoint.*

By analyzing the preceding proofs we easily see that Theorem 4 and Corollary 5 remain valid if we replace in them "join" by "meet" everywhere; we also notice that in every lattice $\mathfrak{A} = \langle A, \leq \rangle$ without 0 the conclusions of Lemma 3 (with $\beta = 0$) hold, and hence there is a join-distributive function on A to A without fixpoints.

If, instead of considering arbitrary lattices, we restrict ourselves to

Boolean algebras, we immediately conclude from Corollary 5 that in every Boolean algebra $\mathfrak{A} = \langle A, \leq \rangle$ in which the set A is (infinitely) denumerable there is a join-distributive function f on A to A without fixpoints. This result can be extended to a wider class of Boolean algebras, in fact to all infinite Boolean algebras with an ordered basis;[6] the proof will not be given here.[7] However, the question remains open whether the result can be extended to arbitrary incomplete or even to arbitrary countably incomplete Boolean algebras (that is, to those which, in our terminology, are not 0-join-complete).

---

[6]For the notion of a Boolean algebra with ordered basis, see [5]. It is well known that every denumerable Boolean algebra has an ordered basis, and that every infinite Boolean algebra with an ordered basis is countably incomplete, but that the converses of these statements do not hold.

[7]The essential property of infinite Boolean algebras with an ordered basis which is involved in this proof is that every such algebra contains a sequence of disjoint non-zero elements $\langle b_\mu;\ \mu < \omega \rangle$ such that, for every element x of the algebra, either the set $\mathsf{E}_\mu[b_\mu \cap x = 0]$ or the set $\mathsf{E}_\mu[b_\mu \cap x \neq 0]$ is finite. The idea of the proof was suggested to the author by an argument in [4, p. 921], where a particular case of the result in question was obtained.

REFERENCES

1. Garrett Birkhoff, *Lattice theory*, revised edition, Amer. Math. Soc. Colloq. Publ. vol. 25, New York, 1948.

2. Anne C. Davis, *A fixpoint criterion for completeness of a lattice* (preliminary report), Bull. Amer. Math. Soc. **57** (1951), 77.

3. Felix Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914 (reprinted, New York, 1949).

4. Bjarni Jónsson and Alfred Tarski, *Boolean algebras with operators*. Part I, Amer. J. Math. **73** (1951), 891-939.

5. Andrzej Mostowski and Alfred Tarski, *Boolesche Ringe mit geordneter Basis*, Fund. Math. **32** (1939), 69-86.

6. Alfred Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math. **5** (1955), 285-309.

UNIVERSITY OF CALIFORNIA, BERKELEY

Pacific Journal of Mathematics

Vol. 5, No. 2 October, 1955