

Pacific Journal of Mathematics

ON A THEOREM OF L. A. SANTALÓ

BRANKO GRÜNBAUM

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1. Introduction. L. A. Santaló [2] proved, as a by-product of other investigations, the following theorem:

Let a set of parallel line segments be given in the plane. If every three of the segments can be intersected (that is, met) by a straight line, then there exists a straight line intersecting all the segments.

This result was rediscovered by M. Dresher and T. E. Harris (cf. [1]). An interesting generalisation was obtained by H. Rademacher and I. J. Schoenberg [1]. Their generalisation—extending a special case due to T. E. Harris—is obtained from Santaló's theorem on replacing in it “three” by “ $m + 2$ ” and “straight line” by “polynomial line $y = \alpha_0 x^m + \dots + \alpha_m$ ”.

The proof of Rademacher and Schoenberg (as well as a proof of Santaló's original theorem by J. Rey Pastor, cf. [2]) is based on Helly's theorem on convex sets.

The principal aim of the present paper is to give a generalisation of Santaló's theorem in a different direction. We shall restrict ourselves to intersections by straight lines, but, on the other hand, shall allow much greater freedom in the choice of sets to be intersected. As far as we are aware, our theorems cannot be deduced from Helly's theorem on convex sets.

We shall use the following customary definition:

Two sets S and S^* in the plane are said to be separated by a straight line L if $S \subset H \cup L$ and $S^* \subset H^* \cup L$, where H and H^* , $H \neq H^*$ are the two open half-planes determined by L . The separation is strict, if $S \subset H$ and $S^* \subset H^*$.

For the sake of brevity we shall use also the following:

DEFINITION. A family of point sets in the plane is said to have property \mathfrak{A} if, either (i) there are three sets belonging to the family which cannot be intersected simultaneously by a straight line, or (ii) there exists a straight

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line intersecting all the sets of the family.

Thus, Santaló's theorem is equivalent to the statement that a family of parallel line segments has property \mathcal{A} .

Our main result is the following:

THEOREM 1. *A finite sequence $\{S_i\}$ ($i = 1, \dots, n$) of connected sets in the plane, such that for every $i = 1, 2, \dots, n-1$, there exists a straight line strictly separating the sets $\bigcup_{j \leq i} S_j$ and $\bigcup_{j > i} S_j$, has property \mathcal{A} .*

If the sets S_i are compact, this theorem remains valid even for infinite sequences $\{S_i\}$ ($i = 1, 2, \dots$); indeed, we have the somewhat more general theorem:

THEOREM 2. *Let A be an ordered set and $\{S_\alpha\}$ ($\alpha \in A$) a family of compact, connected sets in the plane. If for any finite subset $\{\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n\}$ of A , with $\max_{1 \leq i \leq m} \alpha_i < \min_{1 \leq j \leq n} \beta_j$ (strict inequality) there exists a straight line strictly separating $\bigcup_{i=1}^m S_{\alpha_i}$ and $\bigcup_{j=1}^n S_{\beta_j}$, then the family $\{S_\alpha\}$ has property \mathcal{A} .*

If the separating straight lines are parallel, we have stronger results:

THEOREM 3. *Every finite family of connected sets in the plane, any two of which can be separated (not necessarily strictly) by a straight line of fixed direction, has property \mathcal{A} .*

THEOREM 4. *Every family of compact, connected sets in the plane, any two of which can be separated (not necessarily strictly) by a straight line of fixed direction, has property \mathcal{A} .*

In the proofs we shall assume that every three sets may be intersected by a straight line and show the existence of a straight line intersecting all of them.

2. Proof of Theorem 1. 2.1. If P and Q are any two points, we denote by $[P; Q]$ the closed, by $(P; Q)$ the open line segment determined by P and Q .

We need the following

LEMMA. *Let D_0 and D_1 be two directed straight lines in the plane. For $i = 0, 1$, let $X_1^{(i)}, X_2^{(i)}, \dots, X_n^{(i)}$ be a sequence of n , not necessarily different, points on D_i , ordered according to the direction of D_i and such that $(X_j^{(0)}; X_j^{(1)}) \cap (X_k^{(0)}; X_k^{(1)}) = \emptyset$ for $j \neq k$. (\emptyset denotes the void set.) Then there exists*

a family $D(t)$, $0 \leq t \leq 1$, of straight lines, depending continuously on t , such that

$$D(0) = D_0; D(1) = D_1; D(t) \cap [X_j^{(0)}; X_j^{(1)}] \neq \phi \quad (0 \leq t \leq 1; j = 1, 2, \dots, n).$$

Proof of the Lemma. The Lemma is obvious if either D_0 and D_1 are parallel, or

$$D_0 \cap [X_1^{(1)}; X_n^{(1)}] = D_1 \cap [X_1^{(0)}; X_n^{(0)}] = \phi.$$

In the remaining case we may assume $D_0 \cap [X_1^{(1)}; X_n^{(1)}] \neq \phi$. Let D_i^+ and D_i^- denote the two open half-lines determined by the intersection point $P = D_0 \cap D_1$ on D_i , with D_i^- preceding D_i^+ according to the direction given on D_i . The points $X_j^{(0)}$ and $X_j^{(1)}$ are said to be *wrongly paired* if either $X_j^{(0)} \in D_0^-$ and $X_j^{(1)} \in D_1^+$, or $X_j^{(0)} \in D_0^+$ and $X_j^{(1)} \in D_1^-$.

If there are no wrongly paired points the Lemma is obvious, since D_0 may be rotated about P till it coincides with D_1 .

If there are wrongly paired points, it is easily seen that

- (i) they are consecutive, that is, they are $X_j^{(0)}, X_j^{(1)}$ for $1 \leq k \leq j \leq k+r \leq n$; and
- (ii) either $X_k^{(0)} = X_{k+1}^{(0)} = \dots = X_{k+r}^{(0)}$ or $X_k^{(1)} = X_{k+1}^{(1)} = \dots = X_{k+r}^{(1)}$.

Without loss of generality we may assume $X_k^{(0)} = \dots = X_{k+r}^{(0)} \in D_0^-$. We pass from D_0 to D_1 by rotating D_0 first about $X_k^{(0)}$ until it contains the segment $[X_{k+r}^{(0)}; X_{k+r}^{(1)}]$ and then about $X_{k+r}^{(1)}$ until it coincides with D_1 . This completes the proof of the Lemma.

2.2. Now we shall prove Theorem 1 under the restrictive assumption that the sets S_i are *compact and convex*.

We shall proceed by induction on the number of sets, n . Since the case $n = 3$ is trivial, we may assume that the theorem holds for sequences consisting of not more than n , $n \geq 3$, sets, and prove its validity for sequences consisting of $n + 1$ sets.

By the induction hypothesis there exist straight lines intersecting all the sets S_i , $i = 1, 2, \dots, n$. Let K be the set of all the points in the plane through which there passes at least one straight line intersecting all the sets S_i , $i = 1, 2, \dots, n$. Since the S_i are compact, K is closed. We shall prove that $K \cap S_{n+1} \neq \phi$.

Since the sets S_i are convex, it is clear from the assumption that to every straight line intersecting them a direction may be given so that the intersection of the straight line with S_j precedes that with S_k for $j < k$. From the convexity of the sets S_i it follows then by the Lemma that, given two straight lines both intersecting all S_i , $i = 1, 2, \dots, n$, it is possible to move one of them continuously until it coincides with the other, in such a way that in all intermediate positions, it intersects all the S_i , $i = 1, 2, \dots, n$.

Let L_i ($i = 1, 2, \dots, n$) be straight lines strictly separating the sets $\bigcup_{j < i} S_j$ and $\bigcup_{j > i} S_j$. For linguistic convenience we shall assume L_n vertical and the half-plane H_n , determined by L_n and containing $\bigcup_{j \leq n} S_j$, situated on the left side of L_n . We put $I = K \cap L_n$. Now $I \neq L_n$ (since we may assume L_1 not vertical and then $L_1 \cap L_n \notin I$). Since I is closed, there are in view of the Lemma four possible, mutually exclusive cases:

- (a) $I = \phi$;
- (b) I is a closed segment (possibly reducing to a single point);
- (c) I is a closed half-line;
- (d) I consists of two closed, disjoint half-lines.

Case (a). This case may be easily reduced to one of the remaining cases. In fact, we remark:

- (i) since $K \neq \phi$, it contains at least one straight line, which is parallel to L_n (since $I = \phi$);
- (ii) any straight line L'_n , obtained from L_n by a sufficiently small rotation about one of its points, separates strictly the sets S_{n+1} and $\bigcup_{j \leq n} S_j$. But then $K \cap L'_n \neq \phi$, that is, we have one of the cases (b), (c), (d). (Actually, K consists in case (a) of a single straight line, but we do not use this in our proof.)

Case (b). Let E denote the lower end-point of I and let T denote a straight line, which passes through E and intersects all S_i , $i = 1, 2, \dots, n$. There exists only one such line T and, moreover, there exist two sets, S_p and S_q , $p < q$, such that T separates them, S_p being contained in the lower closed half-plane determined by T .

Indeed, if there were two different straight lines T_1 and T_2 , through E , both intersecting all S_i , $i = 1, 2, \dots, n$, with $T_2 \cap H_n$ below $T_1 \cap H_n$, the straight line T_3 passing through any point of $T_1 \cap S_1$ and any point of $T_2 \cap S_n$ would also intersect all S_i , $i = 1, 2, \dots, n$, but its intersection with L_n would be below E , in contradiction to the definition of E .

Similarly, at least one S_i must be contained in the upper closed half-plane determined by T , since otherwise a suitable translate of T would still intersect all the S_i , $i = 1, 2, \dots, n$, while intersecting L_n below E —again a contradiction.

Now let q be the greatest index such that S_q has no points below T . If all S_i , $i < q$, had points above T , it would be possible to rotate T about any point of $T \cap S_q$ in such a way that, while intersecting all S_i , $i = 1, 2, \dots, n$, it would intersect L_n below E —again a contradiction. This proves our statements about T .

Similarly we see that through E' , the upper end-point of l , there passes a unique straight line T' , intersecting all S_i , $i = 1, 2, \dots, n$, and separating at least two of them, say $S_{p'}$ and $S_{q'}$, $p' < q'$, with $S_{p'}$ contained in the upper closed half-plane determined by T' . Moreover, the slope of T' is not smaller than that of T , since T is a separating common tangent to S_p and S_q , while T' intersects them. (We note that S_p and S_q may coincide with $S_{p'}$ and $S_{q'}$.)

H_n^* being the half-plane to the right of L_n , its subset M consisting of all the points not above T' and not below T is well defined. From the definition of l and the lemma of 2.1 it follows immediately that $K \cap H_n^* \supset M$. (Actually $K \cap H_n^* = M$, as may be seen from the sequel, but we do not use this.) The complement of M in H_n^* consists of two open, disjoint sets, one of them being below, the other above M .

Now $S_{n+1} \subset H_n^*$ and it cannot lie entirely below (resp. above) M since in the first (resp. second) case no straight line would intersect S_p , S_q (resp. $S_{p'}$, $S_{q'}$) and S_{n+1} , contradicting the assumption about the existence of a straight line intersecting any three of the sets. But S_{n+1} is connected, therefore $S_{n+1} \cap M \neq \phi$, and since $M \subset K$, this completes the proof in case (b).

Case (c). Assuming the half-line l directed upwards, let E denote its end-point. As in Case (b), we may prove the existence of a unique straight line T passing through E , intersecting all S_i , $i = 1, 2, \dots, n$, and separating at least two of them, S_p and S_q , $p < q$, S_p having no points above T . Let M be the set of all points to the right of L_n and above or on T . As in Case (b), $M \subset K$. But $M \cap S_{n+1} \neq \phi$, since otherwise it would be impossible to intersect S_p , S_q and S_{n+1} , in contradiction to the assumptions. Therefore we have $K \cap S_{n+1} \neq \phi$, completing the proof in Case (c).

Case (d) l consists of two closed, disjoint half-lines on L_n . Denote by E_1 and E_2 , with E_1 below E_2 , the end-points of these half-lines. As in Case (b) we may establish the existence of unique straight lines T_j , $j = 1, 2$, passing

through E_j and intersecting all S_i , $i = 1, 2, \dots, n$, and separating at least one pair of sets, say S_{p_j} and S_{q_j} , such that $T_j \cap S_{q_j}$ is nearer to E_j than $T_j \cap S_{p_j}$. It is easy to see that T_2 has a greater slope than T_1 .

Let N be the subset of H_n^* consisting of all the points which are either in the lower closed half-plane determined by T_1 , or in the upper one determined by T_2 . As in case (b) we see that $K \cap H_n^* \supset N$.

Now, S_{n+1} and $\bigcup_{i \leq n} S_n$ are strictly separated by L_n . Also, if $\min(p_2, q_1) \geq \max(p_1, q_2)$ (resp. $\min(p_1, q_2) \geq \max(p_2, q_1)$) there exists a straight line — for example, L_{p_1} (resp. L_{p_2}) — strictly separating S_{p_1} (resp. S_{p_2}) and $S_{q_1} \cup S_{n+1}$ (resp. $S_{q_2} \cup S_{n+1}$). Now if S_{n+1} is above T_1 (resp. below T_2), there cannot exist a straight line intersecting S_{p_1}, S_{q_1} (resp. S_{p_2}, S_{q_2}) and S_{n+1} . Hence, by the assumption about the intersectability of any three sets, S_{n+1} must have points in common with N , and thus also with K . This completes the proof of Theorem 1 for compact and convex S_i .

2.3. The assumption of compactness of the sets S_i may be dropped by a well-known method, used also in [1] in the proof of Helly's theorem.

Suppose it possible to intersect every three of the convex sets S_i by a suitable straight line. To every triple of sets S_i take one such line and on it one point in each of the three sets. In every set S_i a finite number of points is thus obtained. Obviously, the convex hull of these points, S_i^* , is compact and $S_i^* \subset S_i$, $i = 1, 2, \dots, n$. The sequence $\{S_i^*\}$ clearly satisfies all the assumptions of 2.2, so that we can conclude that there exists a straight line intersecting all S_i^* , and, a fortiori, all the sets S_i .

2.4. Finally, we drop the assumption of convexity by the simple remark that, if two sets are (strictly) separated by a straight line, so are their convex hulls, and if a straight line intersects the convex hull of a connected set, it intersects the set itself as well. We can, therefore, from a given sequence of connected sets pass to the sequence of their convex hulls, and after applying to these the result of 2.3, return to the sets themselves.

This completes the proof of Theorem 1.

3. Proof of Theorem 2. Let $\alpha_1, \beta_1 \in A$ with $\alpha_1 < \beta_1$. There exists a straight line strictly separating S_{α_1} and S_{β_1} . Following a method used in [1] we introduce in the plane an orthogonal coordinate system XY in such a way that the Y axis strictly separates S_{α_1} and S_{β_1} . To every straight line $y = \xi x + \eta$ in the XY plane corresponds the point (ξ, η) in another, ΞH , plane.

For any finite subset $\{\alpha'\}$ of A , it follows from Theorem 1 that there exist straight lines intersecting all S_{α} . If $\{\alpha_1, \beta_1\} \subset \{\alpha'\}$ then the set of points (ξ, η) corresponding to all such straight lines is a compact set in the ΞH plane. Since the intersection of any finite number of such sets is not void, it follows from a well-known theorem of F. Riesz that the intersection of all of them is not void. But a straight line corresponding to a point common to all these sets intersects all the sets S_{α} , $\alpha \in A$. Q.e.d.

4. Proof of Theorem 3. Without loss of generality we may assume that the straight lines separating the sets are vertical.

As in the proof of Theorem 1 we assume the sets S_i convex and compact. It is easy to get rid of these restrictions exactly as in 2.3 and 2.4. Our proof is again inductive, we assume the theorem for some $n \geq 3$ and prove it for $n+1$.

Every one of the $n+1$ sets is contained in a minimal strip, bounded by two vertical straight lines (the two lines may coincide or one of them may be at infinity). Strips containing different sets have an intersection which is either void or consists of a single vertical line belonging to the boundary of the strips.

It is possible, therefore, to enumerate the sets proceeding, say, from left to right and, in case of indetermination—that is, when a vertical line contains more than one set, —in an arbitrary fashion. Let V_i be a vertical line separating the sets S_i and S_{i+1} , $i = 1, 2, \dots, n$. It follows from the method of enumeration that V_i separates $\cup_{j \leq i} S_j$ and $\cup_{j > i} S_j$.

If for some s , $1 \leq s \leq n-2$, $V_s \equiv V_{s+2}$, the sets S_{s+1} and S_{s+2} are situated entirely on the straight line $V_s \equiv V_{s+1} \equiv V_{s+2}$. Then either

(i) $S_{s+1} \cap S_{s+2} = \phi$ and V_s is the only straight line intersecting S_{s+1} and S_{s+2} , hence, since every three sets can be intersected by a straight line, all the sets S_i have to intersect V_s and the theorem is proved; —or

(ii) $S' = S_{s+1} \cap S_{s+2} \neq \phi$ and then it is easily seen (either directly or by the Lemma of 2.1) that the n sets $S_1, \dots, S_s, S', S_{s+3}, \dots, S_{n+1}$ satisfy all the conditions of the theorem so that, by the induction hypothesis, there exists a straight line intersecting all of them, hence, a fortiori, intersecting all S_i , $i = 1, 2, \dots, n+1$.

Thus we may assume $V_i \neq V_{i+2}$, $i = 1, 2, \dots, n-2$, therefore a straight line intersecting all the sets S_i , $i = 1, 2, \dots, n$, cannot be vertical, hence it must intersect V_n . Let K denote the set of all points through which there pass straight lines intersecting all S_i , $i = 1, 2, \dots, n$. From the lemma of 2.1 it follows that $I = K \cap V_n$ is a closed segment (possibly reducing to a point). Using the

same method as in the proof of Case (b) of Theorem 1 we may see that there exist straight lines T and T' and sets $S_p, S_{p'}, S_q, S_{q'}$, having all the properties of the sets so denoted in 2.2. As in Case (b) of Theorem 1 it follows that S_{n+1} must contain points not below T and not above T' and hence that $S_{n+1} \cap K \neq \phi$, completing the proof of Theorem 3.

5. Proof of Theorem 4. We again assume the separating straight lines vertical. By Theorem 3 to every finite subfamily of the family $\{S_\alpha\}_{(\alpha \in A)}$ there corresponds a straight line intersecting all sets of the subfamily. If to every finite subfamily there corresponds a vertical intersecting line, it is easily seen that all the sets S_α , (except possibly one or two), lie entirely on one vertical line (which intersects also the possibly remaining one or two) and the theorem is established in this case.

If there is at least one finite subfamily of $\{S_\alpha\}$ for which there is no vertical straight line intersecting all the sets of the subfamily, then there exist two sets S_{α_1} and S_{β_1} in the subfamily, which can be separated by a vertical straight line. We then complete the proof exactly in the way in which we proved Theorem 2.

6. Remarks. Simple examples show that some of the assumptions made in Theorems 1-4 are essential.

6.1. The connectedness of the sets S_i required in the theorems cannot be dropped entirely. Indeed, denoting by $[P; Q]$ the closed segment with endpoints P and Q , let

$$\begin{aligned} S_1 &= [(0, 0); (0, 3)], & S_2 &= [(1, 0); (1, 2)], \\ S_3 &= (2, 3) \cup [(2, 0); (2, 1)], & S_4 &= [(3, 2); (3, 3)]. \end{aligned}$$

Then, except that S_3 is not connected, the sequence S_1, S_2, S_3, S_4 satisfies the assumptions of all four theorems and yet it does not have property \mathfrak{A} .

It is possible however to substitute for the assumption of connectedness the somewhat weaker one that if both open half-planes determined by a straight line L contain points of S then $L \cap S \neq \phi$. As a matter of fact, in 2.4 and the corresponding part of the proof of Theorem 3 only the above property of connected sets was used.

6.2. The strict separation required in Theorems 1 and 2 cannot be dropped. Indeed, let

$$S_1 = [(-3, -1); (0, -1)], S_2 = [(-1, 0); (-1, 1)],$$

$$S_3 = [(1, 0); (1, 1)], S_4 = [(0, -1); (3, -1)].$$

Except for the impossibility to separate strictly S_1 and S_4 , the sequence S_1, S_2, S_3, S_4 satisfies all the assumptions of Theorems 1 and 2, although it does not have property \mathfrak{A} .

6.3. The compactness assumption in Theorems 2 and 4 cannot be dropped either. Indeed, if S_0 is the open interval with endpoints $(0, 0)$ and $(0, 1)$, and $S_i = [(i, 0); (i, 1/i)]$, $i = 1, 2, 3, \dots$, the sequence $\{S_i\}$ ($i = 0, 1, 2, \dots$) satisfies all the assumptions of Theorems 2 and 4 except for the noncompactness of S_0 , and yet it does not have property \mathfrak{A} .

Added in proof. Since the submission of this paper a slightly weaker version of Theorems 3 and 4 appeared in a paper by V. L. Klee, Jr., *Common secants for plane convex sets*, Proc. Amer. Math. Soc., **5** (1954), 639-641.

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