ON THE CHANGE OF INDEX FOR SUMMABLE SERIES

DIETER GAIER
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1. Introduction. Assume we have given a series

\[(1.1) \quad a_0 + a_1 + a_2 + \cdots + a_n + \cdots\]

and consider

\[(1.2) \quad b_0 + b_1 + b_2 + \cdots + b_n + \cdots \quad \text{with} \quad b_0 = 0 \quad \text{and} \quad b_n = a_{n-1} \quad (n \geq 1);\]

denote the partial sums by \(s_n\) and \(t_n\), respectively. Since \(s_n = t_{n+1}\), the convergence of (1.1) is equivalent to that of (1.2). However, if a method of summability \(V\) is applied to both series, the statements

\[(1.3) \quad (a) \quad V - \sum a_n = s \quad \quad (b) \quad V - \sum b_n = s^1\]

need not be equivalent (for example, if \(V\) is the Borel method; see [4, p. 183]). If \(V(x; s_V)\) and \(V(x; t_V)\) denote the \(V\)-transforms of the sequences \(\{s_n\}\) and \(\{t_n\}\), respectively, it is therefore interesting to investigate, for which methods \(V\) and under what restrictions on \(\{a_n\}\) the relations

\[(1.4) \quad (a) \quad V(x; s_V) \simeq K \cdot x^q \quad \quad (b) \quad V(x; t_V) \simeq K \cdot x^q\]

\((x \rightarrow x_0, K \text{ constant}; \quad q \geq 0, \text{ fixed})^2\)

are equivalent.

The cases \(V = C_k\) (Cesàro) and \(V = A\) (Abel) are quickly disposed of (§2), while \(V = E\) (general Euler transform) and \(V = B\) (Borel) present some interest (§§3-5).

2. Theorem 1. The statements (1.4a) and (1.4b) are equivalent for

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1 We shall always let \(\sum_{n=0}^{\infty} a_n = \sum a_n.\)

2 \(x \rightarrow x_0\) through values depending on the method \(V.\)

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\( V = C_k (k > -1) \) and \( V = A \).

**Proof.** If

\[
S_n^{(k)} = C_k(n; s_\nu) \cdot \binom{n + k}{n}
\]

and

\[
T_n^{(k)} = C_k(n; t_\nu) \cdot \binom{n + k}{n},
\]

we have by definition of the Cesàro means

\[
(2.1) \quad (1 - x)^{k+1} \sum T_n^{(k)} x^n = \sum b_n x^n = x \cdot \sum a_n x^n = x (1 - x)^{k+1} \sum S_n^{(k)} x^n,
\]

the series being convergent for \( |x| < 1 \). The proof of Theorem 1 now follows from the inner equality in (2.1) and the relation

\[
\frac{T_n^{(k)}}{\binom{n + k}{n}} = \frac{S_{n-1}^{(k)}}{\binom{n + k}{n}} \quad \text{as} \quad \frac{S_{n-1}^{(k)}}{\binom{n - 1 + k}{n - 1}} \quad \text{as} \quad (n \to \infty).
\]

**3.** Let \( g(w) = \sum \gamma_n w^n \) be regular and schlicht in \( |w| \leq 1 \), and assume \( g(0) = 0 \), \( g(1) = 1 \). Then the \( E \)-transforms of \( \sum a_n \) and \( \sum b_n \) are obtained by the formal relations [5]

\[
\sum a_n z^n = \sum a_n [g(w)]^n = \sum \alpha_n w^n; \quad E(n; s_\nu) = \sum_{\nu=0}^{n} \alpha_\nu \quad (n = 0, 1, \ldots),
\]

\[
\sum b_n z^n = \sum b_n [g(w)]^n = \sum \beta_n w^n; \quad E(n; t_\nu) = \sum_{\nu=0}^{n} \beta_\nu
\]

**Theorem 2.** The statements (1.4.a) and (1.4.b) are equivalent for \( V = E \).

**Proof.** First we note that if either

\[
E(n; s_\nu) = O(n^q) \quad \text{or} \quad E(n; t_\nu) = O(n^q) \quad (n \to \infty),
\]

\(^3\) For \( q = 0 \) see [4, p. 102].
then the formal relations (3.1) are actually valid for $|w| < 1$ and also

\[
(3.2) \quad \sum \beta_n w^n = \sum b_n [g(w)]^n = g(w) \cdot \sum a_n [g(w)]^n = g(w) \cdot \sum \alpha_n w^n \quad (|w| < 1).
\]

Denote by $A_n$, $B_n$, $C_n$ the partial sums of $\sum \alpha_n$, $\sum \beta_n$, $\sum \gamma_n$, respectively. We assume first

\[
E(n; s) = A_n \approx K \cdot n^q \quad (n \to \infty).
\]

Then, since by (3.2) $\sum \beta_n$ is the Cauchy product of $\sum \alpha_n$ and $\sum \gamma_n$, we have

\[
E(n; t) = B_n = \gamma_0 A_0 + \gamma_1 A_1 + \cdots + \gamma_{n-1} A_{n-1}
\]

and for $n \geq 1$

\[
(3.3) \quad \frac{B_n}{n^q} = \frac{\gamma_0}{n^q} A_0 + \frac{\gamma_1}{n^q} \frac{A_1}{1^q} + \cdots + \frac{\gamma_{n-1}}{n^q} \frac{A_{n-1}}{(n-1)^q}. \tag{3.3}
\]

For the matrix $c_{n, \nu}$ in this transformation of the convergent sequence $\{A_n n^{-q}\}$ we have clearly

\[
\lim_{n \to \infty} c_{n, \nu} = 0 \quad (\nu = 0, 1, \ldots).
\]

Furthermore

\[
\sum_{\nu} |c_{n, \nu}| = \sum_{\nu=1}^{n-1} \frac{\frac{\gamma_\nu}{n^q}}{n^q} \leq \sum_{\nu=1}^{n} \frac{\gamma_\nu}{n^q} \leq \sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{n^q} = M < \infty;
\]

finally we prove

\[
\lim_{n \to \infty} \sum_{\nu=0}^{n-1} c_{n, \nu} = 1.
\]

For $q = 0$ this follows from

\[
\sum_{\nu=0}^{n-1} c_{n, \nu} = \sum_{\nu=1}^{n} \gamma_\nu \to g(1) = 1 \quad (n \to \infty);
\]
for \( q > 0 \)

\[
\sum_{\nu=0}^{n-1} c_{n\nu} = \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} \gamma_{n-\nu} \cdot \frac{\nu^q}{n^q} = \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} \gamma_{\nu} \left( \frac{n - \nu}{n} \right)^q
\]

\[
= \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} C_{\nu} \left[ \left( \frac{n - \nu}{n} \right)^q - \left( \frac{n - \nu - 1}{n} \right)^q \right],
\]

and the last term is a positive regular transformation of the sequence \( \{ C_n \} \) tending to \( g(1) = 1 \), whence

\[
\sum_{\nu} c_{n\nu} \to 1 \quad (n \to \infty).
\]

Therefore the transformation (3.3) of \( \{ A_n n^{-q} \} \) converges to \( K \), which proves

\( B_n \sim K \cdot n^q \) \((n \to \infty)\).

Assume on the other hand \( B_n \sim Kn^q \) \((n \to \infty)\). Putting \( w = 0 \) in (3.2), one obtains \( \beta_0 = 0 \), so that

\[
\sum \alpha_n w^n = [g(w)]^{-1} \sum \beta_n w^n = w [g(w)]^{-1} \sum \beta_{n+1} w^n
\]

is regular in \( |w| < 1 \). Furthermore the expansion of the function \( w [g(w)]^{-1} \) for \( w = 1 \) converges absolutely to 1, since \( w = 0 \) is the only zero of \( g(w) \) in \( |w| \leq 1 \). An argument similar to the one above shows then that \( B_{n+1} \sim Kn^q \) \((n \to \infty)\) implies \( A_n \sim Kn^q \) \((n \to \infty)\), which completes the proof of Theorem 2.

We add a few remarks about the assumptions on the function \( z = g(w) \) by which the \( E \)-method is defined.

a. Theorem 2 becomes false if only regularity of \( g(w) \) in \( |w| < 1 \), and continuity and schlichtness in \( |w| \leq 1 \) are assumed. For there exist such functions \( g(w) \) whose power series do not converge absolutely on \( |w| = 1 \) (cf. [2]). Therefore in (3.2) one could find a convergent \( \sum \alpha_n \) whose transform \( \sum \beta_n \) diverges.

b. All that was used about the function \( g(w) \) in the proof of Theorem 2 was that the power series of \( g(w) \) and of \( w [g(w)]^{-1} \) converge absolutely to the value 1 for \( w = 1 \). This can be guaranteed by the weaker assumption that \( g(w) \) with \( g(1) = 1 \) and \( g(0) = 0 \) is regular in \( |w| < 1 \), continuous and schlicht
in \(|w| \leq 1\), and that the image of \(|w| = 1\) under the mapping \(g(w)\) is a rectifiable Jordan curve. Because then

\[
\int_0^{2\pi} |g'(e^{i\phi})| \, d\phi < \infty
\]

and hence \(\sum |\gamma_n| < \infty\) [8, p. 158]; on the other hand also

\[
\int_0^{2\pi} |G'(e^{i\phi})| \, d\phi < \infty,
\]

where

\[
G'(w) = \left[ \frac{w}{g(w)} \right]' = \frac{g'(w) - wg'(w)}{[g(w)]^2},
\]

so that also the power series of \(G(w)\) converges absolutely to the value 1 for \(w = 1\).

c. If

\[
g(w) = w [(p + 1) - pw]^{-1} \quad (p \geq 0, \text{ fixed})
\]

one has \(E = E_p\) as the familiar Euler method of order \(p\), for which Theorem 2 is known in the case \(q = 0\) [4, p. 180].

d. The function

\[
g(w) = (2 - w) - 2(1 - w)^{5/2} \quad (g(0) = 0)
\]

leads to the method of Mersman [6], as Scott and Wall showed [7, p. 270]. Here Theorem 2 is also applicable, since the more general conditions about \(g(w)\) in remark (b) are satisfied, as is readily seen.

4. The Borel method is defined by the transformation

\[
B(x; s_\nu) = e^{-x} \sum \frac{s_\nu x^\nu}{\nu!} \quad (x \geq 0),
\]

where the power series is assumed to define an entire function. It is known that \(B(x; s_\nu) \rightarrow K (x \rightarrow \infty)\) implies \(B(x; t_\nu) \rightarrow K (x \rightarrow \infty)\), but not conversely [4, p. 183]. We now prove more generally
**Theorem 3.** The relation

\[ B(x; s_v) \sim K x^q \]

implies

\[ B(x; t_v) \sim K x^q \]

**Proof.** We have for \( x > 0 \) [4, p. 196]

\[
x^{-q} B(x; t_v) = x^{-q} e^{-x} \sum_{\nu} \frac{t_{\nu} x^{\nu}}{\nu!} = x^{-q} e^{-x} \sum_{\nu} \frac{s_{\nu} x^{\nu+1}}{(\nu + 1)!}
\]

\[
= x^{-q} e^{-x} \sum_{\nu} \frac{s_{\nu} t^{\nu}}{\nu!} dt = x^{-q} \int_0^x e^{-x(t-t)} t^q \frac{B(t; s_v)}{t^q} dt.
\]

This transformation of the convergent function \( B(t; s_v) t^{-q} (t \to \infty) \) by means of the 'matrix'

\[
c(x, t) = e^{-(x-t)} \left( \frac{t}{x} \right)^q
\]

is regular, since

\[
\int_{t_1}^{t_2} |c(x, t)| dt \to 0 \quad \text{as} \quad (x \to \infty; t_1, t_2 > 0, \text{fixed})
\]

and

\[
\int_0^x |c(x, t)| dt = \int_0^x c(x, t) dt = e^{-x} \int_0^x e^t \left( \frac{t}{x} \right)^q dt \to 1 \quad \text{as} \quad (x \to \infty).
\]

Therefore \( B(x; t_v) \sim K x^q (x \to \infty). \)

We discuss now the converse of Theorem 3.

**Theorem 4.** The relation

\[ B(x; t_v) \sim K x^q \]

implies

\[ B(x; s_v) \sim K x^q \]
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\[ B(x; s) \simeq Kx^q \quad (x \to \infty), \]

if

\[ (4.2) \limsup |a_n|^{1/n} < \infty, \]

that is, if the series \( \sum a_n x^n \) has a positive radius of convergence.

Proof. Using (4.1) we have for \( x > 0 \)

\[ F(x) = x^{-q} B(x; s) = x^{-q} e^{-x} \int_0^x e^t B(t; s) dt. \]

Consider now \( F(x) \) as function of the complex variable \( x \) for \( \Re(x) \geq 1 \). Then (4.2) implies \( |t_n| \leq M^n \) for some constant \( M > 0 \) and hence in \( \Re(x) \geq 1 \)

\[ |B(x; t_n)| \leq e^{-1} \sum \frac{M^n |x|^n}{n!} = e^{-1} M|x|, \]

and also

\[ (4.3) \quad |F(x)| \leq \alpha e^\beta|x| \quad \Re(x) \geq 1 \]

for positive constants \( \alpha \) and \( \beta \). Hence one knows that

\[ F(x) \to K \quad (x \to +\infty) \]

implies

\[ F'(x) \to 0 \quad (x \to +\infty), \]

that is,

\[ x^{-q} B(x; s) + \int_0^x e^t B(t; s) dt \left[ -1 - \frac{q}{x} \right] e^{-x} x^{-q} \]

\[ = x^{-q} B(x; s) - K + o(1) = o(1) \quad (x \to +\infty), \]

from which the result follows.

5. We now show that Theorem 4 is best possible in a certain sense.

\[ ^4 \text{If } F(x) \text{ is regular in } \Re(x) \geq 1 \text{ and (4.3) holds, then } F(x) \to A \ (x \to +\infty) \text{ implies } F'(x) \to 0 \ (x \to +\infty). \text{ This lemma was used also in [3], where Theorem 4 was proved for } q = 0. \]
Theorem 5. In Theorem 4 the Condition (4.2) cannot be replaced by

\[(5.1) \quad \limsup n^{-\varepsilon} |a_n|^{1/n} < \infty \quad (\varepsilon > 0).\]

For the proof we need the following

Lemma. For every \(\beta > 1\), there exists an entire function \(f(z)\) of order \(\beta\) satisfying

\[(5.2) \quad f(x) \to 0 \quad (x \to +\infty), \quad f'(x) \to 0 \quad (x \to +\infty) \quad (z = x + iy).\]

Proof. Put \(\alpha = \beta^{-1}\) and consider the Mittag-Leffler function

\[E_\alpha(z) = \sum \frac{z^n}{\Gamma(1 + \alpha n)},\]

which is an entire function of order \(\alpha^{-1} = \beta\). Let \(m\) be the integer with

\[\frac{\alpha}{1 - \alpha} \leq m < \frac{\alpha}{1 - \alpha} + 1.\]

We first study the derivatives of \(E_\alpha(z)\) of order \(1, 2, \ldots, m\) on the line \(\arg z = \alpha \pi/2\) for large \(|z|\). For these \(z\) (assume for definiteness \(|z| > 2\)) one has

[1, pp. 272-275]

\[(5.3) \quad E_\alpha(z) = \frac{1}{2\pi i \alpha} \int_L e^{t^{1/\alpha}} \frac{dt}{t-z} + \frac{1}{\alpha} e^{z^{1/\alpha}},\]

the path \(L\) being

\[t = re^{-i\phi_0} \quad (\infty > r \geq 1, \alpha \pi > \phi_0 > \frac{\pi \phi_0}{2}), \quad t = e^{i\phi} \quad (-\phi_0 \leq \phi \leq +\phi_0),\]

\[t = re^{i\phi_0} \quad (1 \leq r < \infty);\]

\(t^{1/\alpha}\) is the branch which is positive for \(t > 0\). The \(k\)th derivative of the integral part in (5.3) can then be estimated as follows

\[\left|\frac{1}{2\pi i \alpha} \int_L e^{t^{1/\alpha}} \frac{k!}{(t-z)^{k+1}} dt\right| \leq \frac{k!}{2\pi \alpha |z|^{k+1}} \int_L |e^{t^{1/\alpha}}| \left|\frac{dt}{|1-(t/z)|^{k+1}}\right| = O(|z|^{-k-1}) = o(1) \quad (|z| \to \infty),\]
since for our values of $z$ one has $|1 - (t/z)| \geq \delta > 0$ and on the straight line segments of $L$

$$|e^{t^{1/\alpha}}| = e^{1^{1/\alpha} \cos \phi_0/\alpha} \quad \text{with} \quad \cos \frac{\phi_0}{\alpha} < 0.$$ 

Therefore

$$E_{\alpha}^*(z) = o(1) + \frac{1}{\alpha^2} e^{z^{1/\alpha}} z^{1/\alpha - 1}$$

$$E_{\alpha}''(z) = o(1) + \frac{1}{\alpha^3} e^{z^{1/\alpha}} z^{-1/\alpha + 1/2}$$

(5.4)

$$E_{\alpha}^{(m-1)}(z) = o(1) + \frac{1}{\alpha^m} e^{z^{1/\alpha}} z^{-(1/\alpha - 1)(m-1)}$$

$$E_{\alpha}^{(m)}(z) = o(1) + \frac{1}{\alpha^{m+1}} e^{z^{1/\alpha}} z^{-(1/\alpha - 1)m}.$$

Now we consider the function

$$F(z) = \frac{1}{z} [E_{\alpha}^{(m-1)}(z) - E_{\alpha}^{(m-1)}(0)],$$

which is again an entire function of order $\alpha^{-1}$. For $|z| \to \infty$ on $\text{arg} \ z = \alpha \pi/2$ we have by (5.4)

$$F(z) = o(1) + \frac{1}{\alpha^m} e^{z^{1/\alpha}} z^{-(1/\alpha - 1)(m-1)-1} = o(1);$$

however

$$F'(z) = o(1) + \frac{1}{\alpha^{m+1}} e^{z^{1/\alpha}} z^{-(1/\alpha - 1)m-1},$$

and herein $|e^{z^{1/\alpha}}| = 1$ and $(1/\alpha) - 1)m - 1 \geq 0$, so that $F'(z) \to 0$ ($|z| \to \infty$ on $\text{arg} \ z = \alpha \pi/2$). For the lemma it is therefore sufficient to take

$$f(z) = F(ze^{i \alpha \pi/2}).$$
Proof of Theorem 5. Define the \( \{ a_n \} \) of (1.1) by

\[
f(x) = \int_0^x e^{-t} \sum_{\nu=0}^{\infty} \frac{a_\nu t^{\nu}}{\nu!} \, dt = \int_0^x e^{-t} a(t) \, dt,
\]

with the \( f(x) \) of the above lemma and \( \beta = (1-\varepsilon)^{-1} \). Since \( f(x) \) is of order \( \beta > 1 \), so is \( a(t) \), and therefore [1, p. 238] \(^5\)

\[
\limsup n^{1/\beta} \left| \frac{a_n}{n!} \right|^{1/n} = e \limsup n^{-\varepsilon} |a_n|^{1/n} < \infty,
\]

that is, (5.1) is fulfilled. Furthermore

\[
f(x) \to 0 \quad (x \to +\infty),
\]

which is equivalent to

\[
B(x; t_\nu) \to 0 \quad (x \to +\infty).
\]

However, in order that

\[
B(x; s_\nu) \to 0 \quad (x \to +\infty),
\]

it would be necessary and sufficient to have [4, pp. 182-183]

\[
e^{-x} a(x) = f'(x) \to 0 \quad (x \to +\infty),
\]

which by our lemma is not fulfilled. So we have given an example of a series \( \sum a_n \) for which \( B(x; t_\nu) \to 0 \) \( (x \to +\infty) \) does not imply \( B(x; s_\nu) \to 0 \) \( (x \to +\infty) \) and for which (5.1) holds.

\(^5\)Prof. Lösch (Stuttgart) suggested to me the relation to the coefficient problem for entire functions.

References


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