Pacific Journal of Mathematics

# ON THE CHANGE OF INDEX FOR SUMMABLE SERIES

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Vol. 5, No. 4 December 1955

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**1. Introduction.** Assume we have given a series

$$(1.1) \qquad a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

and consider

$$(1.2) \qquad b_0 + b_1 + b_2 + \cdots + b_n + \cdots \text{ with } b_0 = 0 \text{ and } b_n = a_{n-1} \quad (n \geq 1);$$

denote the partial sums by $s_n$ and $t_n$, respectively. Since $s_n = t_{n+1}$, the convergence of (1.1) is equivalent to that of (1.2). However, if a method of summability $V$ is applied to both series, the statements

$$(1.3) \qquad \text{(a)} \quad V - \sum a_n = s \qquad\qquad \text{(b)} \quad V - \sum b_n = s$$[1]

need not be equivalent (for example, if $V$ is the Borel method; see [**4**, p. 183]). If $V(x; s_\nu)$ and $V(x; t_\nu)$ denote the $V$-transforms of the sequences $\{s_n\}$ and $\{t_n\}$, respectively, it is therefore interesting to investigate, for which methods $V$ and under what restrictions on $\{a_n\}$ the relations

$$(1.4) \qquad \text{(a)} \quad V(x; s_\nu) \cong K \cdot x^q \qquad\qquad \text{(b)} \quad V(x; t_\nu) \cong K \cdot x^q$$

$$(x \longrightarrow x_0,\ K \text{ constant};\ q \geq 0, \text{ fixed})$$[2]

are equivalent.

The cases $V = C_k$ (Cesàro) and $V = A$ (Abel) are quickly disposed of (§ 2), while $V = E$ (general Euler transform) and $V = B$ (Borel) present some interest (§§ 3-5).

2. THEOREM 1. *The statements* (1.4.a) and (1.4.b) *are equivalent for*

[1] We shall always let $\sum_{n=0}^{\infty} a_n = \sum a_n$.

[2] $x \longrightarrow x_0$ through values depending on the method $V$.

Received December 1, 1953. This work has been sponsored, in part, by the Office of Naval Research under contract N5ori-07634.

*Pacific J. Math.* 5 (1955), 529-539

$V = C_k (k > -1)$ *and* $V = A$.[3]

*Proof.* If

$$S_n^{(k)} = C_k(n; s_\nu) \cdot \binom{n+k}{n}$$

and

$$T_n^{(k)} = C_k(n; t_\nu) \cdot \binom{n+k}{n},$$

we have by definition of the Cesàro means

$$(2.1) \qquad (1-x)^{k+1} \sum T_n^{(k)} x^n = \sum b_n x^n = x \cdot \sum a_n x^n = x(1-x)^{k+1} \sum S_n^{(k)} x^n,$$

the series being convergent for $|x| < 1$. The proof of Theorem 1 now follows from the inner equality in (2.1) and the relation

$$\frac{T_n^{(k)}}{\binom{n+k}{n}} = \frac{S_{n-1}^{(k)}}{\binom{n+k}{n}} \cong \frac{S_{n-1}^{(k)}}{\binom{n-1+k}{n-1}} \qquad (n \longrightarrow \infty).$$

**3. Let** $g(w) = \sum \gamma_n w^n$ be regular and schlicht in $|w| \leq 1$, and assume $g(0) = 0$, $g(1) = 1$. Then the $E$-transforms of $\sum a_n$ and $\sum b_n$ are obtained by the formal relations [5]

$$(3.1) \qquad \begin{aligned} &\sum a_n z^n = \sum a_n [g(w)]^n = \sum \alpha_n w^n; \quad E(n; s_\nu) = \sum_{\nu=0}^{n} \alpha_\nu \\ &\sum b_n z^n = \sum b_n [g(w)]^n = \sum \beta_n w^n; \quad E(n; t_\nu) = \sum_{\nu=0}^{n} \beta_\nu \end{aligned} \qquad (n = 0, 1, \cdots).$$

THEOREM 2. *The statements* (1.4.a) *and* (1.4.b) *are equivalent for* $V = E$.

*Proof.* First we note that if either

$$E(n; s_\nu) = O(n^q) \text{ or } E(n; t_\nu) = O(n^q) \qquad (n \longrightarrow \infty),$$

[3] For $q = 0$ see [4, p. 102].

then the formal relations (3.1) are actually valid for $|w| < 1$ and also

$$(3.2)\qquad \sum \beta_n w^n = \sum b_n[g(w)]^n = g(w)\cdot \sum a_n[g(w)]^n = g(w)\cdot \sum \alpha_n w^n \qquad (|w| < 1).$$

Denote by $A_n$, $B_n$, $C_n$ the partial sums of $\sum \alpha_n$, $\sum \beta_n$, $\sum \gamma_n$, respectively. We assume first

$$E(n; s_\nu) = A_n \cong K \cdot n^q \qquad (n \longrightarrow \infty).$$

Then, since by (3.2) $\sum \beta_n$ is the Cauchy product of $\sum \alpha_n$ and $\sum \gamma_n$, we have

$$E(n; t_\nu) = B_n = \gamma_n A_0 + \gamma_{n-1} A_1 + \cdots + \gamma_1 A_{n-1}$$

and for $n \geq 1$

$$(3.3)\qquad \frac{B_n}{n^q} = \frac{\gamma_n}{n^q} A_0 + \gamma_{n-1}\frac{1^q}{n^q}\cdot\frac{A_1}{1^q} + \cdots + \gamma_1 \frac{(n-1)^q}{n^q}\cdot\frac{A_{n-1}}{(n-1)^q}.$$

For the matrix $c_{n\nu}$ in this transformation of the convergent sequence $\{A_n n^{-q}\}$ we have clearly

$$\lim_{n\to\infty} c_{n\nu} = 0 \qquad (\nu = 0, 1, \cdots).$$

Furthermore

$$\sum_\nu |c_{n\nu}| = \sum_{\nu=1}^{n-1} |\gamma_{n-\nu}|\cdot\frac{\nu^q}{n^q} + \frac{|\gamma_n|}{n^q} \leq \sum_{\nu=1}^{n} |\gamma_\nu| \leq \sum_{\nu=1}^{\infty} |\gamma_\nu| = M < \infty;$$

finally we prove

$$\lim_{n\to\infty} \sum_{\nu=0}^{n-1} c_{n\nu} = 1.$$

For $q = 0$ this follows from

$$\sum_{\nu=0}^{n-1} c_{n\nu} = \sum_{\nu=1}^{n} \gamma_\nu \longrightarrow g(1) = 1 \qquad (n \longrightarrow \infty);$$

for $q > 0$

$$\sum_{\nu=0}^{n-1} c_{n\nu} = \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} \gamma_{n-\nu} \cdot \frac{\nu^q}{n^q} = \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} \gamma_\nu \left(\frac{n-\nu}{n}\right)^q$$

$$= \frac{\gamma_n}{n^q} + \sum_{\nu=1}^{n-1} C_\nu \left[\left(\frac{n-\nu}{n}\right)^q - \left(\frac{n-\nu-1}{n}\right)^q\right],$$

and the last term is a positive regular transformation of the sequence $\{C_n\}$ tending to $g(1) = 1$, whence

$$\sum_\nu c_{n\nu} \longrightarrow 1 \qquad (n \longrightarrow \infty).$$

Therefore the transformation (3.3) of $\{A_n n^{-q}\}$ converges to $K$, which proves $B_n \cong K \cdot n^q$ $(n \longrightarrow \infty)$.

Assume on the other hand $B_n \cong Kn^q$ $(n \longrightarrow \infty)$. Putting $w = 0$ in (3.2), one obtains $\beta_0 = 0$, so that

$$\sum \alpha_n w^n = [g(w)]^{-1} \sum \beta_n w^n = w[g(w)]^{-1} \sum \beta_{n+1} w^n$$

is regular in $|w| < 1$. Furthermore the expansion of the function $w[g(w)]^{-1}$ for $w = 1$ converges absolutely to 1, since $w = 0$ is the only zero of $g(w)$ in $|w| \leq 1$. An argument similar to the one above shows then that $B_{n+1} \cong Kn^q$ $(n \longrightarrow \infty)$ implies $A_n \cong Kn^q$ $(n \longrightarrow \infty)$, which completes the proof of Theorem 2.

We add a few remarks about the assumptions on the function $z = g(w)$ by which the $E$-method is defined.

a. Theorem 2 becomes false if only regularity of $g(w)$ in $|w| < 1$, and continuity and schlichtness in $|w| \leq 1$ are assumed. For there exist such functions $g(w)$ whose power series do not converge absolutely on $|w| = 1$ (cf. [2]). Therefore in (3.2) one could find a convergent $\sum \alpha_n$ whose transform $\sum \beta_n$ diverges.

b. All that was used about the function $g(w)$ in the proof of Theorem 2 was that the power series of $g(w)$ and of $w[g(w)]^{-1}$ converge absolutely to the value 1 for $w = 1$. This can be guaranteed by the weaker *assumption that* $g(w)$ *with* $g(1) = 1$ *and* $g(0) = 0$ *is regular in* $|w| < 1$, *continuous and schlicht* *in* $|w| \leq 1$, *and that the image of* $|w| = 1$ *under the mapping* $g(w)$ *is a rectifiable Jordan curve.* Because then

$$\int_0^{2\pi} |g'(e^{i\phi})|\, d\phi < \infty$$

and hence $\sum |\gamma_n| < \infty$ [**8**, p. 158]; on the other hand also

$$\int_0^{2\pi} |G'(e^{i\phi})|\, d\phi < \infty,$$

where

$$G'(w) = \left[\frac{w}{g(w)}\right]' = \frac{g(w) - wg'(w)}{[g(w)]^2},$$

so that also the power series of $G(w)$ converges absolutely to the value 1 for $w = 1$.

c. If

$$g(w) = w[(p+1) - pw]^{-1} \qquad (p \geq 0, \text{ fixed})$$

one has $E = E_p$ as the familiar Euler method of order $p$, for which Theorem 2 is known in the case $q = 0$ [**4**, p. 180].

d. The function

$$g(w) = (2 - w) - 2(1 - w)^{1/2} \qquad (g(0) = 0)$$

leads to the method of Mersman [**6**], as Scott and Wall showed [**7**, p. 270]. Here Theorem 2 is also applicable, since the more general conditions about $g(w)$ in remark (b) are satisfied, as is readily seen.

**4. The Borel method** is defined by the transformation

$$B(x; s_\nu) = e^{-x} \sum \frac{s_\nu x^\nu}{\nu!} \qquad (x \geq 0),$$

where the power series is assumed to define an entire function. It is known that $B(x; s_\nu) \longrightarrow K$ $(x \longrightarrow \infty)$ implies $B(x; t_\nu) \longrightarrow K$ $(x \longrightarrow \infty)$, but not conversely [**4**, p. 183]. We now prove more generally

THEOREM 3. *The relation*

$$B(x; s_\nu) \cong Kx^q \qquad (x \longrightarrow \infty)$$

*implies*

$$B(x; t_\nu) \cong Kx^q \qquad (x \longrightarrow \infty).$$

*Proof.* We have for $x > 0$ [4, p. 196]

$$(4.1) \qquad x^{-q} B(x; t_\nu) = x^{-q} e^{-x} \sum \frac{t_\nu x^\nu}{\nu!} = x^{-q} e^{-x} \sum \frac{s_\nu x^{\nu+1}}{(\nu+1)!}$$

$$= x^{-q} e^{-x} \int_0^x \sum \frac{s_\nu t^\nu}{\nu!} \, dt = x^{-q} \int_0^x e^{-(x-t)} t^q \frac{B(t; s_\nu)}{t^q} \, dt.$$

This transformation of the convergent function $B(t; s_\nu) t^{-q}$ $(t \longrightarrow \infty)$ by means of the 'matrix'

$$c(x, t) = e^{-(x-t)} \left( \frac{t}{x} \right)^q \qquad (0 \le t \le x)$$

is regular, since

$$\int_{t_1}^{t_2} |c(x, t)| \, dt \longrightarrow 0 \qquad (x \longrightarrow \infty;\ t_1, t_2 > 0, \text{ fixed})$$

and

$$\int_0^x |c(x, t)| \, dt = \int_0^x c(x, t) \, dt = e^{-x} \int_0^x e^t \left( \frac{t}{x} \right)^q dt \longrightarrow 1 \qquad (x \longrightarrow \infty).$$

Therefore $B(x; t_\nu) \cong Kx^q$ $(x \longrightarrow \infty)$.

We discuss now the converse of Theorem 3.

THEOREM 4. *The relation*

$$B(x; t_\nu) \cong Kx^q \qquad (x \longrightarrow \infty)$$

*implies*

$$B(x; s_\nu) \cong Kx^q \qquad (x \longrightarrow \infty),$$

*if*

(4.2) $$\limsup |a_n|^{1/n} < \infty,$$

*that is, if the series $\sum a_n z^n$ has a positive radius of convergence.*

*Proof.* Using (4.1) we have for $x > 0$

$$F(x) = x^{-q} B(x; t_\nu) = x^{-q} e^{-x} \int_0^x e^t B(t; s_\nu)\, dt.$$

Consider now $F(x)$ as function of the complex variable $x$ for $\Re(x) \geq 1$. Then (4.2) implies $|t_n| \leq M^n$ for some constant $M > 0$ and hence in $\Re(x) \geq 1$

$$|B(x; t_\nu)| \leq e^{-1} \sum \frac{M^n |x|^n}{n!} = e^{-1+M|x|},$$

and also

(4.3) $$|F(x)| \leq \alpha e^{\beta|x|} \qquad \Re(x) \geq 1$$

for positive constants $\alpha$ and $\beta$. Hence one knows that

$$F(x) \longrightarrow K \qquad (x \longrightarrow +\infty)$$

implies

$$F'(x) \longrightarrow 0 \qquad (x \longrightarrow +\infty),$$ [4]

that is,

$$x^{-q} B(x; s_\nu) + \int_0^x e^t B(t; s_\nu)\, dt \left[-1 - \frac{q}{x}\right] e^{-x} x^{-q}$$

$$= x^{-q} B(x; s_\nu) - K + o(1) = o(1) \qquad (x \longrightarrow +\infty),$$

from which the result follows.

**5. We now show** that Theorem 4 is best possible in a certain sense.

[4] If $F(x)$ is regular in $\Re(x) \geq 1$ and (4.3) holds, then $F(x) \longrightarrow A$ $(x \longrightarrow +\infty)$ implies $F'(x) \longrightarrow 0$ $(x \longrightarrow +\infty)$. This lemma was used also in [3], where Theorem 4 was proved for $q = 0$.

THEOREM 5. *In Theorem 4 the Condition (4.2) cannot be replaced by*

$$\limsup n^{-\epsilon} |a_n|^{1/n} < \infty \qquad (\epsilon > 0). \tag{5.1}$$

For the proof we need the following

LEMMA. *For every $\beta > 1$, there exists an entire function $f(z)$ of order $\beta$ satisfying*

$$f(x) \longrightarrow 0 \ (x \longrightarrow +\infty),\ f'(x) \not\longrightarrow 0 \ (x \longrightarrow +\infty) \qquad (z = x + iy). \tag{5.2}$$

*Proof.* Put $\alpha = \beta^{-1}$ and consider the Mittag-Leffler function

$$E_\alpha(z) = \sum \frac{z^n}{\Gamma(1+\alpha n)},$$

which is an entire function of order $\alpha^{-1} = \beta$. Let $m$ be the integer with

$$\frac{\alpha}{1-\alpha} \le m < \frac{\alpha}{1-\alpha} + 1.$$

We first study the derivatives of $E_\alpha(z)$ of order $1, 2, \cdots, m$ on the line $\arg z = \alpha\pi/2$ for large $|z|$. For these $z$ (assume for definiteness $|z| > 2$) one has [1, pp. 272-275]

$$E_\alpha(z) = \frac{1}{2\pi i \alpha} \int_L e^{t^{1/\alpha}} \frac{dt}{t-z} + \frac{1}{\alpha} e^{z^{1/\alpha}}, \tag{5.3}$$

the path $L$ being

$$t = re^{-i\phi_0} \left( \infty > r \ge 1, \alpha\pi > \phi_0 > \frac{\pi\alpha}{2} \right), \quad t = e^{i\phi} (-\phi_0 \le \phi \le +\phi_0),$$

$$t = re^{i\phi_0} \quad (1 \le r < \infty);$$

$t^{1/\alpha}$ is the branch which is positive for $t > 0$. The $k$th derivative of the integral part in (5.3) can then be estimated as follows

$$\left| \frac{1}{2\pi i \alpha} \int_L e^{t^{1/\alpha}} \frac{k!}{(t-z)^{k+1}} dt \right| \le \frac{k!}{2\pi\alpha |z|^{k+1}} \int_L |e^{t^{1/\alpha}}| \frac{|dt|}{|1-(t/z)|^{k+1}}$$

$$= O(|z|^{-k-1}) = o(1) \quad (|z| \longrightarrow \infty),$$

since for our values of $z$ one has $|1-(t/z)| \geq \delta > 0$ and on the straight line segments of $L$

$$|e^{t^{1/\alpha}}| = e^{|t|^{1/\alpha} \cdot \cos \phi_0/\alpha} \text{ with } \cos \frac{\phi_0}{\alpha} < 0.$$

Therefore

$$\begin{aligned} E'_\alpha(z) &= o(1) + \frac{1}{\alpha^2} e^{z^{1/\alpha}} z^{1/\alpha - 1} \\ E''_\alpha(z) &= o(1) + \frac{1}{\alpha^3} e^{z^{1/\alpha}} z^{(1/\alpha-1)2} \\ E_\alpha^{(m-1)}(z) &= o(1) + \frac{1}{\alpha^m} e^{z^{1/\alpha}} z^{(1/\alpha-1)(m-1)} \\ E_\alpha^{(m)}(z) &= o(1) + \frac{1}{\alpha^{m+1}} e^{z^{1/\alpha}} z^{(1/\alpha-1)m}. \end{aligned} \tag{5.4}$$

Now we consider the function

$$F(z) = \frac{1}{z} [E_\alpha^{(m-1)}(z) - E_\alpha^{(m-1)}(0)],$$

which is again an entire function of order $\alpha^{-1}$. For $|z| \longrightarrow \infty$ on $\arg z = \alpha\pi/2$ we have by (5.4)

$$F(z) = o(1) + \frac{1}{\alpha^m} e^{z^{1/\alpha}} z^{(1/\alpha-1)(m-1)-1} = o(1);$$

however

$$F'(z) = o(1) + \frac{1}{\alpha^{m+1}} e^{z^{1/\alpha}} z^{(1/\alpha-1)m-1},$$

and herein $|e^{z^{1/\alpha}}| = 1$ and $((1/\alpha)-1)m - 1 \geq 0$, so that $F'(z) \not\rightarrow 0$ ($|z| \longrightarrow \infty$ on $\arg z = \alpha\pi/2$). For the lemma it is therefore sufficient to take

$$f(z) = F(ze^{i\alpha\pi/2}).$$

*Proof of Theorem* 5. Define the $\{a_n\}$ of (1.1) by

$$f(x) = \int_0^x e^{-t} \sum \frac{a_\nu t^\nu}{\nu!} \, dt = \int_0^x e^{-t} a(t) dt,$$

with the $f(x)$ of the above lemma and $\beta = (1-\epsilon)^{-1}$. Since $f(x)$ is of order $\beta > 1$, so is $a(t)$, and therefore [**1**, p. 238][5]

$$\limsup n^{1/\beta} \left| \frac{a_n}{n!} \right|^{1/n} = e \limsup n^{-\epsilon} |a_n|^{1/n} < \infty,$$

that is, (5.1) is fulfilled. Furthermore

$$f(x) \longrightarrow 0 \qquad (x \longrightarrow +\infty),$$

which is equivalent to

$$B(x; t_\nu) \longrightarrow 0 \qquad (x \longrightarrow +\infty).$$

However, in order that

$$B(x; s_\nu) \longrightarrow 0 \qquad (x \longrightarrow +\infty),$$

it would be necessary and sufficient to have [**4**, pp. 182-183]

$$e^{-x} a(x) = f'(x) \longrightarrow 0 \qquad (x \longrightarrow +\infty),$$

which by our lemma is not fulfilled. So we have given an example of a series $\sum a_n$ for which $B(x; t_\nu) \longrightarrow 0$ $(x \longrightarrow +\infty)$ does not imply $B(x; s_\nu) \longrightarrow 0$ $(x \longrightarrow +\infty)$ and for which (5.1) holds.

[5]Prof. Lösch (Stuttgart) suggested to me the relation to the coefficient problem for entire functions.

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Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

*During the absence of E.G. Straus.

UNIVERSITY OF CALIFORNIA PRESS · BERKELEY AND LOS ANGELES

# Pacific Journal of Mathematics

Vol. 5, No. 4 December, 1955