ON HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH ARBITRARY CONSTANT COEFFICIENTS

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Let \( K \) be an arbitrary ordinary differential field—for our purposes it is sufficient to consider an arbitrary (algebraic) field \( K \) which is converted into a differential field by setting \( c' = 0 \) for every \( c \in K \). Let \( u \) be a differential indeterminate over \( K \) and let \( u = u_0, u_1, \ldots \) represent the successive derivatives of \( u \). Further, let \( c_0, \ldots, c_m \) be arbitrary constants over the field \( K(u_0, u_1, \ldots) \), that is, \( m + 1 \) further indeterminates with which we compute in the usual way, setting \( c_i' = 0 \). In addition to the ring \( R = K[u] = K[u_0, u_1, \ldots] \), we will also be interested in the rings \( R_t + m = K[u_0, u_1, \ldots, u_t + m] \). Theorems referring to some one of these rings \( R_t + m \) may, if convenient, be regarded as belonging to ordinary, rather than differential, algebra, but we will still apply the operation of differentiation to elements of \( R_t + m \) (not involving \( u_t + m \)). This then amounts to a convenience in writing formulas.

Let \( l_0 = c_0 u_0 + \cdots + c_m u_m \). This element generates a prime differential ideal \( [l_0] = (l_0, l_1, \ldots) \) in \( S = K(c)\{u\} \), where \( l_i = c_0 u_i + \cdots + c_m u_i + m \). We are interested in having explicitly a basis for \( [l_0] \cap K\{u\} \). If \( \Delta(u) \) is the determinant of coefficients of any \( m + 1 \) of the \( l_i \) regarded as linear forms in the \( c_j \), then clearly \( \Delta(u) \in [l_0] \cap K\{u\} \) and Theorem 2 below asserts that the \( \Delta(u) \) obtained from all choices of the \( l_i \) form the required basis.

Let us confine ourselves to the rings \( R_t + m \) and \( S_t + m = K(c)[u_0, \ldots, u_t + m] \). In \( S_t + m \), let \( p = (l_0, \ldots, l_t) \).

**Lemma 1.** \( p = (l_0, \ldots, l_t) \) is an \( m \)-dimensional prime ideal in \( S_t + m \).

**Proof.** Let \( G(u_0, \ldots, u_t + m) \in S_t + m \). Eliminating successively \( u_t + m, u_{t+m-1}, \ldots, u_m \) mod \( (l_0, \ldots, l_t) \), we may write \( G(u_0, \ldots, u_t + m) = G_1(u_0, \ldots, u_{m-1}) \mod (l_0, \ldots, l_t) \), where \( G_1 \in S_t + m \) is a polynomial in the indicated variables. Moreover, starting with indeterminate values \( \xi_i \) for \( u_i, \ i = 0, \ldots, m - 1 \), we can build up a zero \( (\xi_0, \ldots, \xi_t + m) \) of \( p \) by defining \( \xi_m \) from the condition

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$l_0(\xi) = 0$, and defining $\xi_{m+i}$ successively from the condition $l_i(\xi) = 0$. Then $(\xi_0, \ldots, \xi_{t+m})$ is clearly a general point of $p$, whence $p$ is prime and $m$-dimensional.

**Lemma 2.** Let $p \cap R_{t+m} = P$; and let $t \geq m - 1$. Then $P$ is a $2m$-dimensional prime ideal in $R_{t+m}$.

*Proof.* Consider the equations:

\[
\begin{align*}
  c_0 \xi_0 + \cdots + c_m \xi_m &= 0 \\
  c_0 \xi_1 + \cdots + c_m \xi_{1+m} &= 0 \\
  &\vdots \\
  c_0 \xi_{m-1} + \cdots + c_m \xi_{2m-1} &= 0.
\end{align*}
\]

From these we are going to solve successively for the $c_i$, $i = 0, \ldots, m - 1$. Since $\xi_0 \neq 0$, we can solve for $c_0$ and find $c_0 \in K(c_1, \ldots, c_m, \xi_0, \ldots, \xi_m)$. Suppose in this way, solving successively for the $c_i$, we find

\[
c_0, \ldots, c_i \in K(c_{i+1}, \ldots, c_{m}, \xi_0, \ldots, \xi_{m+i}), \quad i < m - 1.
\]

In fact, assume we have found inductively that

\[
(A_i) \quad c_0, \ldots, c_i \in K(\xi_0, \ldots, \xi_{2i+1}) \cdot c_{i+1} + K(\xi_0, \ldots, \xi_{2i+2}) \cdot c_{i+2} + \cdots + K(\xi_0, \ldots, \xi_{i+m}) \cdot c_m.
\]

Since

\[
dt K(c_0, \ldots, c_m, \xi_0, \ldots, \xi_{m+i})/K(c_0, \ldots, c_m) = m
\]

and

\[
dt K(c_0, \ldots, c_m)/K = m + 1,
\]

we have

\[
dt K(c_0, \ldots, c_m, \xi_0, \ldots, \xi_{m+i})/K = 2m + 1
\]

\[
= dt K(c_{i+1}, \ldots, c_m, \xi_0, \ldots, \xi_{m+i})/K,
\]

where $dt$ stands for “degree of transcendency”. From this we see that $\xi_0, \ldots, \xi_{m+i}$ are algebraically independent over $K$ (since the set $c_{i+1}, \ldots, \xi_{m+i}$ has
2m + 1 members), in particular they are not zero. The coefficient of \( c_{i+1} \) in \( l_{i+1}(\xi) \) is \( \xi_{2(i+1)} \) plus a term in \( K(\xi_0, \ldots, \xi_{2(i+1)}) \) arising from \( c_0 \xi_{i+1} + \cdots + c_i \xi_{2i+1} \), and since \( i + 1 < m \), we have \( 2(i + 1) < m + i + 1 \) and \( \xi_{2(i+1)} \notin K(\xi_0, \ldots, \xi_{2i+1}) \). Hence \( c_{i+1} \in K(\xi_{i+2}, \ldots, \xi_{m+i+1}) \); also \( A_{i+1} \) holds. Continuing, we have \( c_0, \ldots, c_{m-1} \in K(\xi_m, \xi_{i+1}, \xi_{2m-1}) \). Hence \( \xi_0, \ldots, \xi_{2m-1} \) are algebraically independent over \( K \). Thus \( P \) is at least \( 2m \)-dimensional.

Let \( \Delta_i(\xi), \ i \geq m, \) be the determinant of the coefficients of the forms \( l_0(\xi), \ldots, l_{m-1}(\xi), \ l_i(\xi) \) regarded as linear forms in \( c_0, \ldots, c_m \); that is,

\[
\Delta_i(\xi) = \begin{vmatrix}
\xi_0 & \cdots & \xi_m \\
\xi_1 & \cdots & \xi_{1+m} \\
& \cdots & \\
\xi_{m-1} & \cdots & \xi_{2m-1} \\
\xi_i & \cdots & \xi_{i+m}
\end{vmatrix}
\]

Then one finds \( c_j \Delta_i(\xi) = 0, \) so that \( \Delta_i(\xi) = 0 \). The coefficient of \( \xi_{i+m} \) in this equation is a polynomial in the indeterminates \( \xi_0, \ldots, \xi_{2m-1} \); this coefficient contains the term \( \xi_0 \xi_2 \cdots \xi_{2m-2} \) and hence is not zero (therefore also \( l_0(\xi), \ldots, l_{m-1}(\xi) \) are linearly independent over \( K(\xi) \)). Thus \( P \) is at most \( 2m \)-dimensional, and hence exactly \( 2m \)-dimensional, Q.E.D.

**Lemma 3.** Let \( M = M(u) \) be the matrix:

\[
\begin{vmatrix}
\underbrace{u_0 \cdots u_m}_{t \geq m} \\
\underbrace{u_1 \cdots u_{1+m}} \\
& \cdots \\
\underbrace{u_m \cdots u_{2m}} \\
& \cdots \\
\underbrace{u_t \cdots u_{t+m}}
\end{vmatrix}
\]

Let \( A \) be the ideal generated in \( R_{t+m} \) by the \((m + 1) \times (m + 1)\) subdeterminants of \( M(u) \). Then \( A \subseteq P \).

**Proof.** Since \( l_0(\xi), \ldots, l_{m-1}(\xi) \) are linearly independent over \( K(\xi) \) (and in fact over any field containing \( K(\xi) \)) but \( l_0(\xi), \ldots, l_{m-1}(\xi), l_1(\xi) \) are linearly dependent over \( K(\xi) \), the matrix \( M(\xi) \) has rank \( m \). Hence \( A \subseteq P \).

We want to prove \( A = P \), in particular that \( A \) is prime. Conversely, if we
knew that \( A \) were prime, we could conclude immediately that \( A = P \). In fact, suppose \( A \) is prime and let \( \eta_0, \ldots, \eta_{t+m} \) be a general point of \( A \). Since \( A \) has a basis of forms of degree \( m + 1 \), no form of degree \( m \) vanishes at \( \eta \). Hence all \( m \times m \) subdeterminants of \( M(\eta) \) differ from zero, and it follows that \( A \) is \( 2m \)-dimensional, whence \( A = P \).

In proving \( A = P \), we proceed by induction on \( m \), the assertion being clearly true for \( m = 0 \). For given \( m \), we proceed by induction on \( t (t \geq m) \). For \( t = m \), we have to prove the following lemma.

**Lemma 4.** Let \( D \) be the determinant

\[
\begin{vmatrix}
 u_0 \cdots u_m \\
 u_1 \cdots u_{1+m} \\
 \vdots \\
 u_m \cdots u_{2m}
\end{vmatrix}
\]

Then \( D \) is different from zero and is irreducible in \( R_{2m} \).

*Proof.* By induction on \( m \), being trivial for \( m = 0 \). \( D \) is linear in \( u_0 \), the coefficient \( \delta \) of \( u_0 \) being different from zero and irreducible by induction; in particular, therefore, \( D \neq 0 \). Also \( D \) is linear in \( u_{2m} \) and the coefficient \( \delta' \) of \( u_{2m} \) is irreducible. \( D \) is reducible if and only if \( \delta \) is a factor of \( D - u_0 \delta \), hence of \( D \). Similarly for \( \delta' \). Now \( \delta \) and \( \delta' \) are not associates, since they are of different degree in \( u_0 \). So \( D \) is reducible if and only if it is divisible by \( \delta \delta' \). For \( m = 1 \), this means if and only if \( u_0 u_2 - u_1^2 \) is divisible by \( u_0 u_2 \). This is not the case. For \( m > 1 \), \( D \) is reducible only if it is of degree at least \( 2m \), whereas it is of degree \( m + 1 \). Hence for every \( m \), \( D \) is irreducible.

**Definition.** An ideal is called homogeneous if it has a basis of forms. Similarly we call an ideal *isobaric* if it has a basis of isobaric polynomials.

**Lemma 5.** \( A \) and \( P \) are homogeneous and isobaric.

*Proof.* \( A \) is clearly homogeneous. Moreover consider one of the \((m + 1) \times (m + 1)\) subdeterminants of \( M(u) \), say one involving the \( i \)th and \( j \)th rows, \( i < j \). Then \( u_{i+k-2} \) is the element in the \( i \)th row and \( k \)th-column and \( u_{j+l-2} \) is the element in the \( j \)th row and \( l \)th column. Suppose \( k > l \). The determinant in question has together with a term \( \pi \cdot u_{i+k-2} u_{j+l-2} \) also a term \( \pm \pi \cdot u_{i+l-2} u_{j+k-2} \), which is of the same weight. Hence if rows \( i_0, \ldots, i_m \) are involved, each term has the weight of the term \( u_{i_0} u_{i_1 + 1} u_{i_2 + 2} \cdots u_{i_m + m} \), that is, the determinant is
isobaric. Thus \( A \) is isobaric. As for \( P \), we know that \( p \) is homogeneous, and from this and the fact that \( P = p \circ R_{t+m} \) one concludes immediately that \( P \) also is homogeneous. To see that \( P \) is isobaric, let \( g(u) \in P \) and write \( g(u) = g_r(u) + g_{r+1}(u) + \cdots \), where \( g_j(u) \) is zero or isobaric of weight \( j \). It is clearly sufficient to prove \( g_r(u) \in P \), assuming \( g_r \neq 0 \). Since \( g(u) \in P \), we have

\[
h(c) g(u) = \sum A_i(c, u) l_i(c, u),
\]

where \( h(c) \) is a polynomial in the \( c_i \) alone, and the \( A_i \) are polynomials in the \( c_i \) and \( u_j \). We assign to \( c_i \) the weight \( m - i \). Let \( h(c) = h_s(c) + h_{s+1}(c) + \cdots \), where \( h_j(c) \) is zero or isobaric of weight \( j \) and \( h_s(c) \neq 0 \). Observe that the \( l_i(c, u) \) are isobaric. Comparing terms of like weight on both sides of the above equation we see that \( h_s(c) g_r(u) = \sum A_i^r(c, u) l_i(c, u) \). Hence \( g_r(u) \in p \).

**Theorem 1.** \( A = P \). In particular, therefore, for \( m > 0 \), \( A \vdash u_0 = A \).

**Proof.** We proceed by induction on \( m \) and \( t \), and first show that \( A \vdash u_0 = A \). Let \( \xi_0, \ldots, \xi_{t+m} \) be the general zero of \( P \) introduced above. Let \( D(u) \) be the determinant occurring in Lemma 4. From \( D(\xi) = 0 \) we see that \( \xi_{2m} \) can be written as a quotient of two polynomials in the indeterminates \( \xi_0, \ldots, \xi_{2m-1} \), with the denominator being

\[
\begin{vmatrix}
\xi_0 & \cdots & \xi_{m-1} \\
\vdots & \ddots & \vdots \\
\xi_{m-1} & \cdots & \xi_{2m-2}
\end{vmatrix}
\]

which is irreducible by Lemma 4. Hence we see that

\[
\begin{vmatrix}
\xi_2 & \cdots & \xi_{m+1} \\
\vdots & \ddots & \vdots \\
\xi_{m+1} & \cdots & \xi_{2m}
\end{vmatrix} \neq 0,
\]

(for were it zero, then \( \xi_{2m} \) could be written as a quotient of two irreducible polynomials in \( \xi_1, \ldots, \xi_{2m-1} \), the denominator this time not being an associate of the other denominator). Hence \( \xi_0 \) is algebraic over \( K(\xi_1, \ldots, \xi_{t+m}) \). Hence \( \xi_1, \ldots, \xi_{t+m} \) defines a \( 2m \)-dimensional prime ideal \( P_1 \) in \( K[u_1, \ldots, u_{t+m}] \); and \( P_1 \) is generated by the \((m + 1) \times (m + 1)\) subdeterminants of \( M(u) \) which do not involve the first row of \( M(u) \). Designating also by \( P_1 \), the extension of \( P_1 \) to \( K[u_0, \ldots, u_{t+m}] \), we see that \( P_1 \subseteq A \). Let now \( u_0 g(u) \in A \). We write
\[ u_0g(u) = \sum A_i(u) \Delta_i(u), \] where the \( \Delta_i(u) \) are the \((m + 1) \times (m + 1)\) sub-
determinants of \( M(u) \), and the \( A_i \) are polynomials. We write \( A_i = A_i' + u_0 A_i'' \), where \( A_i' \) does not involve \( u_0 \). We then have \( u_0(g(u) - \sum A_i'' \Delta_i(u)) = \sum A_i' \Delta_i(u) \).

The right hand side here is of degree at most one in \( u_0 \), hence \( g_1 = g(u) - \sum A_i'' \Delta_i(u) \) does not involve \( u_0 \); \( g_1 = g(u_1, \ldots, u_{t+m}) \). Now \( g(u) \) and \( \Delta_i(u) \) vanish at \( \xi_1, \ldots, \xi_{m+t} \), hence so does \( g_1 \); that is, \( g_1 \) vanishes at \( \xi_1, \ldots, \xi_{m+t} \). Hence, \( g_1 \in P_1 \), whence \( g \in A \). Hence \( A : u_0 = A \).

As a corollary to the above we get that \( A : f = A \) for any polynomial \( f \in R_{m+t} \) containing a term \( du_0^f \), \( d \in K, d \neq 0 \) \((m > 0)\). For suppose \( fg \in A \): to prove \( g \in A \). We may suppose \( f \) and \( g \) isobaric; and also homogeneous. We then get \( du_0^g g \in A \), whence \( g \in A \).

We proceed to prove that \( A \) is prime. Let \( \overline{l_i} = l_i / u_0 = c_0 v_i + \cdots + c_m v_{i+m} \), where \( v_i = u_i / u_0 \). We pass to the rings \( \overline{R}_{t+m} = K[v_1, \ldots, v_{t+m}] \) and \( \overline{S}_{t+m} = K(c)[v] \). Observe that \( v_1, \ldots, v_{t+m} \) are algebraically independent over \( K \).

Let \( \overline{M} \) be the matrix of the coefficients of the \( \overline{l_i} \), that is, the matrix:

\[
\begin{pmatrix}
1 & v_1 & v_2 & \cdots & v_m \\
v_1 & v_2 & v_3 & \cdots & v_{1+m} \\
\ddots & \ddots & \ddots \\
v_t & v_{t+1} & v_{t+2} & \cdots & v_{t+m}
\end{pmatrix}
\]

and let \( A \) be the ideal generated in \( \overline{R}_{t+m} \) by the \((m + 1) \times (m + 1)\) subdeterminants of \( \overline{M}(v) \). Each such subdeterminant is a power of \( u_0 \) times an \((m + 1) \times (m + 1)\) subdeterminant of \( M(u) \); and vice-versa. It would therefore be sufficient to prove \( \overline{A} \) prime, in fact it would be sufficient to prove that the extension of \( A \) to the quotient ring \( Q \) of \( \overline{R}_{t+m} \) relative to the ideal \( (v_1, \ldots, v_{t+m}) \) is prime.

For suppose this proved and \( g(u) h(u) \in A \), where we assume without loss of generality that \( g(u), h(u) \) are homogeneous. Dividing by appropriate powers of \( u_0 \) and setting

\[ g(u)/u_0^f = \overline{g}(v), \quad h(u)/u_0^g = \overline{h}(v), \]

we get \( \overline{g}(v) \overline{h}(v) \in \overline{A} \), whence by assumption \( \overline{f}(v) \overline{g}(v) \) or \( \overline{f}(v) \overline{h}(v) \), say \( \overline{f} \) \( \overline{g} \) is in \( \overline{A} \) for some \( \overline{f}(v) \in \overline{R}_{t+m} \), \( \overline{f} \notin \overline{(v_1, \ldots, v_m)} \). Multiplying by a power of \( u_0 \) we find \( u_0^g f(u) g(u) \in A \), where \( f(u) \) contains a term \( d u_0^c \). Hence \( g(u) \in A \).

The ideal \( A \) in \( \overline{R}_{t+m} \) has \( \xi_1/\xi_0, \ldots, \xi_{t+m}/\xi_0 \) as a zero, hence is at least \((2m - 1)\)-dimensional. Also \( \overline{A} \) remains at least \((2m - 1)\)-dimensional upon extension to \( Q \). In fact, if \( \xi_1/\xi_0, \ldots, \xi_{t+m}/\xi_0 \) determines \( \overline{P} \) in \( \overline{R}_{t+m} \), then
\[ P \subseteq (v_1, \ldots, v_{t+m}), \text{ as one sees from the fact that } \xi_0, \ldots, \xi_{t+m} \text{ determines a homogeneous and isobaric ideal } P \text{ and } u_0 \notin P. \]

Subtracting \( v_i \) times the first row from the \((i + 1)\)th row of \( \widetilde{M} \), we get the matrix
\[
\begin{pmatrix}
1 & v_1 & v_2 & \cdots & v_m \\
0 & v_2 - v_1v_1 & v_3 - v_1v_2 & \cdots & v_{m+1} - v_1v_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & v_{t+1} - v_tv_1 & v_{t+2} - v_tv_2 & \cdots & v_{t+m} - v_tv_m \\
\end{pmatrix}
\]

Each \((m + 1) \times (m + 1)\) subdeterminant of this matrix is also an \((m + 1) \times (m + 1)\) subdeterminant of \( M \). Hence one sees that every \( m \times m \) subdeterminant of the matrix
\[
\begin{pmatrix}
v_2 & v_3 & \cdots & v_{1+m} \\
\vdots & \vdots & \ddots & \vdots \\
v_{t+1} & v_{t+2} & \cdots & v_{t+m} \\
\end{pmatrix}
\]
is a leading-form of an element in \( Q \cdot \overline{A} \). These \( m \times m \) subdeterminants generate, by induction, a \( 2(m - 1) \)-dimensional prime ideal in \( K[v_2, \ldots, v_{t+m}] \), and hence a \( (2m - 1) \)-dimensional prime ideal \( \overline{q} \) in \( K[v_1, \ldots, v_{t+m}] \). The leading form ideal of \( \overline{A} \) contains or equals \( \overline{q} \). If it contained \( \overline{q} \) properly, it would be of dimension less than \( 2m - 1 \). But an ideal and its leading form ideal have the same dimension [1; Satz 8]. Hence \( \overline{q} \) is the leading-form ideal of \( \overline{A} \) and \( \overline{A} \) is \( (2m - 1) \)-dimensional.

Moreover \( \overline{A} \) is prime. For quite generally in a local ring, if an ideal \( \overline{A} \) has a prime ideal \( \overline{q} \) as leading form ideal, it must itself be prime. In fact, suppose \( gh \in \overline{A}, \ g \notin \overline{A}, \ h \notin \overline{A}. \) Then the leading form ideal \( LFI(\overline{A},g) \) of \( (\overline{A},g) \) contains \( \overline{q} \) properly, and likewise for \( (\overline{A},h) \). But \( LFI(\overline{A},g) \times LFI(\overline{A},h) \subseteq LFI((\overline{A},g) \times (\overline{A},h)) \subseteq LFI(\overline{A}) = \overline{q}, \) a contradiction. Hence \( \overline{A} \) is prime, and the proof is complete.

The following theorem is an immediate consequence of Theorem 1.

**Theorem 2.** A basis for \( [I_0] \cap K[u] \) is given by the \((m + 1) \times (m + 1)\) subdeterminants of the \( \infty \times (m + 1) \) matrix
\[
\begin{pmatrix}
u_0 & u_1 & \cdots & u_m \\
u_1 & u_2 & \cdots & u_{1+m} \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]
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