ON HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH ARBITRARY CONSTANT COEFFICIENTS

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Let $K$ be an arbitrary ordinary differential field—for our purposes it is sufficient to consider an arbitrary (algebraic) field $K$ which is converted into a differential field by setting $c' = 0$ for every $c \in K$. Let $u$ be a differential indeterminate over $K$ and let $u = u_0, u_1, \ldots$ represent the successive derivatives of $u$. Further, let $c_0, \ldots, c_m$ be arbitrary constants over the field $K(u) = K(u_0, u_1, \ldots)$, that is, $m + 1$ further indeterminates with which we compute in the usual way, setting $c_i' = 0$. In addition to the ring $R = K[u] = K[u_0, u_1, \ldots]$, we will also be interested in the rings $R^t = K[u_0, u_1, \ldots, u_{t+m}]$. Theorems referring to some one of these rings $R^t$ may, if convenient, be regarded as belonging to ordinary, rather than differential, algebra, but we will still apply the operation of differentiation to elements of $R^t$ (not involving $u_{t+m}$). This then amounts to a convenience in writing formulas.

Let $l_0 = c_0 u_0 + \cdots + c_m u_m$. This element generates a prime differential ideal $[l_0] = (l_0, l_1, \ldots)$ in $S = K(c)[u]$, where $l_i = c_0 u_i + \cdots + c_m u_i + m$. We are interested in having explicitly a basis for $[l_0] \cap K[u]$. If $\Delta(u)$ is the determinant of coefficients of any $m + 1$ of the $l_i$ regarded as linear forms in the $c_j$, then clearly $\Delta(u) \in [l_0] \cap K[u]$ and Theorem 2 below asserts that the $\Delta(u)$ obtained from all choices of the $l_i$ form the required basis.

Let us confine ourselves to the rings $R^t$ and $S^t = K(c)[u_0, \ldots, u_{t+m}]$. In $S^t$, let $p = (l_0, \ldots, l_t)$.

**Lemma 1.** $p = (l_0, \ldots, l_t)$ is an $m$-dimensional prime ideal in $S^t$.

**Proof.** Let $G(u_0, \ldots, u_{t+m}) \in S^t$. Eliminating successively $u_{t+m}$, $u_{t+m-1}, \ldots, u_m$ mod $(l_0, \ldots, l_t)$, we may write $G(u_0, \ldots, u_{t+m}) = G_1(u_0, \ldots, u_{m-1}) \bmod (l_0, \ldots, l_t)$, where $G_1 \in S^t$ is a polynomial in the indicated variables. Moreover, starting with indeterminate values $\xi_i$ for $u_i$, $i = 0, \ldots, m-1$, we can build up a zero $(\xi_0, \ldots, \xi_{t+m})$ of $p$ by defining $\xi_m$ from the condition

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$l_0(\xi) = 0$, and defining $\xi_{m+i}$ successively from the condition $l_i(\xi) = 0$. Then $(\xi_0, \cdots, \xi_{t+m})$ is clearly a general point of $p$, whence $p$ is prime and $m$-dimensional.

**Lemma 2.** Let $p \cap R_{t+m} = P$; and let $t \geq m - 1$. Then $P$ is a $2m$-dimensional prime ideal in $R_{t+m}$.

**Proof.** Consider the equations:

\[
c_0 \xi_0 + \cdots + c_m \xi_m = 0
\]
\[
c_0 \xi_1 + \cdots + c_m \xi_{1+m} = 0
\]
\[\vdots
\]
\[
c_0 \xi_{m-1} + \cdots + c_m \xi_{2m-1} = 0.
\]

From these we are going to solve successively for the $c_i$, $i = 0, \cdots, m - 1$. Since $\xi_0 \neq 0$, we can solve for $c_0$ and find $c_0 \in K(c_1, \cdots, c_m, \xi_0, \cdots, \xi_m)$. Suppose in this way, solving successively for the $c_i$, we find

\[
c_0, \cdots, c_i \in K(c_{i+1}, \cdots, c_m, \xi_0, \cdots, \xi_{m+i}), \quad i < m - 1.
\]

In fact, assume we have found inductively that

\[(A_i) \quad c_0, \cdots, c_i \in K(\xi_0, \cdots, \xi_{2i+1}) \cdot c_{i+1}
\]

\[+ K(\xi_0, \cdots, \xi_{2i+2}) \cdot c_{i+2} + \cdots + K(\xi_0, \cdots, \xi_i) \cdot c_m.
\]

Since

\[
dt K(c_0, \cdots, c_m, \xi_0, \cdots, \xi_{m+i})/K(c_0, \cdots, c_m) = m \quad \text{and}
\]
\[
dt K(c_0, \cdots, c_m)/K = m + 1,
\]
we have

\[
dt K(c_0, \cdots, c_m, \xi_0, \cdots, \xi_{m+i})/K = 2m + 1
\]
\[= dt K(c_{i+1}, \cdots, c_m, \xi_0, \cdots, \xi_{m+i})/K,
\]

where $dt$ stands for "degree of transcendency". From this we see that $\xi_0, \cdots, \xi_{m+i}$ are algebraically independent over $K$ (since the set $c_{i+1}, \cdots, \xi_{m+i}$ has
2m + 1 members), in particular they are not zero. The coefficient of \( c_{i+1} \) in \( l_{i+1}(\xi) \) is \( \xi_{2(i+1)} \) plus a term in \( K(\xi_0, \ldots, \xi_{2i+1}) \) arising from \( c_0 \xi_{i+1} + \cdots + c_i \xi_{2i+1} \), and since \( i + 1 < m \), we have \( 2(i + 1) < m + i + 1 \) and \( \xi_{2(i+1)} \notin K(\xi_0, \ldots, \xi_{2i+1}) \). Hence \( c_{i+1} \in K(c_{i+2}, \ldots, \xi_{m+i+1}) \); also \( A_{i+1} \) holds. Continuing, we have \( c_0, \ldots, c_m \in K(c_m, \xi_0, \ldots, \xi_{2m-1}) \). Hence \( \xi_0, \ldots, \xi_{2m-1} \) are algebraically independent over \( K \). Thus \( P \) is at least 2m-dimensional.

Let \( \Delta_i(\xi) \), \( i \geq m \), be the determinant of the coefficients of the forms \( l_0(\xi), \ldots, l_{m-1}(\xi) \), \( l_i(\xi) \) regarded as linear forms in \( c_0, \ldots, c_m \); that is,

\[
\Delta_i(\xi) = \begin{vmatrix}
\xi_0 & \cdots & \xi_m \\
\xi_1 & \cdots & \xi_{1+m} \\
\cdots \\
\xi_{m-1} & \cdots & \xi_{2m-1} \\
\xi_i & \cdots & \xi_{i+m}
\end{vmatrix}
\]

Then one finds \( \xi \Delta_i(\xi) = 0 \), so that \( \Delta_i(\xi) = 0 \). The coefficient of \( \xi_{i+m} \) in this equation is a polynomial in the indeterminates \( \xi_0, \ldots, \xi_{2m-1} \); this coefficient contains the term \( \xi_0 \xi_2 \cdots \xi_{2m-2} \) and hence is not zero (therefore also \( l_0(\xi), \ldots, l_{m-1}(\xi) \) are linearly independent over \( K(\xi) \)). Thus \( P \) is at most 2m-dimensional, and hence exactly 2m-dimensional, Q.E.D.

**Lemma 3.** Let \( M = M(u) \) be the matrix:

\[
\begin{vmatrix}
u_0 & \cdots & u_m \\
u_1 & \cdots & u_{1+m} \\
\cdots \\
u_m & \cdots & u_{2m} \\
\cdots \\
u_t & \cdots & u_{t+m}
\end{vmatrix}, \ t \geq m.
\]

Let \( A \) be the ideal generated in \( R_{t+m} \) by the \((m + 1) \times (m + 1)\) subdeterminants of \( M(u) \). Then \( A \subseteq P \).

**Proof.** Since \( l_0(\xi), \ldots, l_{m-1}(\xi) \) are linearly independent over \( K(\xi) \) (and in fact over any field containing \( K(\xi) \)) but \( l_0(\xi), \ldots, l_{m-1}(\xi), l_i(\xi) \) are linearly dependent over \( K(\xi) \), the matrix \( M(\xi) \) has rank \( m \). Hence \( A \subseteq P \).

We want to prove \( A = P \), in particular that \( A \) is prime. Conversely, if we
knew that \( A \) were prime, we could conclude immediately that \( A = P \). In fact, suppose \( A \) is prime and let \( \eta_0, \ldots, \eta_{t+m} \) be a general point of \( A \). Since \( A \) has a basis of forms of degree \( m+1 \), no form of degree \( m \) vanishes at \( \eta \). Hence all \( m \times m \) subdeterminants of \( M(\eta) \) differ from zero, and it follows that \( A \) is \( 2m \)-dimensional, whence \( A = P \).

In proving \( A = P \), we proceed by induction on \( m \), the assertion being clearly true for \( m = 0 \). For given \( m \), we proceed by induction on \( t (t \geq m) \). For \( t = m \), we have to prove the following lemma.

**Lemma 4.** Let \( D \) be the determinant

\[
\begin{vmatrix}
u_0 \cdots u_m \\
u_1 \cdots u_{1+m} \\
\cdots \\
u_m \cdots u_{2m} \\
\end{vmatrix}
\]

Then \( D \) is different from zero and is irreducible in \( R_{2m} \).

**Proof.** By induction on \( m \), being trivial for \( m = 0 \). \( D \) is linear in \( u_0 \), the coefficient \( \delta \) of \( u_0 \) being different from zero and irreducible by induction: in particular, therefore, \( D \neq 0 \). Also \( D \) is linear in \( u_{2m} \) and the coefficient \( \delta' \) of \( u_{2m} \) is irreducible. \( D \) is reducible if and only if \( \delta \) is a factor of \( D - u_0 \delta \), hence of \( D \). Similarly for \( \delta' \). Now \( \delta \) and \( \delta' \) are not associates, since they are of different degree in \( u_0 \). So \( D \) is reducible if and only if it is divisible by \( \delta \delta' \). For \( m = 1 \), this means if and only if \( u_0 u_2 - u_1^2 \) is divisible by \( u_0 u_2 \). This is not the case. For \( m > 1 \), \( D \) is reducible only if it is of degree at least \( 2m \), whereas it is of degree \( m + 1 \). Hence for every \( m \), \( D \) is irreducible.

**Definition.** An ideal is called homogeneous if it has a basis of forms. Similarly we call an ideal *isobaric* if it has a basis of isobaric polynomials.

**Lemma 5.** \( A \) and \( P \) are homogeneous and isobaric.

**Proof.** \( A \) is clearly homogeneous. Moreover consider one of the \( (m+1) \times (m+1) \) subdeterminants of \( M(u) \), say one involving the \( i \)th and \( j \)th rows, \( i < j \). Then \( u_{i+k-2} \) is the element in the \( i \)th row and \( k \)th-column and \( u_{j+l-2} \) is the element in the \( j \)th row and \( l \)th column. Suppose \( k > l \). The determinant in question has together with a term \( \pi \cdot u_{i+k-2} u_{j+l-2} \) also a term \( \pm \pi \cdot u_{i+l-2} u_{j+k-2} \), which is of the same weight. Hence if rows \( i_0, \ldots, i_m \) are involved, each term has the weight of the term \( u_{i_0} u_{i_1+1} u_{i_2+2} \cdots u_{i_m+m} \), that is, the determinant is
isobaric. Thus $A$ is isobaric. As for $P$, we know that $p$ is homogeneous, and from this and the fact that $P = p \cap R_{t+m}$ one concludes immediately that $P$ also is homogeneous. To see that $P$ is isobaric, let $g(u) \in P$ and write $g(u) = g_r(u) + g_{r+1}(u) + \cdots$, where $g_j(u)$ is zero or isobaric of weight $j$. It is clearly sufficient to prove $g_r(u) \in P$, assuming $g_r \neq 0$. Since $g(u) \in P$, we have

$$h(c)g(u) = \sum A_i(c, u)l_i(c, u),$$

where $h(c)$ is a polynomial in the $c_i$ alone, and the $A_i$ are polynomials in the $c_i$ and $u_j$. We assign to $c_i$ the weight $m - i$. Let $h(c) = h_s(c) + h_{s+1}(c) + \cdots$, where $h_j(c)$ is zero or isobaric of weight $j$ and $h_s(c) \neq 0$. Observe that the $l_i(c, u)$ are isobaric. Comparing terms of like weight on both sides of the above equation we see that $h_s(c)g_r(u) = \sum A_i^*(c, u)l_i(c, u)$. Hence $g_r(u) \in p$.

**Theorem 1.** $A = P$. In particular, therefore, for $m > 0$, $A: u_0 = A$.

**Proof.** We proceed by induction on $m$ and $t$, and first show that $A: u_0 = A$. Let $\xi_0, \cdots, \xi_{t+m}$ be the general zero of $P$ introduced above. Let $D(u)$ be the determinant occurring in Lemma 4. From $D(\xi) = 0$ we see that $\xi_{2m}$ can be written as a quotient of two polynomials in the indeterminates $\xi_0, \cdots, \xi_{2m-1}$, with the denominator being

$$\begin{vmatrix}
\xi_0 & \cdots & \xi_{m-1} \\
\vdots & \ddots & \vdots \\
\xi_{m-1} & \cdots & \xi_{2m-2}
\end{vmatrix}$$

which is irreducible by Lemma 4. Hence we see that

$$\begin{vmatrix}
\xi_2 & \cdots & \xi_{m+1} \\
\vdots & \ddots & \vdots \\
\xi_{m+1} & \cdots & \xi_{2m}
\end{vmatrix} \neq 0,$$

(for were it zero, then $\xi_{2m}$ could be written as a quotient of two irreducible polynomials in $\xi_1, \cdots, \xi_{2m-1}$, the denominator this time not being an associate of the other denominator). Hence $\xi_0$ is algebraic over $K(\xi_1, \cdots, \xi_{t+m})$. Hence $\xi_1, \cdots, \xi_{t+m}$ defines a $2m$-dimensional prime ideal $P_1$ in $K[u_1, \cdots, u_{t+m}]$, and $P_1$ is generated by the $(m + 1) \times (m + 1)$ subdeterminants of $M(u)$ which do not involve the first row of $M(u)$. Designating also by $P_1$, the extension of $P_1$ to $K[u_0, \cdots, u_{t+m}]$, we see that $P_1 \subseteq A$. Let now $u_0g(u) \in A$. We write
\[ u_0 g(u) = \sum A_i(u) \Delta_i(u), \] where the \( \Delta_i(u) \) are the \( (m + 1) \times (m + 1) \) subdeterminants of \( M(u) \), and the \( A_i \) are polynomials. We write \( A_i = A_i' + u_0 A_i'' \), where \( A_i' \) does not involve \( u_0 \). We then have \( u_0 (g(u) - \sum A_i'' \Delta_i(u)) = \sum A_i' \Delta_i(u) \). The right hand side here is of degree at most one in \( u_0 \), hence \( g_1 = g(u) - \sum A_i'' \Delta_i(u) \) does not involve \( u_0 \): \( g_1 = g_1(u_1, \ldots, u_{t+m}) \). Now \( g(u) \) and \( \Delta_i(u) \) vanish at \( \xi_0, \ldots, \xi_{m+t} \), hence so does \( g_1 \); that is, \( g_1 \) vanishes at \( \xi_1, \ldots, \xi_{m+t} \). Hence, \( g_1 \in P_1 \), whence \( g \in A \). Hence \( A : u_0 = A \).

As a corollary to the above we get that \( A : f = A \) for any polynomial \( f \in R_{m+t} \) containing a term \( d u_0^r \), \( d \in K \), \( d \neq 0 \) \( (m > 0) \). For suppose \( f g \in A \): to prove \( g \in A \). We may suppose \( f \) and \( g \) isobaric; and also homogeneous. We then get \( d u_0^r g \in A \), whence \( g \in A \).

We proceed to prove that \( A \) is prime. Let \( \bar{l}_i = l_i/u_0 = c_0 v_i + \cdots + c_m v_{i+m} \), where \( v_i = u_i/u_0 \). We pass to the rings \( \bar{R}_{t+m} = K[v_1, \ldots, v_{t+m}] \) and \( S_{t+m} = K(c)[v] \). Observe that \( v_1, \ldots, v_{t+m} \) are algebraically independent over \( K \). Let \( \bar{M} \) be the matrix of the coefficients of the \( \bar{l}_i \), that is, the matrix:

\[
\begin{bmatrix}
1 & v_1 & v_2 & \cdots & v_m \\
v_1 & v_2 & v_3 & \cdots & v_{1+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_t & v_{t+1} & v_{t+2} & \cdots & v_{t+m}
\end{bmatrix},
\]

and let \( A \) be the ideal generated in \( \bar{R}_{t+m} \) by the \( (m + 1) \times (m + 1) \) subdeterminants of \( M(v) \). Each such subdeterminant is a power of \( u_0 \) times an \( (m + 1) \times (m + 1) \) subdeterminant of \( M(u) \); and vice-versa. It would therefore be sufficient to prove \( \bar{A} \) prime, in fact it would be sufficient to prove that the extension of \( A \) to the quotient ring \( Q \) of \( \bar{R}_{t+m} \) relative to the ideal \( (v_1, \ldots, v_{t+m}) \) is prime. For suppose this proved and \( g(u) h(u) \in A \), where we assume without loss of generality that \( g(u), h(u) \) are homogeneous. Dividing by appropriate powers of \( u_0 \) and setting

\[
g(u)/u_0^r = \bar{g}(v), \quad h(u)/u_0^s = \bar{h}(v),
\]

we get \( \bar{g}(v) \bar{h}(v) \in A \), whence by assumption \( \bar{f}(v) \bar{g}(v) \) or \( \bar{f}(v) \bar{h}(v) \), say \( \bar{f} \bar{g} \) is in \( A \) for some \( \bar{f}(v) \in \bar{R}_{t+m} \), \( \bar{f} \not\in (v_1, \ldots, v_m) \). Multiplying by a power of \( u_0 \) we find \( u_0^r f(u) g(u) \in A \), where \( f(u) \) contains a term \( du_0^s \). Hence \( g(u) \in A \).

The ideal \( \bar{A} \) in \( \bar{R}_{t+m} \) has \( \xi_1/\xi_0, \ldots, \xi_{t+m}/\xi_0 \) as a zero, hence is at least \( (2m-1) \)-dimensional. Also \( \bar{A} \) remains at least \( (2m-1) \)-dimensional upon extension to \( Q \). In fact, if \( \xi_1/\xi_0, \ldots, \xi_{t+m}/\xi_0 \) determines \( \bar{P} \) in \( \bar{R}_{t+m} \), then
\( \overline{P} \subseteq (v_1, \ldots, v_{t+m}) \), as one sees from the fact that \( \xi_0, \ldots, \xi_{t+m} \) determines a homogeneous and isobaric ideal \( P \) and \( u_0 \notin P \).

Subtracting \( v_i \) times the first row from the \((i+1)\)th row of \( \overline{M} \), we get the matrix

\[
\begin{vmatrix}
1 & v_1 & v_2 & \cdots & v_m \\
0 & v_2 - v_1 v_1 & v_3 - v_1 v_2 & \cdots & v_{m+1} - v_1 v_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & v_{t+1} - v_t v_1 & v_{t+2} - v_t v_2 & \cdots & v_{2m} - v_t v_m \\
\end{vmatrix}
\]

Each \((m+1) \times (m+1)\) subdeterminant of this matrix is also an \((m+1) \times (m+1)\) subdeterminant of \( M \). Hence one sees that every \( m \times m \) subdeterminant of the matrix

\[
\begin{vmatrix}
v_2 & v_3 & \cdots & v_{1+m} \\
\vdots & \vdots & \ddots & \vdots \\
v_{t+1} & v_{t+2} & \cdots & v_{t+m} \\
\end{vmatrix}
\]

is a leading-form of an element in \( Q \cdot \overline{A} \). These \( m \times m \) subdeterminants generate, by induction, a \( 2(m-1) \)-dimensional prime ideal in \( K[v_2, \ldots, v_{t+m}] \), and hence a \( (2m-1) \)-dimensional prime ideal \( \overline{q} \) in \( K[v_1, \ldots, v_{t+m}] \). The leading form ideal of \( \overline{A} \) contains or equals \( \overline{q} \). If it contained \( \overline{q} \) properly, it would be of dimension less than \( 2m-1 \). But an ideal and its leading form ideal have the same dimension \([1; \text{ Satz 8}]\). Hence \( \overline{q} \) is the leading-form ideal of \( \overline{A} \) and \( \overline{A} \) is \((2m-1)\)-dimensional.

Moreover \( A \) is prime. For quite generally in a local ring, if an ideal \( \overline{A} \) has a prime ideal \( \overline{q} \) as leading form ideal, it must itself be prime. In fact, suppose \( gh \in \overline{A} \), \( g \notin \overline{A} \), \( h \notin \overline{A} \). Then the leading form ideal \( LFI(\overline{A}, g) \) of \( (\overline{A}, g) \) contains \( \overline{q} \) properly, and likewise for \( (\overline{A}, h) \). But \( LFI(\overline{A}, g) \times LFI(\overline{A}, h) \subseteq LFI((\overline{A}, g) \times (\overline{A}, h)) \subseteq LFI(\overline{A}) = \overline{q} \), a contradiction. Hence \( \overline{A} \) is prime, and the proof is complete.

The following theorem is an immediate consequence of Theorem 1.

**Theorem 2.** A basis for \([l_0] \cap K[u]\) is given by the \((m+1) \times (m+1)\) subdeterminants of the \( \infty \times (m+1) \) matrix

\[
\begin{vmatrix}
u_0 & u_1 & \cdots & u_m \\
u_1 & u_2 & \cdots & u_{1+m} \\
\vdots & \vdots & \ddots & \vdots \\
\end{vmatrix}
\]
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