

Pacific Journal of Mathematics

ON SECOND-ORDER NON-LINEAR OSCILLATION

F. V. ATKINSON

ON SECOND-ORDER NON-LINEAR OSCILLATIONS

F. V. ATKINSON

1. In this paper we establish criteria regarding the behaviour, oscillatory or otherwise, near $x = \infty$ of the solutions of

$$(1.1) \quad y'' + fy^{2n-1} = 0,$$

where $f = f(x)$ is positive and continuous for $x \geq 0$ and n is an integer greater than 1. A solution, not identically zero, will be said to be *oscillatory* if it has infinitely many zeros for $x \geq 0$.

The three possibilities to be distinguished are that the solutions of (1.1) might be (i) all oscillatory, (ii) some oscillatory and some not, and (iii) all nonoscillatory. We give here a necessary and sufficient condition for (i) to hold, and a sufficient condition for (iii).

In the linear case, $n=1$, a number of criteria have been found for cases (i) and (iii); in the linear case (ii) is impossible. A very sensitive procedure is afforded by the chain of logarithmic tests studied by J. C. P. Miller [3], P. Hartman [1], and W. Leighton [2]; some further developments in this field have been given recently by Ruth L. Potter [4], who has in particular a result [Theorem 5.1] bearing on the limitations of this procedure. There does not, however, seem to have been found any simple necessary and sufficient condition for (i) to hold in the linear case, so it is noteworthy that such a criterion exists in the nonlinear case.

2. The result in question is:

THEOREM 1. *Let $f = f(x)$ be positive and continuous for $x \geq 0$, and let n be an integer greater than unity. Then a necessary and sufficient condition for all solutions of (1.1) to be oscillatory is*

$$(2.1) \quad \int_0^{\infty} xf dx = \infty.$$

We remark that in the linear case the criterion is necessary but not sufficient.

It should be mentioned that no solution of (1.1) becomes infinite for any finite positive x -value; this is ensured by the positiveness of $f(x)$.

We prove first that if (1.1) has a nonoscillatory solution, then (2.1) cannot hold; this will prove the sufficiency of the criterion.

Received October 29, 1953.

Let then y denote a nonoscillatory solution of (1.1); y will then be ultimately of one sign, which we may without loss of generality take to be positive. It follows from (1.1) that y'' will be ultimately negative, so that y' will tend either to a positive limit, or to zero, or to a negative limit, or to $-\infty$. The last two cases can be excluded since they would imply that y is ultimately negative. Thus y must be ultimately monotonic increasing, and y' must tend to a finite nonnegative limit.

We next integrate (1.1) over $(0, x)$, getting

$$(2.2) \quad y'(x) - y'(0) + \int_0^x f y^{2n-1} dt = 0 .$$

Since $y'(x)$ tends to a limit as $x \rightarrow \infty$, this implies that the integral on the left of (2.2) converges as $x \rightarrow \infty$; we may therefore integrate (1.1) over (x, ∞) , getting now

$$y'(\infty) - y'(x) + \int_x^\infty f y^{2n-1} dt = 0 ,$$

whence, since $y'(\infty) \geq 0$,

$$(2.3) \quad y'(x) \geq \int_x^\infty f y^{2n-1} dt$$

Still with the assumption that y is ultimately positive, let a be an x -value such that $y(x) > 0$ for $x \geq a$. We integrate (2.3) over (a, x) , where $x > a$, and get

$$y(x) - y(a) \geq \int_a^x du \int_u^\infty f y^{2n-1} dt = \int_a^x (t-a) f y^{2n-1} dt + (x-a) \int_x^\infty f y^{2n-1} dt ,$$

and hence, for $x > a$,

$$y(x) \geq \int_a^x (t-a) f y^{2n-1} dt ,$$

which we re-write in the form

$$(2.4) \quad (x-a) f y^{2n-1} \left\{ \int_a^x (t-a) f y^{2n-1} dt \right\}^{1-2n} \geq (x-a) f .$$

We now take any x_1, x_2 such that $a < x_1 < x_2$, and integrate (2.4) over (x_1, x_2) . This gives

$$(2-2n)^{-1} \left[\left(\int_a^x (t-a) f y^{2n-1} dt \right)^{2-2n} \right]_{x_1}^{x_2} \geq \int_{x_1}^{x_2} (x-a) f dx .$$

If now we make $x_2 \rightarrow \infty$, the left side remains finite; this proves that

$$\int_{x_1}^\infty (x-a) f dx < \infty ,$$

which is equivalent to

$$(2.5) \quad \int_0^{\infty} x f dx < \infty ,$$

in contradiction of (2.1). Thus the sufficiency of the criterion is proved.

As to the necessity, we shall show that if (2.5) is the case then for any prescribed value of $y(\infty)$, for example 1, there exists a solution of (1.1) such that

$$(2.6) \quad y(\infty)=1 , \quad y'(\infty)=0 ,$$

which is obviously nonoscillatory.

It is easily verified that if the integral equation

$$(2.7) \quad y(x)=1-\int_x^{\infty} (t-x)f(t)\{y(t)\}^{2n-1}dt$$

has a solution y which is continuous and uniformly bounded as $x \rightarrow \infty$, then it is also a solution of (1.1) with the supplementary conditions (2.6). The existence of a bounded continuous solution of (2.7) may be established by the Picard method of successive approximation. We define a sequence of functions

$$y_m(x) \quad (m=0, 1, \dots), \quad x \geq 0 ,$$

by

$$y_0(x) \equiv 0 ,$$

$$y_{m+1}(x) = 1 - \int_x^{\infty} (t-x)f(t)\{y_m(t)\}^{2n-1}dt \quad (m=0, 1, \dots) .$$

The remainder of the argument need only be sketched. We can prove by induction that if x is so large that

$$\int_x^{\infty} (t-x)f(t)dt < 1 ,$$

assuming now (2.5), then $0 \leq y_m(x) \leq 1$. We have also

$$y_{m+2}(x) - y_{m+1}(x) = \int_x^{\infty} (t-x)f(t)\{(y_m(t))^{2n-1} - (y_{m+1}(t))^{2n-1}\}dt ,$$

whence, for sufficiently large x ,

$$|y_{m+2}(x) - y_{m+1}(x)| \leq (2n-1) \max_{t \geq x} |y_m(t) - y_{m+1}(t)| \int_x^{\infty} (t-x)f(t)dt .$$

From this we deduce the convergence of the sequence $y_m(x)$ ($m=0, 1, \dots$), for x so large that

$$(2n-1) \int_x^\infty (t-x)f(t)dt < 1 ;$$

the continuity of the limiting function is easily established. This proves the existence of a nonoscillatory solution of (1.1) for sufficiently large x , which is enough for our purpose.

This completes the proof of Theorem 1.

3. We conclude with a simple sufficient criterion for nonoscillatory solutions which happens also to be true in the linear case [4, Lemma 1.2].

THEOREM 2. *Let $f(x)$ be positive and continuously differentiable for $x \geq 0$, and let $f' \leq 0$. Let also*

$$(3.1) \quad \int_0^\infty x^{2n-1} f dx < \infty .$$

Then (1.1) has no oscillatory solutions.

We observe first of all that the result

$$\frac{d}{dx} \left\{ \frac{1}{2} y'^2 + \frac{1}{2n} f y^{2n} \right\} = \frac{1}{2n} f' y^{2n} \leq 0$$

implies that, for any solution, y' remains bounded as $x \rightarrow \infty$.

Supposing if possible that (1.1) had an oscillatory solution, let x_0, x_1, \dots be its successive zeros. Let x_m be for convenience a zero for which $y'(x_m) > 0$, and let x'_m be the unique zero of y' in (x_m, x_{m+1}) . Integrating (1.1) over (x_m, x'_m) , we have

$$y'(x'_m) - y'(x_m) + \int_{x_m}^{x'_m} f y^{2n-1} dx = 0 ,$$

or

$$(3.2) \quad y'(x_m) = \int_{x_m}^{x'_m} f y^{2n-1} dx .$$

Now y' is positive and decreasing in (x_m, x'_m) , and $y(x_m) = 0$; hence for $x_m \leq x \leq x'_m$ we have

$$0 \leq y \leq y'(x_m)(x - x_m) .$$

Thus from (3.2) we derive

$$y'(x_m) \leq \{y'(x_m)\}^{2n-1} \int_{x_m}^{x'_m} f(x)(x - x_m)^{2n-1} dx ,$$

and so

$$1 \leq \{y'(x_m)\}^{2n-2} \int_{x_m}^{\infty} f x^{2n-1} dx .$$

This however becomes impossible as x_m becomes large, since $y'(x_m)$ has been proved to remain bounded as $x_m \rightarrow \infty$, while by (3.1) we have

$$\int_{x_m}^{\infty} f x^{2n-1} dx \rightarrow 0 .$$

Since we have obtained a contradiction it follows that (1.1) has under these assumptions no oscillatory solutions. This proves the theorem.

REFERENCES

1. P. Hartman, *On the linear logarithmico-exponential differential equation of the second order*, Amer. J. Math. **70** (1948), 764–779.
2. W. Leighton, *The detection of the oscillations of solutions of a second-order linear differential equation*, Duke Math. J. **17** (1950), 57–61.
3. J. C. P. Miller, *On a criterion for oscillatory solutions of a linear differential equation of the second order*, Proc. Cambridge Phil. Soc. **36** (1940), 283–287.
4. Ruth Lind Potter, *On self-adjoint differential equations of second order*, Pacific J. Math. **3** (1953), 467–491.

UNIVERSITY COLLEGE,
IBADAN, NIGERIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. L. ROYDEN
Stanford University
Stanford, California

E. HEWITT
University of Washington
Seattle 5, Washington

R. P. DILWORTH
California Institute of Technology
Pasadena 4, California

A. HORN*
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

H. BUSEMANN

HERBERT FEDERER

MARSHALL HALL

P. R. HALMOS

HEINZ HOPF

ALFRED HORN

R. D. JAMES

BORGE JESSEN

PAUL LÉVY

GEORGE PÓLYA

J. J. STOKER

KOSAKU YOSIDA

SPONSORS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA, BERKELEY
UNIVERSITY OF CALIFORNIA, DAVIS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD RESEARCH INSTITUTE
STANFORD UNIVERSITY
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *

AMERICAN MATHEMATICAL SOCIETY
HUGHES AIRCRAFT COMPANY
SHELL DEVELOPMENT COMPANY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, Alfred Horn at the University of California Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50; back numbers (Volumes 1, 2, 3) are available at \$2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.) No. 10 1-chome Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

UNIVERSITY OF CALIFORNIA PRESS · BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics

Vol. 5, No. 5

BadMonth, 1955

Henry A. Antosiewicz, <i>A theorem on alternatives for pairs of matrice</i>	641
F. V. Atkinson, <i>On second-order non-linear oscillation</i>	643
Frank Herbert Brownell, III, <i>Fourier analysis and differentiation over real separable Hilbert spac</i>	649
Richard Eliot Chamberlin, <i>Remark on the averages of real function</i>	663
Philip J. Davis, <i>On a problem in the theory of mechanical quadrature</i>	669
Douglas Derry, <i>On closed differentiable curves of order n in n-spac</i>	675
Edwin E. Floyd, <i>Boolean algebras with pathological order topologie</i>	687
George E. Forsythe, <i>Asymptotic lower bounds for the fundamental frequency of convex membrane</i>	691
Israel Halperin, <i>On the Darboux proptert</i>	703
Theodore Edward Harris, <i>On chains of infinite orde</i>	707
Peter K. Henrici, <i>On certain series expansions involving Whittaker functions and Jacobi polynomial</i>	725
John G. Herriot, <i>The solution of Cauchy's problem for a third-order linear hyperbolic differential equation by means of Riesz integral</i>	745
Jack Indritz, <i>Applications of the Rayleigh Ritz method to variational problem</i>	765
E. E. Jones, <i>The flexure of a non-uniform bea</i>	799
Hukukane Nikaidô and Kazuo Isoda, <i>Note on non-cooperative convex game</i>	807
Raymond Moos Redheffer and W. Wasow, <i>On the convergence of asymptotic solutions of linear differential equation</i>	817
S. E. Warschawski, <i>On a theorem of L. Lichtenstei</i>	835
Philip Wolfe, <i>The strict determinateness of certain infinite game</i>	841