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REMARK ON THE AVERAGES OF REAL FUNCTION

RICHARD ELIOT CHAMBERLIN

REMARK ON THE AVERAGES OF REAL FUNCTIONS

R. E. CHAMBERLIN

1. Introduction. Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. It is known that if for each x in the open interval (a, b) there is a positive number t such that

$$[x-t, x+t] \subset (a, b) \quad \text{and} \quad f(x) = \frac{f(x-t) + f(x+t)}{2}$$

then $f(x)$ is linear (see [2, p. 253]). The same method of proof shows that if there is such a t for each $x \in (a, b)$ with

$$f(x) = \frac{1}{2t} \int_{x-t}^{x+t} f(s) ds$$

then $f(x)$ is linear. Suppose $f(x)$ is such that for each $x \in (a, b)$ there exists a t with $[x-t, x+t] \subset (a, b)$ and

$$(1) \quad \frac{f(x+t) + f(x-t)}{2} = \frac{1}{2t} \int_{x-t}^{x+t} f(s) ds.$$

Is $f(x)$ necessarily linear? On page 231 of [1] it is shown that if (1) holds for each x and all t such that $[x-t, x+t] \subset (a, b)$ then $f(x)$ is linear. The question arises whether or not one can relax the requirement that (1) holds for all t in the above intervals and still conclude that $f(x)$ is linear.

In this note a continuously differentiable non-linear function $f(x)$ is given which satisfies (1) for every $x \in (a, b)$ and an infinity of t 's. The values of t depend on x but they may be chosen arbitrarily small for each x . Conditions which together with (1) make $f(x)$ linear are given and the note is concluded with some remarks on the approximation to a function by its averages

$$f(x, t) = \frac{1}{2t} \int_{x-t}^{x+t} f(s) ds.$$

DEFINITION. A continuous function $f(x)$ on $[a, b]$ will be said to have property (1) if for each $x \in (a, b)$ there are arbitrarily small values of t for which (1) is true.

2. An example. We give an example of a continuously differentiable function having property (1) which is not linear. Let

$$(2) \quad f(x) = \sum_{n=1}^{\infty} \frac{\cos 10^{2n} \pi x}{n^2 \cdot 10^{2n}}$$

It is clear that $f(x)$ is not linear and is continuously differentiable. To show that $f(x)$ has property (1) we begin with the following

LEMMA. For every x ,

$$\lim_{n \rightarrow \infty} |\cos 10^{2n} \pi x| \geq 10^{-3}.$$

Since the functions $\cos 10^{2n} \pi x$ ($n \geq 1$) all have 1 as a period it is clear we need only consider $x \in [0, 1]$ in the proof of this lemma. Since there is no loss in generality we assume hereafter that we are dealing with the interval $[0, 1]$ and x is in this interval.

Let the decimal expansion of x be $.a_1 a_2 \dots$. Then

$$10^{2n} x = a_1 a_2 \dots a_{2n} + .a_{2n+1} a_{2n+2} \dots \text{ and } |\cos 10^{2n} \pi x| = |\cos(.a_{2n+1} a_{2n+2} \dots) \pi|.$$

Suppose $|\cos 10^{2n} \pi x| < 10^{-3}$. Set $.a_{2n+1} a_{2n+2} \dots = .5 + r_n$ where $|r_n| < .5$. Then

$$10^{-3} \geq |\cos(.a_{2n+1} a_{2n+2} \dots) \pi| = |\sin r_n \pi| = \sin |r_n \pi| \geq \frac{2}{\pi} |r_n \pi|,$$

that is $\frac{1}{2 \cdot 10^3} \geq |r_n|$. Hence there is an integer b with $0 \leq b \leq 5$ such that $|r_n| = .000b \dots$. Therefore,

$$|\cos 10^{2(n+1)} \pi x| = |\cos(.0b \dots) \pi| \geq \left(1 - \frac{(.1\pi)^2}{2}\right) > .9.$$

Thus for every x and every n_0 there are integers $n > n_0$ such that $|\cos 10^{2n} \pi x| \geq 10^{-3}$. This proves the lemma.

For the function (2) we have

$$(3) \quad g(x, t) = \frac{1}{2} [f(x+t) + f(x-t)] - \frac{1}{2t} \int_{x-t}^{x+t} f(s) ds$$

$$= \sum_{n=1}^{\infty} \frac{1}{2 \cdot 10^{2n} \cdot n^2} [\cos 10^{2n} \pi(x+t) + \cos 10^{2n} \pi(x-t)]$$

$$- \frac{1}{2t} \sum_{n=1}^{\infty} \frac{\sin 10^{2n} \pi(x+t) - \sin 10^{2n} \pi(x-t)}{10^{2n} \cdot n^3 \cdot 10^{2n} \pi}.$$

From elementary trigonometric identities we now obtain

$$g(x, t) = \sum_{n=1}^{\infty} \frac{1}{10^{2n} n^2} \cos 10^{2n} \pi x \left[\cos 10^{2n} \pi t - \frac{\sin 10^{2n} \pi t}{10^{2n} \pi t} \right].$$

We investigate in detail the last expression for $g(x, t)$ in (3).

Given x , let $\overline{\lim}_{n \rightarrow \infty} |\cos 10^{2n} \pi x| = d$. From the lemma it is clear there are an infinity of integers k with the following properties:

- (a) $|\cos 10^{2k} \pi x| > .99d$.
- (b) $|\cos 10^{2n} \pi x| < 1.01 d$ for $n \geq [k/3]$
- (c) $k \geq 10$.

For these values of k we show that the sign of $g(x, t)$ in (3) is determined by the sign of the k -th term in its series expansion if t is chosen properly. We assume hereafter that k is subject to conditions (a), (b) and (c).

For the given x and subject to conditions (a), (b) and (c) pick k large enough so that for $t = 2 \cdot 10^{-2k}$, $[x-t, x+t] \subset [0, 1]$. Then

$$(4) \quad g(x, 10^{-2k}) = \sum_{n=1}^{\infty} \frac{\cos 10^{2n} \pi x}{n^2 10^{2n}} \left[\cos 10^{2(n-k)} \pi - \frac{\sin 10^{2(n-k)} \pi}{10^{2(n-k)} \pi} \right]$$

$$= \sum_{n=1}^{k-1} \frac{\cos 10^{2n} \pi x}{n^2 10^{2n}} \left(-\frac{\pi^2}{6} 10^{4(n-k)} + \theta_n \cdot 10^{6(n-k)} \right) + (-1) \cdot \frac{\cos 10^{2k} \pi x}{k^2 10^{2k}} + \sum_{n=k+1}^{\infty} \frac{\cos 10^{2n} \pi x}{n^2 10^{2n}}$$

where $|\theta_n| < 2$. Now

$$(5) \quad \left| \sum_{n=1}^{k-1} \frac{\cos 10^{2n} \pi x}{n^2 10^{2n}} \left(-\frac{\pi^2}{6} 10^{4(n-k)} + \theta_n 10^{6(n-k)} \right) \right|$$

$$\leq \frac{10}{3} \cdot \frac{1}{10^{2k}} \cdot \sum_{n=1}^{k-1} \frac{|\cos 10^{2n} \pi x|}{n^2} 10^{2(n-k)}$$

$$\leq \frac{10}{3} 10^{-2k} \left(\sum_{n=1}^{[k/3]-1} \frac{1}{n^2} 10^{2(n-k)} \right) \cdot 10^3 d + \left(\frac{10}{3} \right) 10^{-2k} \left(\sum_{n=[k/3]}^{k-1} \frac{1}{n^2} 10^{2(n-k)} \right) 1.01 d$$

where we have used the lemma and property (b) of k to get the last inequality.

For the first sum in the last inequality of (5) we have

$$(6) \quad \sum_{n=1}^{[k/3]-1} \frac{1}{n^2} 10^{2(n-k)} < \sum_{n=1}^{[k/3]-1} 10^{2(n-k)} \leq 10^{-4/3(k-1)} \frac{1 - (10^{-2})^{(k/3)}}{1 - 10^{-2}}$$

$$< (1.01) 10^{-4/3(k-1)}.$$

To get an estimate on the second part of the last inequality of (5),

recall that if $s_n = \sum_{i=1}^n \alpha_i$

then

$$\sum_{n=r}^m \alpha_n \beta_n = \sum_{n=r}^m s_n (\beta_n - \beta_{n+1}) - s_{r-1} \beta_r + s_m \beta_{m+1}.$$

Letting $\alpha_n = 10^{2(n-k)}$, $\beta_n = 1/n^2$ we get

$$(7) \quad \sum_{n=[k/3]}^{k-1} \frac{10^{2(n-k)}}{n^2} = \sum_{n=[k/3]}^{k-1} 10^{-2(k-n)} \left(\frac{1-10^{-2n}}{1-10^{-2}} \right) \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \\ - 10^{-2(k-[k/3]-1)} \left(\frac{1-10^{-2[k/3]}}{1-10^{-2}} \right) \cdot 1/[k/3]^2 + 10^{-2} \left(\frac{1-10^{-2(k-1)}}{1-10^{-2}} \right) \cdot \frac{1}{k^2} < \frac{2}{10^2} \frac{1}{k^2}$$

at least for $k \geq 10$. Using the estimates obtained in (6) and (7) we get

$$(8) \quad \frac{10}{3} \cdot 10^{-2k} \sum_{n=1}^{k-1} \frac{|\cos 10^{2n} \pi x|}{n^2 10^{2n}} \leq \frac{d}{10^{2k}} \left[\frac{1.01}{3} 10^{-4/3(k-1)+4} + \frac{10}{3} \cdot (1.01) \cdot \frac{2}{10^2 k^2} \right] \\ < \frac{2}{10} d \cdot \frac{1}{k^2 10^{2k}} \quad \text{for } k \geq 10.$$

Furthermore

$$(9) \quad \left| \sum_{n=k+1}^{\infty} \frac{\cos 10^{2n} \pi x}{10^{2n} n^2} \left[\cos 10^{2(n-k)} \pi - \frac{\sin 10^{2(n-k)} \pi}{10^{2(n-k)} \pi} \right] \right| \\ \leq 1.01 d \sum_{n=k+1}^{\infty} \frac{1}{10^{2n} n^2} < \frac{1.01 d}{(k+1)^2} \cdot \frac{1}{10^{2(k+1)}} \cdot \frac{1}{1-10^{-2}} < \frac{1}{10} \frac{d}{k^2 10^{2k}}.$$

From (8) and (9) we see that the k -th term of the series for $g(x, 10^{-2k})$ is greater in absolute value than the sum of the remaining terms. Hence the signs of $g(x, 10^{-2k})$ and $-10^{-2k} k^{-2} \cos 10^{2k} \pi x$ are the same. For $t = 2 \cdot 10^{-2k}$ the k -th term of the series for $g(x, 2 \cdot 10^{-2k})$ is $10^{-2k} k^{-2} \cos 10^{2k} \pi x$ and in the same manner as above one can show that the signs of $g(x, 2 \cdot 10^{-2k})$ and the k -th term are the same. Since $10^{-2k} k^{-2} \cos 10^{2k} \pi x$ and $-10^{-2k} k^{-2} \cos 10^{2k} \pi x$ are of opposite signs, $g(x, t)$ vanishes for some $t \in (10^{-2k}, 2 \cdot 10^{-2k})$. But for $g(x, t)$ to vanish means that $f(x)$ satisfies (1). Since for each x there are an infinity of k 's satisfying (a), (b) and (c), there are (for each x) arbitrarily small values of t for which the $f(x)$ of (2) satisfies (1). Hence this $f(x)$ has the property (1).

3. Sufficient conditions for a function to be linear.

LEMMA 1. *If $f(x)$ is continuously differentiable and $f''(x_0) \neq 0$, then $g(x_0, t)$ is of one sign for some open interval $(0, t_0)$ ($t_0 > 0$).*

Under the stated conditions we may represent $f(x)$ by

$$(10) \quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + o[(x-x_0)^2].$$

Using (10) and the definition of $g(x, t)$ gives

$$\begin{aligned}
 (11) \quad g(x_0, t) &= \frac{f(x_0+t) + f(x_0-t)}{2} - \frac{1}{2t} \int_{x_0-t}^{x_0+t} f(u) du \\
 &= \left\{ f(x_0) + \frac{f''(x_0)}{2} t^2 + o(t^2) \right\} - \frac{1}{2t} \int_{x_0-t}^{x_0+t} \left\{ f(x_0) + f'(x_0)(u-x_0) \right. \\
 &\quad \left. + \frac{f''(x_0)}{2} (u-x_0)^2 + o[(u-x_0)^2] \right\} du = \frac{1}{3} f''(x_0) t^2 + o(t^2).
 \end{aligned}$$

Thus if $f''(x_0) \neq 0$, it is clear $g(x_0, t)$ is one-signed for sufficiently small values of t .

THEOREM 1. *If $f(x)$ has property (1) and $f'(x)$ is absolutely continuous then $f(x)$ is linear.*

For $f''(x)$ exists almost everywhere and by Lemma 1 it is zero everywhere it exists because $f(x)$ has property (1). Hence $f'(x)$ is a constant and $f(x)$ is linear.

THEOREM 2. *If $f'(x)$ is continuous, monotone increasing and not constant in any sufficiently small symmetric interval about x_0 then $g(x_0, t)$ is one-signed in an interval $(0, t_0)$.*

One has

$$f(x_0+t) = f(x_0-t) + \int_{x_0-t}^{x_0+t} f'(u) du$$

and for any $x \in (x_0-t, x_0+t)$ we get

$$(12) \quad f(x) \leq f(x_0-t) + f'(x)(x-x_0+t), \quad f(x_0+t) \geq f(x) + f'(x)(x_0+t-x)$$

where at least one of the inequalities is strict by the hypothesis of Theorem 2. From (12) one obtains

$$(13) \quad \frac{(x-x_0+t)f(x_0+t) + (x_0-x+t)f(x_0-t)}{2t} > f(x).$$

It is obvious from (13) that $g(x_0, t)$ is positive. Clearly this result with the inequality reversed holds if $f'(x)$ is monotone decreasing.

We do not know if property (1) and bounded variation of $f'(x)$ imply linearity for $f(x)$. In view of the two preceding theorems it seems quite likely.

4. Remarks on the approximation of a function by its averages.

Suppose $f(x)$ is a continuous function defined on the interval $(a-\delta, b+\delta)$ ($\delta > 0$). We make some remarks on the approximation to

$f(x)$ by its averages

$$f(x, t) = \frac{1}{2t} \int_{x-t}^{x+t} f(u) du \quad (0 < t < \delta), \quad x \in [p, b].$$

If $f(x)$ is linear then $f(x, t) \equiv f(x)$. If $f(x)$ is not linear in any subinterval then there is an everywhere dense subset of points x at which the approximating functions are all either above or below $f(x)$. Otherwise the conditions of the theorem of [2, p. 253] are met and $f(x)$ would be linear.

One might ask if there are necessarily points at which $f(x, t)$ approaches $f(x)$ monotonely. From the results of § 2 above this can be seen to be false. For $t > 0$, $f(x, t)$ is continuously differentiable function of t and

$$f_t(x, t) = \frac{1}{t} \left\{ \frac{f(x+t) + f(x-t)}{2} - \frac{1}{2t} \int_{x-t}^{x+t} f(u) du \right\} = \frac{1}{t} g(x, t).$$

From this it is clear the function of § 2 gives an example of a continuously differentiable function which at no point is approximated monotonely by its averages.

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