ON CLOSED DIFFERENTIABLE CURVES OF ORDER $n$ IN $n$-SPAC

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1. Introduction. Let \( C_n \) be a closed curve in real projective \( n \)-space \( S_n \) whose coordinates \( x_i (1 \leq i \leq n+1) \) are given in the parametric form

\[
x_i = x_i(s), \quad 1 \leq i \leq n+1, \quad q \leq s < q+1,
\]

where \( x_i(s) \) are real continuous periodic functions of period 1, and \( q \) is any real number. The point with coordinates \( x_i(s) (1 \leq i \leq n+1) \) will be designated by its defining number \( s \).

The curve \( C_n \) is to satisfy the following order condition.

No hyperplane of \( S_n \) contains more than \( n \) points of \( C_n \).

A simple consequence of the above condition is that any \( k+1 \) \((0 \leq k \leq n)\) distinct curve points \( s_1, s_2, \ldots, s_{k+1} \) span a linear \( k \)-subspace \([s_1, s_2, \ldots, s_{k+1}]\). (The square-bracket symbol \([A, B, \ldots]\) will be used throughout to designate the linear subspace spanned by the sets \( A, B, \ldots \).)

The curve \( C_n \) is to satisfy the following differentiability condition.

For each point \( s \) of \( C_n \) and for each integer \( k \) \((0 \leq k \leq n-1)\) a linear \( k \)-subspace \((k, s)\), known as the osculating \( k \)-space at \( s \), exists for which \([s_1, s_2, \ldots, s_{k+1}]\) converges to \((k, s)\) as \( s_1, s_2, \ldots, s_{k+1} \) all approach \( s \) in any way whatsoever.

The curves \( C_3 \) were considered by A. Kneser [2] who studied properties which are invariant to certain continuous displacements. One of his results is that the set of planes of the projective space each of which contains exactly \( k \) \((k=1 \text{ or } 3)\) points of a \( C_3 \) builds a connected set. In the present paper the methods used by Kneser are adapted to study the properties of the curves \( C_n \). All the proofs make use of those lines \( l \) each point of which is included in \( n \) distinct \((n-1, s)\). Thus the paper is, in a sense, a study of this line system. Among

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the results is a generalization of the foregoing Kneser result to $n$ dimensions. This in turn leads to the result that those hyperplanes which contain less than $n$ points of $C_n$ are exactly those hyperplanes which contain at least one line $l$. This result is related to a result, implicit in a paper of Scherk [4], which states that the above hyperplanes are exactly those hyperplanes which contain certain limiting positions of the lines $l$.

2. Multiplicities. As all the critical boundary cases involve multiple intersection points, these points will have special importance. In this section we record the definition for multiplicity and note some known results which we shall use.

**Definition 1.** A linear subspace $Q$ is defined to intersect $C_n$ exactly $k$-fold $(0 < k \leq n - 1)$ at $s$ if $(k - 1, s) \subseteq Q$, $(k, s) \nsubseteq Q$, and $n$-fold if $(n - 1, s) = Q$.

A point $P$ is defined to be included in $(n - 1, s)$ exactly $k$-fold $(0 < k \leq n - 1)$ if $P \in (n - k, s)$, $P \nsubseteq (n - k - 1, s)$, and $n$-fold if $P = (0, s)$.

The following multiplicity convention will be assumed throughout. Let $s_1, s_2, \ldots, s_j$ be any point system, and let $s_i$ occur $k_i$-times $(1 \leq i \leq j)$ in this system. A linear subspace $Q$ is said to contain this system provided $(k_i - 1, s_i) \subseteq Q$ $(1 \leq i \leq j)$. A point $P$ is said to be included in the system $(n - 1, s_1)$, $(n - 1, s_2)$, $\ldots$, $(n - 1, s_j)$ provided $P \in (n - k_i, s_i) (1 \leq i \leq j)$. Unless otherwise stated the points of any given set are not necessarily all distinct.

For reference we state the easily proved:

**Lemma 1.** For $n \geq 2$, the projection of $C_n$ from one of its curve points $s'$ is a $C_{n-1}$. The space $(k, s)$, $s \Rightarrow s'$, $0 \leq k \leq n - 2$, projects into the space $(k, s')$ of the projected $C_{n-1}$ and the space $(k, s)$, $1 \leq k \leq n - 1$, into the space $(k - 1, s')$ of $C_{n-1}$.

By use of Lemma 1, it can be proved by induction that $C_n$ satisfies the sharpened order condition, that no hyperplane cuts $C_n$ in more than $n$ curve points where multiple intersections are now counted with their proper multiplicity. This leads to the fact that the system $s_1$, $s_2$, $\ldots$, $s_{k+1}$ $(0 < k \leq n - 1)$ is included in a unique $k$-space which we designate by $[s_1, s_2, \ldots, s_{k+1}]$. We note without proof that $C_n$ satisfies the sharpened differentiability condition that $[s_1, s_2, \ldots, s_{k+1}]$ converges to $(k, s)$ as $s_1, s_2, \ldots, s_{k+1}$ all approach $s$.

Use will be made of the duality theorem of Scherk [3] which states that all the $(n - 1, s)$ build the dual of a $C_n$. This implies that
no point $P$ is contained within more than $n$ $(n-1, s)$ and also that the
intersection of $(n-1, s_1), (n-1, s_2), \ldots, (n-1, s_k) \ (1 \leq k \leq n)$
approaches $(n-k, s)$ as $s_1, s_2, \ldots, s_k$ all approach $s$ in any way whatsoever.

3. Notation. Throughout the paper the symbols $l, l^r$ will be
tacitly assumed to represent lines each of the points of which is with-
in $n$ distinct $(n-1, s)$ of a given $C_n$; $L, L^r$ will be assumed to represent
the $(n-2)$-spaces with the property that every hyperplane through
such a space cuts $C_n$ in $n$ distinct points.

Where a proof involves both $C_n$ and $C_{n-1}$ the symbol $(k, s)_{n-1}$ will
be used to designate the osculating $k$-space of the curve $C_{n-1}$.

4. A construction for the lines $l$.

**Theorem 1.** If, for $n \geq 2$, $A$ and $B$ are any two distinct points of
a given line $l$, then curve points $s_i, t_i$ of $C_n$ exist so that $A \in (n-1, s_i),
B \in (n-1, t_i) \ (1 \leq i \leq n)$ and $s_1 < t_1 < s_2 < \cdots < s_n < t_n < s_1 + 1 =-s_{n+1}$.

Conversely if $A$ and $B$ are points for which $A \in (n-1, s_i),
B \in (n-1, t_i), s_1 < t_1 < s_2 < \cdots < s_n < t_n < s_1 + 1 =-s_{n+1}$, then $AB$ is a line $l$.

**Proof.** Let $P(s)$ be the intersection $l \cap (n-1, s)$. Note that $l \cap (n-1, s)$; for otherwise $l$ would contain a point of $(n-2, s)$, which
point would be within $(n-1, s)$ at least twice contrary to the definition
of $l$. Therefore $P(s)$ is defined uniquely for all $s$. As $s$ moves
continuously on $C_n$ in a fixed direction, $P(s)$ moves continuously on $l$
because $(n-1, s)$ is continuous. Also, $P(s)$ moves continuously in a
fixed direction; for if $P(s)$ were to experience a reversal of direction
at $P(s_0)$ then, in every curve neighborhood of $s_0$, points $s_L, s_R$
would exist so that $s_L < s_0 < s_R$. Then, as $P(s)$ is continuous,

$$P(s_0) \in \lim_{s_L \to s_0, \ s_R \to s_0} (n-1, s_L) \cap (n-1, s_R)=(n-2, s_0)$$

and $l$ would contain a point not in $n$ distinct $(n-1, s)$ contrary to the
hypothesis. Let $(n-1, s_i) \ (1 \leq i \leq n; s_1 < s_2 < \cdots < s_n < s_1 + 1 =-s_{n+1})$ be the
complete set of $(n-1, s)$ which contain $A$. As $s$ increases continuously
from $s_1$ to $s_n$, $P(s)$ makes one complete circuit of $l$ in a fixed direction.
Consequently it crosses the point $B$ exactly once. Hence $t_i$ exists on
$C_n$ so that $B \in (n-1, t_i) \ (s_i < t_i < s_2)$. Likewise within each arc $s_i < s < s_{i+1} \ (2 \leq i \leq n)$, a point $t_i$ exists on $C_n$ so that $s_i < t_i < s_{i+1}$, $B \in (n-1, t_i)$. Thus the theorem is proved.

To prove the converse, let $C$ be any interior point of one of the
segments $AB$ of the line through $A$ and $B$, and $D$ any interior point
of the other segment. As $P(s)$ is continuous and

$$P(s_1) = A, \quad P(t_1) = B,$$

at least one solution $P(s) = C$, or $P(s) = D$ must exist for which $s_1 < s < t_1$. Likewise each of the $2n$ arcs $s_i < s < t_i, \quad t_i < s < s_{i+1} (1 < i < n)$ contains at least one solution $P(s) = C$ or $P(s) = D$. But as $C$ is contained in at most $n (n - 1, s)$ there must be exactly $n$ solutions $P(s) = C$. As these are all distinct and $C$ is arbitrary, $AB$ is a line $l$. The proof is now complete.

This proof of the converse, due to Dr. P. Scherk, replaces a more complicated one of my own. I should like to take the opportunity to thank him for many helpful suggestions which have contributed to the readability of the paper.

5. Hyperplanes with a given number of curve points.

Lemma 2. If, for $n \geq 3$, $C_{n-1}$ is the projection of $C_n$ from one of its points $s$, then a line $l$ of $C_n$ is projected into a line $l$ of $C_{n-1}$.

This is proved in [1].

Lemma 3. For $n \geq 3$, the projection of a $C_n$ from a line $l$ is a $C_{n-2}$.

Proof. No hyperplane through $l$ can cut $C_n$ in more than $n - 2$ points. This is true for $n = 2$ as it is equivalent to the fact that a line $l$ of $C_2$ cannot contain any curve points. Assume the assertion is true for $C_{n-1} (n > 2)$. Let $H$ be a hyperplane which contains $l$. The result is clear if $H$ contains no points of $C_n$. Let $\bar{s}$ be a point of $C_n$ within $H$. Project from $s$. Then $C_n$ is projected into a $C_{n-1}$ by Lemma 1, and $l$ into a line $l$ of $C_{n-1}$, by Lemma 2, which is within the projection $\bar{H}$ of $H$. By the induction assumption $\bar{H}$ contains at most $n - 3$ points of $C_{n-1}$. Therefore $H$, which contains the points $C_n$ into which these are projected together with $s$ contains at most $n - 2$ points of $C_n$.

The space of all 2-spaces through $l$ is an $(n - 2)$-space $S_{n-2}$ whose hyperplanes are the hyperplanes of the original space which contain $l$. The elements $[l, s]$ of $S_{n-2}$ build a curve $C$, and $C$ has order $n - 2$ by the result of the previous paragraph. This implies

$$[l, s'] \approx [l, s''] \quad \text{if} \quad s' \approx s''.$$
Thus there is a one-to-one correspondence between the points of \( C_n \) and those of \( C \). Where \( 0 \leq k \leq n-2 \), let

\[ [l, s_1], [l, s_2], \ldots, [l, s_{k+1}] \]

be given curve points of \( C \). Because of the order condition these points span a \((k+2)\)-space \( Q \) which contains \( l \). If \( s_1, s_2, \ldots, s_{k+1} \) all approach \( s \), then \( Q \rightarrow [l, (k, s)] \) because of the differentiability condition. Thus the set of elements \([l, s]\) of \( S_{n-2} \) is a \( C_{n-2} \) with osculating \( k \)-spaces \([l, (k, s)]\). As this set is equivalent to the projection of \( C_n \) from \( l \), the lemma is established.

Most induction proofs for the curves \( C_n \) make use of Lemma 1; in the following proof Lemma 3 is used for this purpose.

**Theorem 2.** Where \( 0 \leq k \leq n \) (mod \( 2 \)), let \( s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_k \) be any points of \( C_n \); then:

(a) If, for \( n \geq 1 \), \( H_1, H_2 \) be hyperplanes which contain \( s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_k \) respectively, and no additional points of \( C_n \), then hyperplanes \( H(p) \) \((0 \leq p \leq 1)\) exist, continuously dependent on \( p \), each of which contains exactly \( k \) points of \( C_n \) and for which \( H(0)=H_1, H(1)=H_2 \);

(b) If \( s_i = t_i \) \((1 \leq i \leq k)\), then \( H(p) \) can be chosen so that it contains exactly the points \( s_i \) \((1 \leq i \leq k, 0 \leq p \leq 1)\);

(c) if \( n \geq 2 \), \( 0 \leq k \leq n-2 \), for a given line \( l \), a hyperplane \( H^t \) exists so that it contains exactly the points \( s_1, s_2, \ldots, s_k \), together with the line \( l \).

**Proof.** We first prove (c). If \( n=2 \) then \( k=0 \) and the result is equivalent to the fact that \( H^t = l \) does not cut \( C_2 \). Assume the result for for all curves \( C_{n-1} \) \((n \geq 2)\). Project from \( l \). Thus \( C_n \) is projected into a \( C_{n-2} \), by Lemma 3, and \( s_1, s_2, \ldots, s_k \) into points of \( C_{n-2} \) with the same numerical coordinates. If \( k=n-2 \), a unique hyperplane

\[ H' = [s_1, s_2, \ldots, s_k] \]

exists in the projected \((n-2)\)-space through these points. If \( k<n-2 \), then by the induction assumption a hyperplane \( H' \) exists in the projected space which contains exactly the points \( s_1, s_2, \ldots, s_k \) of \( C_{n-2} \). Consequently, if \( H' \) is defined to be the hyperplane of the original space which is projected into \( H' \), this hyperplane contains exactly the points \( s_1, s_2, \ldots, s_k \) of \( C_n \). As \( l \subseteq H' \), (c) is proved for \( C_n \). The proof
can now be completed by induction.

To prove (a) and (b), consider first the case $k=0$. With this restriction neither $H_1$ nor $H_2$ contains points of $C_n$. As the curve is connected, it lies entirely within one of the two open regions of the projective space whose boundary is the set of points of $H_1$ and $H_2$. Hence an affine coordinate system exists so that the equations of $H_1$, $H_2$ are $x_1=0$, $x_1=1$, respectively, and $C_n$ contains no points for which $0 \leq x_1 \leq 1$. Now (a) and (b) follow for $k=0$ if $H(p)$ is defined to be the hyperplane with the equation $x_i=p$, $0 \leq p \leq 1$.

Now let $k=n$; (b) is trivial in this case. Let $f_i(p)$ $(0 \leq p \leq 1$, $1 \leq i \leq n)$ be any real-valued continuous functions for which $f_i(0)=s_i$, $f_i(1)=t_i$. Then (a) follows if $H(p)$ is defined to be the hyperplane spanned by the points with coordinates $f_i(p)$ $(1 \leq i \leq n)$.

In particular this establishes (a) and (b) for $C_1$ and $C_2$. Assume both results for all $C_{n-1}$ $(n>2)$. We may assume $0<k\leq n-2$. Let $l$ be arbitrary. By (c), hyperplanes $H'_1$, $H'_2$ exist which contain exactly the points $s_1$, $s_2$, $\ldots$, $s_k$; $t_1$, $t_2$, $\ldots$, $t_k$, respectively, together with the line $l$. Let $\bar{H}_1$, $\bar{H}_2$, $C_{n-1}$ be the projections of $H_1$, $H'_1$, $C_n$, respectively, from $s_i$. By the induction assumption (b), hyperplanes $\bar{H}(p)$ $(0 \leq p \leq 1)$ exist in the projected space, continuously dependent on $p$, each of which contains exactly the points $s_2$, $\ldots$, $s_k$ of $C_{n-1}$, and for which

$$\bar{H}(0)=\bar{H}_1, \ \bar{H}(1)=\bar{H}_2.$$ 

Let $H(p)$ $(0 \leq p \leq (1/3))$ be the hyperplane of the original space which is projected into $\bar{H}(3p)$. Then $H(p)$ depends continuously on $p$, contains exactly the points $s_1$, $s_2$, $\ldots$, $s_k$ of $C_n$, and $H(0)=H_1$, $H(1/3)=H'_1$. Likewise $H(p)$ $(2/3 \leq p \leq 1)$ exists so that it depends continuously on $p$, contains exactly the points $t_1$, $t_2$, $\ldots$, $t_k$ of $C_n$, and for which

$$H(2/3)=H'_1, \ H(1)=H_2.$$ 

After a projection from $l$, a similar argument can be used to construct a hyperplane $H(p)$ ($(1/3) \leq p \leq (2/3)$) which depends continuously on $p$, contains exactly $k$ points of $C_n$, and for which

$$H(1/3)=H'_1, \ H(2/3)=H'_2.$$ 

This proves (a) for $C_n$. Also (b) is clear if $H(p)$ is defined as above with the additional conditions that

$$H'_1=H'_2=H(p) \quad ((1/3) \leq p \leq (2/3)).$$

The proof can now be completed by induction.
6. **Hyperplanes which do not contain \( n \) points of \( C_n \).**

**Definition 2.** \( \Sigma(C_n) \) is the set of all points included in at least one space \( L \) of the curve \( C_n \) (cf. § 3).

**Lemma 4.** If, for \( n \geq 3 \), \( P \in \Sigma(C_{n-1}) \), where \( P \) is the projection of a point \( s' \) of \( C_n \), \( P \cong s' \), and \( C_{n-1} \) that of \( C_n \), then \( P \in \Sigma(C_n) \).

**Proof.** If \( P \in \Sigma(C_{n-1}) \), then points \( s_1, s_2, \ldots, s_{n-1}; t_1, t_2, \ldots, t_{n-1} \) of the projection \( C_{n-1} \) exist so that

\[
\overline{P} \in [s_1, s_2, \ldots, s_{n-1}] \cap [t_1, t_2, \ldots, t_{n-1}] = L
\]

and

\[
s_1 < t_1 < s_2 < \cdots < t_{n-1} < s_1 + 1,
\]

by the dual of Theorem 1. Moreover,

\[
[s_1, s_2, \ldots, s_{n-1}], [t_1, t_2, \ldots, t_{n-1}]
\]

may be chosen to be any two distinct hyperplanes through \( L \) within the projected \((n-1)\)-space. Therefore these hyperplanes may be chosen so that \( t_{n-1} < s' < s_1 + 1 \). Let the numbers

\( s_1, s_2, \ldots, s_{n-1}, t_1, t_2, \ldots, t_{n-1}, s' \)

now represent points of \( C_n \). Then \( P \in [t_1, t_2, \ldots, t_{n-1}, s'] \). As \( t_1, t_2, \ldots, t_{n-1}, s' \) are represented by linearly independent vectors the intersection

\[
\bigcap_{i=1}^{i=n-1} [t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n-1}, s'] = s'.
\]

Hence, because \( P \cong s' \), at least one value \( i \) exists with

\[
P \notin [t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n-1}, s'] \quad (1 \leq i \leq n-1).
\]

For such a value \( i \)

\[
[t_1, t_2, \ldots, t_{i-1}, P, t_{i+1}, \ldots, t_{n-1}, s'] = [t_1, t_2, \ldots, t_{n-1}, s'].
\]

Let \( t_n \) be a point of \( C_n \) with \( t_n > s' \). Then

\[
[t_1, t_2, \ldots, t_{i-1}, P, t_{i+1}, \ldots, t_{n-1}, t_n]
\]

approaches \([t_1, t_2, \ldots, t_{n-1}, s']\) as \( t_n \) approaches \( s' \). Because of the continuity of the curve points of \( C_n \), \([t_1, t_2, \ldots, t_{i-1}, P, t_{i+1}, \ldots, t_n]\) will contain a point \( t_i' \) of \( C_n \) for which \( s_i < t_i' < s_{i+1} \) provided \( t_n \) is sufficiently
close to $s'$. If $t_n$ is such a point, and $s_n$ is defined as $s'$, then

$$P \in [s_1, s_2, \ldots, s_n] \cap [t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n]$$

and

$$s_1 < t_1 < s_2 < \cdots < s_i < t_i < s_{i+1} < t_{i+1} < s_n < t_n < s_1 + 1.$$ 

It follows from the dual of Theorem 1 and Definition 2 that $P \in \Sigma(C_n)$. The lemma is thus established.

**COROLLARY.** If, for $n \geq 3$, $P$ is a point for which $P \in [(k, s_1), s_2] \ (s_1 = s_2, 0 \leq k \leq n - 3, P = s_2) \ P_{\Phi}(k, s_1)$, then $P \in \Sigma(C_n)$.

**PROOF.** If $n = 3$ then $P \in [s_1, s_2]$ $(s_1 = s_2, P = s_1 = s_2)$. Let $t_1, t_2$ be points of $C_3$ for which $s_1 < t_1 < s_2 < t_2 < s_1 + 1$. Then $P_{\Phi}[t_1, t_2]$; for otherwise $t_1, t_2, s_1, s_2$ would be coplanar in contradiction to the order condition. Hence $[P, t_1, t_2]$ is a plane. This plane must contain a third point $t$ of $C_3$, as $C_3$ is closed. Now $P = t$ because $[s_1, s_2]$ cannot contain a third curve point. If $P$ is the projection of $P$ from $t$ then

$$P_{[s_1, s_2]} \cap [t_1, t_2],$$

where $s_1, s_2, t_1, t_2$ now represent curve points of the projection $C_2$ of $C_3$ from $t$. This implies, by the dual of Theorem 1, that $P_{[s_1, s_2]} \cap [t_1, t_2]$, and so by the Lemma that $P_{[s_1, s_2]} \cap [t_1, t_2]$.

**LEMMA 5.** (a) For $n \geq 2$, $\Sigma(C_n)$ is open. (b) If a boundary point $P$ of $\Sigma(C_n)$ is approached by a sequence $P^u$ of points interior to $\Sigma(C_n)$, and $L$ is the limit of a space sequence $L^u$ for which $P^u \in L^u$, then $(k, s) \ (0 \leq k \leq n - 2)$ exists for which $P \in (k, s) \subseteq L$.

**PROOF.** If $P \in \Sigma(C_n)$ then a space $L$ exists for which $P \in L$. By the dual of Theorem 1, $s_1, s_2, \ldots, s_n; t_1, t_2, \ldots, t_n$ exist so that

$$L \subseteq [s_1, s_2, \ldots, s_n] \cap [t_1, t_2, \ldots, t_n] \text{ and } s_1 < t_1 < s_2 < \cdots < t_n < s_1 + 1.$$ 

If $P'$ is sufficiently close to $P$ then it is contained within an $(n - 2)$-space $L'$ which is so close to $L$ that it has the form

$$[s'_1, s'_2, \ldots, s'_n] \cap [t'_1, t'_2, \ldots, t'_n] \ (s'_1 < t'_1 < s'_2 < \cdots < t'_n < s'_1 + 1).$$
By the dual of Theorem 1, $P' \in \Sigma(C_n)$, and so (a) is proved.

To prove (b), let $H^1_t, H^2_t$ be any two hyperplane sequences with $L^\mu \subseteq H^1_t, L^\mu \subseteq H^2_t$, which converge to two distinct limits $H_1$ and $H_2$, respectively. By the dual of Theorem 1, $s^+_i, s^n_i, \ldots, s^\nu_i; t^+_i, t^n_i, \ldots, t^\nu_i$ exist so that $s^+_i < t^+_i < s^n_i < \cdots < t^n_i < s^\nu_i + 1$ and

$$H^1_t = [s^+_i, s^n_i, \ldots, s^\nu_i], \quad H^2_t = [t^+_i, t^n_i, \ldots, t^\nu_i].$$

As $H^1_t, H^2_t$ converge, the sequences $s^\nu_i, t^\nu_i (1 \leq i \leq n)$ also converge. If $s_i, t_i$ are the respective limits of these sequences,

$$\overline{L} = [s_1, s_2, \ldots, s_n] \cap [t_1, t_2, \ldots, t_n] \text{ and } s_1 \leq t_1 \leq s_2 \leq \cdots \leq t_n \leq s_1 + 1.$$

At least one equality sign must occur in this system, for otherwise $P \in \overline{L}$ and so $P \in \Sigma(C_n)$; this is impossible as $P$ is a boundary point of the open set $\Sigma(C_n)$. We may suppose, after a possible adjustment in the notation, $s_1 = t_1$. Hence $s_1 \in \overline{L}$. If $n = 2$ this proves the Lemma, as

$$P = \overline{L} = s_1 = (0, s_1).$$

Assume it holds for all curves $C_{n-1}$, $n > 2$. If $P = s_1$, then it is already true for $C_n$. If $P = s_1$, project from $s_1$. Let $C_{n-1}$ be the projection of $C_n$ and $P'$ that of $P$. Then $P' \notin \Sigma(C_{n-1})$, for otherwise, by Lemma 4, $P \in \Sigma(C_n)$. Moreover,

$$P' \in [s_2, s_3, \ldots, s_n] \cap [t_2, t_3, \ldots, t_n] = \overline{L}'$$

and this space is approached by the system

$$[s^+_i, s^n_i, \ldots, s^\nu_i] \cap [t^+_i, t^n_i, \ldots, t^\nu_i],$$

where all the numbers now represent points of $C_{n-1}$. Thus $P'$ is a boundary point of $\Sigma(C_{n-1})$. Therefore by the induction assumption $(k, s)_{n-1}$ exists so that

$$P' \in (k, s)_{n-1} \subseteq \overline{L} \quad (0 \leq k \leq n - 3).$$

Consequently, $P \in [s_1, (k, s)] \subseteq \overline{L}$. Because $P = s_1$, it now follows from the Corollary to Lemma 4 that $P \in (k, s)$, or $s = s_1$ and $P \in (k + 1, s)$. Either of these possibilities shows the lemma to be true and so the proof is complete.

**Lemma 6.** If, for $n \geq 3$, $l^\nu$ is a sequence which converges to $\overline{l}$, and $p$ an integer for which $\overline{l} \leq (p, s), \overline{l} \leq (p - 1, s)$ $(0 < p < n - 1)$ then $[l^\nu, (q, s)] \rightarrow (q + 2, s) (p - 1 \leq q \leq n - 3)$.
Proof. The space $[l^\mu, (q, s)]$ is a $(q+2)$-space because $q < n-1$ while $l^\nu$ and $(q, s)$ have no common points. Consider first the case for which $q = n-3, p = n-2$. If the lemma were false then a convergent subsequence of $[l^\nu, (n-3, s)]$ would exist whose limit would be a hyperplane $Q$ for which $Q = (n-1, s)$. As $l^\nu \rightarrow l$,

$$[l, (n-3, s)] = (n-2, s) \subseteq Q.$$  

Consequently $Q$ would cut $C_n$ in $s$ at least $(n-1)$-fold. As $C_n$ is closed, $Q$ would cut $C_n$ in one additional point $s'$, and $s' = s$ as $Q = (n-1, s)$. Hence, if $l^\nu$ is sufficiently close to $l$, $[l^\nu, (n-3, s)]$ would cut $C_n$ in a point $s'$ so close to $s'$ that $s' = s$. Therefore the hyperplane $[l^\nu, (n-3, s)]$ would cut $C_n$ in more than $n-2$ points in contradiction to Lemma 3. Thus $[l^\nu, (n-3, s)]$ must approach $(n-1, s)$, and the lemma is proved in this case. In particular, it is completely proved for $n=3$. Assume it is established for all $C_{n-1}, n>3$.

Consider next the case for which $q < n-3$. Project from any point $t$ of $C_n$ different from $s$. As $t \not\in (p, s)$, $l$ is projected into a line $\overline{l}$, and $l^\nu$ is projected into a line $\overline{l}^\nu$ defined for the projection $C_{n-1}$ of $C_n$ by Lemma 2. Clearly

$$\overline{l} \subseteq (p, s)_{n-1} \text{ and } \overline{l} \equiv (p-1, s)_{n-1},$$

for otherwise

$$l \subseteq [(p-1, s), t] \cap (p, s) = (p-1, s).$$

Therefore, by the induction assumption, $[l^\nu, (q, s)_{n-1}] \rightarrow (q+2, s)_{n-1}$. This implies $[l^\nu, (q, s), t] \rightarrow [(q+2, s), t]$, and, because $t$ is arbitrary, that $[l^\nu, (q, s)] \rightarrow (q+2, s)$. Thus the lemma is proved in this case.

Finally let $q = n-3, p < n-2$. If $[l^\nu, (n-3, s)]$ does not converge to $(n-1, s)$ this set contains a convergent subsequence with limit $Q$, $Q = (n-1, s)$. Now $1 \leq p < n-2$, and so $n \geq 4$. Hence by the result of the previous paragraph $[l^\nu, (n-4, s)] \rightarrow (n-2, s)$. Consequently $(n-2, s) \subseteq Q$. This leads to the contradiction encountered in the first paragraph. Thus $[l^\nu, (n-3, s)] \rightarrow (n-1, s)$, and the lemma is proved for $C_n$. The proof can now be completed by induction.

Definition 3. $\sigma(C_n)$ is the set of all hyperplanes each of which contains at least one line $l$ of the curve $C_n$.

$\sigma(C_n)$ is the dual of the space $\Sigma(C_n)$.

Theorem 3. For $n \geq 2$, $\sigma(C_n)$ consists of all the hyperplanes which do not contain $n$ points of $C_n$.

Proof. By Lemma 3 each member of $\sigma(C_n)$ contains less than $n$
points of $C_n$. It remains to show that every hyperplane which contains less than $n$ points of $C_n$ contains at least one line $l$. Let $H$ be a hyperplane and $s_1, s_2, \ldots, s_h$ be the points of $C_n$ contained in $H$, where $0 \leq h < n$. As $C_n$ is closed, $h \equiv n \pmod{2}$. By Theorem 2 (c), for a given line $l$, a hyperplane $H_l$ exists which contains $l$ and exactly the points $s_1, s_2, \ldots, s_h$ of $C_n$. By Theorem 2 (b), a system $H(p)$ ($0 \leq p \leq 1$) of hyperplanes exists, continuously dependent on $p$, each of which contains exactly the points $s_1, s_2, \ldots, s_h$ of $C_n$ and for which

$$H(0) = H', \ H(1) = H.$$ 

By Definition 3, $H(0) \in \sigma(C_n)$. Assume $H \notin \sigma(C_n)$. By the dual of Lemma 5 (a), $\sigma(C_n)$ is open. Therefore a least value $p$ of $p$ exists for which $H(p) \notin \sigma(C_n)$. Let $p^\mu$ be a sequence for which $p^\mu \to p$, $p^\mu < p$. As $H(p^\mu) \in \sigma(C_n)$, $l^\mu$ exists for which $l^\mu \subseteq H(p^\mu)$. By replacing $p^\mu$ by an appropriate subsequence if necessary we may assume $l^\mu$ converges. If $l$ be the limit of $l^\mu$ then, by the dual of Lemma 5 (b), $(k, s)$ exists so that

$$l \subseteq (k, s) \subseteq H(p) \quad (1 \leq k < n - 1).$$

We may assume $(k + 1, s) \notin H(p)$; for otherwise $(k, s)$ may be replaced by an osculating space of a greater dimension so that this relation holds. Consequently $s$ occurs exactly $(k + 1)$-fold in the set $s_1, s_2, \ldots, s_h$, and $k + 1 \leq h \leq n - 2$. This is impossible if $h \leq 1$ in which case $H \in \sigma(C_n)$. In particular this proves the theorem for $h \leq 3$. We assume therefore $n > 3$. As $k \leq n - 3$ and $l \subseteq (k, s)$, the number $q$ of Lemma 6 may be specialized to $k$. It follows then from this Lemma that $[l^\mu, (k, s)] \to (k + 2, s)$. Hence, as $[l^\mu, (k, s)] \subseteq H(p^\mu)$, $(k + 2, s) \subseteq H(p)$. This contradicts the fact that $s_1, s_2, \ldots, s_h$ are the only points of $C_n$ in $H(p)$ among which $s$ occurs exactly $(k + 1)$-fold. Therefore $H \notin \sigma(C_n)$. Thus the theorem is established.

7. A characterization of the lines $l$.

**Theorem 4.** For $n \geq 2$, a straight line is a line $l$ if, and only if, every hyperplane through $l$ contains less than $n$ points of $C_n$.

**Proof.** Let $m$ be a straight line which is not a line $l$. Then at least one point $P$ exists on $m$ which is not within $n$ distinct $(n - 1, s)$. A sequence of points $P^\mu$ exists with $P^\mu \to P$ for which each $P^\mu$ is within less than $n$ $(n - 1, s)$. (This can be conveniently proved by induction in the dual formulation.) If $A$ is a point of $m$ for which $A \approx P$ then $[A, P^\mu] \to m$. By the dual of Theorem 3, $L^\mu$ (cf. § 3) exists for which $P^\mu \in L^\mu$. Now $[A, L^\mu]$ contains $[A, P^\mu]$ and also $n$ points of $C_n$ by the
definition of $L^n$. The limit of a convergent subsequence of $[A, L^n]$ is a hyperplane which contains $m$ together with $n$ points of $C_n$. This proves that if every hyperplane through a straight line contains less than $n$ points of $C_n$ then every point of the straight line is within $n$ distinct $(n-1, s)$ and so must be a line $l$.

No hyperplane through a line $l$ can contain $n$ points of $C_n$ by Lemma 3. Thus the proof of the theorem is complete.

References


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