BOOLEAN ALGEBRAS WITH PATHOLOGICAL ORDER
TOPOLOGIE

EDWIN E. FLOYD
If $L$ is a partially ordered set, there are a variety of known ways in which $L$ may be given a topology compatible, in some sense, with its partial ordering (see [1, 6]). Examples, by Northam [3] and Floyd and Klee [2], have very recently appeared of complete lattices which are not Hausdorff in their order topologies. It appears, then, that the various topologies will not be central in the study of all complete lattices. The question remains as to whether or not there is some wide and natural class of lattices in which some compatible topology has nice properties. We give a very simple example of a complete Boolean algebra which is not Hausdorff in any topology compatible with the order. We also give an example of a conditionally complete vector lattice in which addition is not continuous in any compatible topology. This is a counterexample to a result of Birkhoff [1, p. 242], who overlooked the possibility that convergence in the order topology differs from order convergence.

**Definition.** Suppose that $(P, \geq)$ is a partially ordered set, and suppose that $T$ is a topology for the set $P$ (that is, $T$ is a collection of subsets of $P$ closed under arbitrary unions and finite intersections, and with $\varnothing \in T, P \in T$). We say that $T$ is $\sigma$-compatible with $\geq$ if and only if whenever $(x_i)$ is a sequence in $P$ with

$$x_1 \geq x_2 \geq \cdots \quad \text{and} \quad \bigwedge_i x_i = x$$

or

$$x_1 \leq x_2 \leq \cdots \quad \text{and} \quad \bigvee_i x_i = x,$$

then the sequence $(x_i)$ $T$-converges to $x$.

**Theorem 1.** Let $L$ denote the complete Boolean algebra of all regular open subsets of the unit interval $I$, partially ordered by inclusion $\supset$. Suppose that $T$ is a topology for $L$ which is $\sigma$-compatible with $\supset$. Then the topology $T$ is not Hausdorff.

**Proof.** Recall that a subset $b$ of $I$ is a regular open set if and only if $b$ is the interior of its closure. $L$ is known to be a complete

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Boolean algebra [1]. Let \( \mathcal{U} \) be a \( T \)-neighbourhood of the empty set \( \phi \in L \). We show that \( I \in \overline{\mathcal{U}} \). Suppose that \( U_1, U_2, \ldots \), is a basis for the open sets of \( I \), with each \( U_i \) nonempty. There exists for each \( i \) a sequence \( (A_j^i | j = 1, 2, \ldots) \) in \( L \) with

\[
A_1^i \subseteq U_i, A_j^i \Rightarrow \phi,
\]

so that \( A_1^i \succ A_2^i \succ \cdots \) and

\[
\bigwedge (A_j^i | j = 1, 2, \ldots) = \phi.
\]

Since \((A_i^i)\) converges to \( \phi \), there exists \( A_i^i \in \mathcal{U} \). Define \( B_1 = A_1^i \). Since the sequence \((B_1 \setminus A_i^i)\) converges to \( B_i \), there exists \( j \) with \( B_i \setminus A_j^i \in \mathcal{U} \). Define \( B_2 = B_1 \setminus A_j^i \). Similarly there exists \( B_3 = B_2 \setminus A_k^i \in \mathcal{U}, \ldots \). Now \((B_i)\) is a sequence in \( \mathcal{U} \) with \( B_1 \prec B_2 \prec \cdots \). Moreover, since the only regular open set containing \( \bigcup B_i \) is \( I \), we have \( \bigvee_i B_i = I \). Hence \( I \in \overline{\mathcal{U}} \) and the theorem follows.

The following remark answers Problem 77 of Birkhoff [1, p. 167].

**Theorem 2.** If \( L \) is the complete Boolean algebra of Theorem 1, then there exist, for \( i = 1, 2, \ldots \), sequences \( (X_{i,j} | j = 1, 2, \ldots) \) with \((X_{i,j})\) order-converging to \( \phi \) for each \( i \) but such that for no function \( j(i) \) is it true that \((X_{i,j(i)})\) order-converges to \( \phi \).

**Proof.** Let \((X_{i,i})\) denote the sequence \((A_i^i)\) of the proof of Theorem 1. Consider any function \( j(i) \), then

\[
\bigvee_{i \geq k} A_j^i = I.
\]

Hence

\[
\bigwedge_{k \geq i} \bigvee_{i \geq j} A_j^i = I.
\]

Hence the sequence \((X_{i,j(i)})\) does not order-converge to \( \phi \).

**Theorem 3.** Let \( L \) be the complete Boolean algebra of Theorem 1, and let \( M \) be a Stone representation space for \( L \). Let \( N \) denote the lattice of all continuous real-valued functions on \( M \). Then \( N \) is a conditionally complete vector lattice in which the function \( x - y \) is not \( T \)-continuous simultaneously in \( x \) and \( y \) for any \( T \)-topology \( T \) for \( N \) which is \( \sigma \)-compatible with \( > \).

**Proof.** It is known [4, 7] that \( N \) is conditionally complete. We may consider \( L \) as identical with the algebra of all open and closed subsets of \( M \). There is a function \( t : L \to N \) which assigns to \( u \in L \) the
characteristic function \( t(u) \) of the open and closed set \( u \). We show that \( t \) is an embedding of \( L \) in \( N \). It is seen that \( t \) is an isotone one-to-one map of \( L \) onto \( t(L) \), and \( t^{-1} \) is an isotone map of \( t(L) \) on \( L \). We prove that if \( K \subset L \) then

\[
\bigvee t(K) = t(\bigvee K),
\]

where \( \bigvee t(K) \) denotes the least upper bound in \( N \). Clearly

\[
t(\bigvee K) \geq \bigvee t(K).
\]

Now \( \bigvee t(K) \) is a nonnegative continuous function whose value is \( \geq 1 \) on the set \( \bigvee K \), and hence \( \geq 1 \) also on its closure. But the closure of \( \bigvee K \) is \( \bigvee K \) [7]. Hence

\[
t(\bigvee K) \leq \bigvee t(K)
\]

and equality holds. The dual also follows. So \( t \) embeds \( L \) in \( N \). It follows that \( t(L) \) is not Hausdorff in the topology \( T \) restricted to \( t(L) \). Hence \( N \) is not Hausdorff in the topology \( T \). But if \( x-y \) is \( T \)-continuous in \( x \) and \( y \), it is known that \( N \) is then regular [5, p. 54] and hence Hausdorff.

**Corollary.** Suppose, in addition to the hypotheses of Theorem 3, that the function \( y \to -y \) on \( N \) is \( T \)-continuous. Then \( x+y \) is not \( T \)-continuous in \( x \) and \( y \) simultaneously.

This answers, in the negative, a part of Problem 4 of Rennie [6, p. 51].

**References**

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