Pacific Journal of Mathematics

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Vol. 5, No. 5 BadMonth 1955

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1. Introduction. Let the bounded, simply connected, open region R of the (x, y)-plane have the boundary curve C. If a uniform ideal elastic membrane of unit density is uniformly stretched upon C with unit tension across each unit length, then λ, the square of the fundamental frequency, satisfies the conditions (subscripts denote differentiation)

$$\text{(1a)} \qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u \quad \text{in} \quad R\,, \\ \lambda = \text{minimum}\,, \end{cases}$$

with the boundary condition

$$\text{(1b)} \qquad u(x, y) = 0 \quad \text{on} \quad C\,.$$

Variational methods of the Rayleigh-Ritz type are frequently used to approximate λ. They always yield upper bounds for λ, and the upper bounds can be made arbitrarily close.

Another common practical method of approximating λ is to calculate the least eigenvalue λ_h of a suitably chosen finite-difference operator Δ_h over a network with small mesh width h. For one choice of Δ_h it was shown by Courant, Friedrichs, and Lewy [3, p. 57] without details that $\lambda_h \to \lambda$ as $h \to 0$. For convex regions R of a special polygonal form the author has shown [4] that a special case of (11) below is valid for a common choice of Δ_h, and hence that λ_h is asymptotically a lower bound for λ as $h \to 0$. For an unusual finite-difference approximation to problem (1) when R is the union of squares of the network, Polya [12] has found that $\lambda_h > \lambda$ for all h, and also for the higher eigenvalues. The author knows of no other study of the sign or order of decrease of $\lambda - \lambda_h$ to 0.

In the present paper the investigation of [4] is extended to a much wider class of regions: those with piecewise analytic boundary curves and convex corners. The new theorems are stated and proved in §§ 3 and 4. Theorem 2 contains the theorem of [4] as a special case. Lemmas used in the proof of Theorem 1 are given in § 5. Identity (31) of Lemma 7 is interesting in itself.

Received December 2, 1953. Presented to the American Mathematical Society September 4, 1953. The preparation of this paper was sponsored in part by the Office of Naval Research, U.S.N.

When C is no longer made up of line segments of the network, it is necessary when using finite-difference methods either to move C or to alter Δ_h near the boundary. The latter procedure is potentially more accurate, and has been adopted in deriving the rather delicate results proved below. The definition of Δ_h given in § 2 is a self-adjoint modification of Mikeladze's approximation [10; 11], and is believed to be new. The cruder approximations to Δ near C proposed by Collatz in 1933 and expounded in [2, p. 357], while easier to compute in practice, appear to introduce an unmanageable term $O(h^2)$ into (19). It is therefore doubted that Theorem 2 would remain valid for these cruder operators.

The technique of the present paper could be applied to study the asymptotic behavior of λ_h also for other difference approximations to Δ in the interior of R—for example, for those associated with a triangular net [2, p. 367].

It is not clear that one could revise the argument of the paper to prove an inequality of the type

$$\frac{\lambda}{\lambda_h} \leq 1 + bh^2 + o(h^2) .$$

2. Definitions. Assume the bounded, simply connected, open region R to have a closed boundary curve C: $x(s)+iy(s)$ $(0 \leq s \leq s_m)$ which is *piecewise analytic*. That is, $x(s)$ and $y(s)$ are real analytic functions of the arc length s of C in each of a finite number m of closed intervals

$$0 = s_0 \leq s \leq s_1, \quad s_1 \leq s \leq s_2, \quad \cdots, \quad s_{m-1} \leq s \leq s_m .$$

Moreover, we demand that the corners of C be convex; that is, at any point $x(s_j)+iy(s_j)$ $(0 \leq j < m)$ where distinct analytic curves meet, the interior angle of C must be less than π.

For $h>0$, let a *net* consist of the lines $x=\mu h$, $y=\nu h$ $(\mu, \nu = 0, \pm 1, \pm 2, \cdots)$. The points $(\mu h, \nu h)$ in R are the *interior nodes* R_h of the net. The *boundary nodes* C_h of the net consist of (i) all points $(\mu h, \nu h)$ on C, and (ii) all *isolated* points of intersection of the net with C. Thus each node $(\mu h, \nu h)$ of R_h has two *neighboring nodes* in $R_h \cup C_h$ on the line $x=\mu h$, and two in $R_h \cup C_h$ on the line $y=\nu h$. Moreover, each node in C_h has at least one neighbor in $R_h \cup C_h$.

We now move toward a definition of the difference operator Δ_h. Let us denote the neighboring nodes of the node

(2) $\quad (x, y)$ of R_h by $(x-h_1, y)$, $(x+h_2, y)$, $(x, y-h_3)$, and $(x, y+h_4)$,

where $0 < h_i \leq h$ for $i=1, 2, 3, 4$. For nodes remote from C_h, all $h_i = h$. Let v be any net function defined on the nodes of $R_h \cup C_h$, vanishing on C_h. Define $D_x^{(h)}v$ as the (constant) second derivative of the quadratic polynomial function of x assuming the three values $v(x-h_1, y)$, $v(x, y)$, and $v(x+h_3, y)$. That is,

$$(3) \qquad D_x^{(h)}v(x, y) = \frac{2}{h_1+h_2}\left[\frac{v(x+h_2, y)-v(x, y)}{h_2} - \frac{v(x, y)-v(x-h_1, y)}{h_1}\right].$$

Also, $D_y^{(h)}v(x, y)$ is defined analogously. We next define

$$(4) \qquad \begin{aligned} \Delta^{(h)}v(x, y) &= D_x^{(h)}v(x, y) + D_y^{(h)}v(x, y) \\ &= -\left(\frac{2}{h_1h_2} + \frac{2}{h_3h_4}\right)v(x, y) \\ &\quad + \frac{2}{h_1(h_1+h_2)}v(x-h_1, y) + \frac{2}{h_2(h_1+h_2)}v(x+h_2, y) \\ &\quad + \frac{2}{h_3(h_3+h_4)}v(x, y-h_3) + \frac{2}{h_4(h_3+h_4)}v(x, y+h_4) . \end{aligned}$$

The operator $\Delta^{(h)}$ is the approximation to Δ recommended in [10]. It linearly transforms the net function v defined over R_h into the net function $\Delta^{(h)}v$, also defined over R_h. But $\Delta^{(h)}$ is not a self-adjoint linear operator; that is, the matrix $A^{(h)}$ of the linear transformation of v into $\Delta^{(h)}v$ is not symmetric.

We define the matrix A_h as the symmetric part of the matrix $A^{(h)}$:

$$(5) \qquad A_h = \tfrac{1}{2}[A^{(h)} + A^{(h)T}] ,$$

where T means transpose. Finally, *we define Δ_h to be the self-adjoint linear operator corresponding to A_h.*

The explicit expressions for Δ_h assume 16 different forms, depending on the location of (x, y) with respect to C_h. Although we shall not need these expressions for the present paper, we describe them briefly. If, in any of the four directions from (x, y), the neighboring node—say $(x-h_1, y)$, for definiteness—is in R_h, then $h_1=h$, and there is another node $(x-h-h_1', y)$ in $R_h \cup C_h$. Then the term $2v(x-h_1, y)/h_1(h_1+h_2)$ of (4) is to be replaced by

$$(6) \qquad \frac{h_1'+2h+h_2}{(h_1'+h)h(h+h_2)}v(x-h, y) .$$

For any (x, y), the expression for Δ_h is obtained from (4) by making replacements like (6) corresponding to all neighbors of (x, y) in R_h.

When (x, y) is more than two nodes away from C_h, so that all $h_i=h_i'=h$, the values of both $\Delta^{(h)}$ and Δ_h reduce to the familiar form used in [4]:

$$(7)\qquad \Delta_h v(x, y) = \Delta^{(h)} v(x, y) = \frac{1}{h^2}[v(x-h, y)+v(x+h, y)+v(x, y-h)+v(x, y+h)-4v(x, y)] .$$

Let λ_h satisfy the following difference equation for a net function v defined in $R_h \cup C_h$:

$$(8a)\qquad \begin{cases} \Delta_h v = -\lambda_h v & \text{in } R_h , \\ \lambda_h = \text{minimum} , \end{cases}$$

where v is extended to satisfy the boundary condition

$$(8b)\qquad v=0 \quad \text{on} \quad C_h .$$

It is readily shown that λ_h is the minimum over all net functions v satisfying (8b) of the quotient

$$\rho_h(v) = \frac{-h^2 \sum_{R_h} v\Delta_h v}{h^2 \sum_{R_h} v^2} .$$

(This is simply the minimum principle for a definite quadratic form.) By (5), we can write $\rho_h(v)$ in the following equivalent form, simpler to use:

$$(9)\qquad \rho_h(v) = \frac{-h^2 \sum_{R_h} v\Delta^{(h)} v}{h^2 \sum_{R_h} v^2} .$$

The reason for not using the least eigenvalue μ_h of $\Delta^{(h)}$ in this investigation is that μ_h does not have the foregoing minimum property and, in fact, might turn out to be complex. On the other hand, it is known [9, p. 27] that $\lambda_h \le \mathscr{R}(\mu_h)$, so that when μ_h is real it could conceivably be a better approximation to λ than λ_h is. The relative magnitude of $|\lambda_h - \lambda|$ to $|\mu_h - \lambda|$ is not known.

3. The results.

The following new result will be proved in § 4:

THEOREM 1. *Let R be a bounded, open, simply connected region bounded by a piecewise analytic curve C whose corners are convex in the sense of* § 2. *Let τ be the angle between the tangent to C and the x axis. Let u solve problem* (1) *for R, and let u_n be the normal derivative of u on C. Define λ_h as in* § 2. *Let*

$$(10)\qquad a = a(R) = \frac{\iint_R (u_{xx}^2 + u_{yy}^2)dxdy + \int_C u_n^2 \sin^2 2\tau\, d\tau}{12\iint_R (u_x^2 + u_y^2)dxdy} .$$

Then $-\infty < a < \infty$ *and, as* $h \to 0$, *one has*

$$\frac{\lambda_h}{\lambda} \leq 1 - ah^2 + o(h^2) \qquad (h \to 0) . \tag{11}$$

In Theorem 1 the quantity a can probably be negative for certain nonconvex R, because $d\tau$ in (10) will be negative at some points of C. But if R is convex we get a stronger result, as an immediate consequence of Theorem 1.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if* R *is also convex, then* $0 < a < \infty$, *and there exists* $h_0 > 0$ *such that* $\lambda_h < \lambda$ *for all* $h < h_0$.

For the operator Δ_h of § 2 the methods of [3] can undoubtedly be followed to show that $\lambda_h \to \lambda$ as $h \to 0$; the author has not attempted to carry through the details. When $\lambda_h \to \lambda$ as $h \to 0$, the lower bounds λ_{h_0} can be made arbitrarily close by choice of h_0 sufficiently small. Thus for these R the Rayleigh-Ritz methods and the finite-difference methods (8) are theoretically complementary, and together could confine λ to an arbitrarily short interval if one knew an upper bound for h_0.

The author has not developed an upper bound for h_0 in Theorem 2, although it would be desirable to do so by estimating the term $o(h^2)$. One could always make an intelligent guess based on the behavior of λ_h for certain h.

The constant a of (10) is the best possible for certain rectangular regions; see [4]. That the corners of C be convex seems essential to the validity of Theorem 1. Indeed, for one nonconvex polygon some heuristics and an experiment mentioned in [4] make it appear that $\lambda_h = \lambda + Ah^{4/3} + o(h^{4/3})$, where $A > 0$. It would be interesting to know the sign of a for the general case of Theorem 1, or in particular when C is a nonconvex analytic curve.

Corners of angle π are frequent in engineering practice, and it would be desirable to know how λ_h behaves when R has such corners. For such corners Lemma 2 is no longer valid. Lewy [7] provides new tools for an attack on corners of angle π.

4. Proof of Theorem 1. Let u henceforth be the solution of problem (1) for the fundamental eigenvalue λ. It is known that

$$\lambda \iint_R u^2 dxdy = \iint_R (u_x^2 + u_y^2) dxdy . \tag{12}$$

The proof of Theorem 1, following [4], consists in setting the values of the function u at the nodes of $R_h \cup C_h$ into the Rayleigh quotient (9) of problem (8). It will be shown that

$$\frac{\rho_h(u)}{\lambda} = 1 - ah^2 + o(h^2) \qquad (h \to 0) . \tag{13}$$

Since $\lambda_h \leq \rho_h(u)$, the theorem follows from (13).

The denominator $h^2 \sum u^2$ of $\rho_h(u)$ differs from a Riemann sum for $\iint_R u^2 dxdy$ at most by the terms corresponding to squares or part-squares at the boundary C. The total contribution of these terms does not exceed the order of magnitude $Lh \max_R u^2$, where L is the length of C. Hence a fortiori

$$h^2 \sum_{R_h} u^2 = \iint_R u^2 dxdy + o(1) \qquad (h \to 0) . \tag{14}$$

Let the nodes of R_h be divided into three classes:

$$\begin{cases} R_h^1: & \text{those within a distance } h \text{ of some corner of } C; \\ R_h^2: & \text{those not in } R_h^1 \text{ but within a distance } h \text{ of } C; \\ R_h^3: & \text{the other nodes of } R_h . \end{cases} \tag{15}$$

Split the numerator of $\rho_h(u)$ accordingly:

$$-h^2 \sum_{R_h} u\Delta^{(h)}u = \sum_{i=1}^{3} \left(-h^2 \sum_{R_h^i} u\, \Delta^{(h)}u \right) \equiv \sum_{i=1}^{3} S_h^i(u) .$$

There are a fixed number of corners, not exceeding m, and at most two nodes of R_h^1 per corner. Moreover $|\nabla u(x, y)|^2 \to 0$ as $(x, y) \to$ a corner of C, by Lemma 1 in § 5. At any node (x, y) of R_h^1 with neighbors denoted as in (2), we find from (3) that

$$h^2 |u\Delta^{(h)} u| \leq \frac{h^2(u-0)}{\min h_i} \sum_{i=1}^{4} \left| \frac{u-u_i}{h_i} \right| \leq 4h^2 \max |\nabla u|^2 ,$$

where the u_i are the values of u at the four neighbors of (x, y), and where the maximum of $|\nabla u|^2$ is taken over all points within a distance $2h$ of some vertex. Hence

$$|S_h^1(u)| \leq 8mh^2 \max |\nabla u|^2 = o(h^2) \qquad (h \to 0) . \tag{16}$$

Using the notation and assertion of Lemma 3, we have

$$S_h^2(u) = -h^2 \sum_{R_h^2} u\Delta u - \frac{2h^3}{3} \sum_{R_h^2} u(\theta_x u'_{xxx} + \theta_y u''_{yyy}) . \tag{17}$$

Since u satisfies (1a),

$$-h^2 \sum_{R_h^2} u\Delta u = \lambda h^2 \sum_{R_h^2} u^2 . \tag{18}$$

By (17), (18), and Lemma 4,

$$|S_h^2(u) - \lambda h^2 \sum_{R_h^2} u^2| \leq \tfrac{2}{3} h^3 \sum_{R_h^2} u(|u'_{xxx}| + |u''_{yyy}|) = o(h^2) \qquad (h \to 0) .$$

Thus

$$S_h^2(u) = \lambda h^2 \sum_{R_h^2} u^2 + o(h^2) \qquad (h \to 0) . \tag{19}$$

Similarly, using the notation and assertion of Lemma 5, and by (1a), we have

$$S_h^3(u) = \lambda h^2 \sum_{R_h^3} u^2 - \frac{h^4}{12} \sum_{R_h^3} u(u'_{xxxx} + u''_{yyyy}) . \tag{20}$$

Now

$$h^2 \sum_{R_h^2 \cup R_h^3} u^2 = h^2 \sum_{R_h} u^2 - h^2 \sum_{R_h^1} u^2 = h^2 \sum_{R_h} u^2 + o(h^2) , \tag{21}$$

since $u(x, y) \to 0$ as $(x, y) \to C$, and since there are at most $2m$ vertices in R_h^1. Adding (19) and (20), and using (21), we find that

$$\begin{aligned} S_h^2(u) + S_h^3(u) &= \lambda h^2 \sum_{R_h} u^2 - \frac{h^4}{12} \sum_{R_h^3} u(u'_{xxxx} + u''_{yyyy}) + o(h^2) \\ &= \lambda h^2 \sum_{R_h} u^2 - \frac{h^4}{12} \iint_R u(u_{xxxx} + u_{yyyy}) dxdy + o(h^2) , \end{aligned}$$

by Lemma 6. Adding $S_h^1(u)$ to the above, and dividing by (14), we find that

$$\begin{aligned} \rho_h(u) &= \frac{\sum_{i=1}^{3} S_h^i(u)}{h^2 \sum_{R_h} u^2} \\ &= \lambda - \frac{h^2}{12} \frac{\iint_R u(u_{xxxx} + u_{yyyy}) dxdy}{\iint_R u^2 dxdy} + o(h^2) . \end{aligned} \tag{22}$$

Finally, dividing (22) by λ, and applying Lemma 7 and (12), one proves (13) and hence Theorem 1.

5. Some lemmas. The following lemmas are basic to the proof of Theorem 1. In all of them R satisfies the conditions stated at the start of § 2, while $u = u(x, y)$ solves problem (1).

LEMMA 1. *The function u is an analytic function of x and y in $R \cup C$, except possibly at the corners of C. Let r be the distance of (x, y) from a corner P with interior angle π/α, $1 < \alpha < \infty$. Then for $m = 0, 1, 2, \cdots$, any partial derivative of u of order m has the local representation*

(23) $$\frac{\partial^m u}{\partial x^\mu \partial y^\gamma} = r^{\alpha-m} f_m(x, y) \qquad (\mu+\gamma=m) ,$$

where f_m is continuous at P.

Proof. By [1, p. 179], u is analytic in R. The representation (27′) below shows that the interior normal derivative u_n is integrable on C. Then the analyticity of u on C (corners excluded) was shown by Hadamard [5, p. 25].[1]

Let $t=\xi+i\eta$ and $z=x+iy$. For each $t \in R$ let $w=\Phi(z, t)$ map R conformally onto the circle $|w|<1$, with $\Phi(t, t)=0$. We may assume without loss of generality that P is at $z=0$, and that $\Phi(0, t)=1$. Lichtenstein [8, pp. 255–256 and footnote 273] showed[2] that for $m=0, 1, 2, \cdots$, and $z \in R$,

(24) $$\frac{\partial^m \Phi(z, t)}{\partial z^m} = z^{\alpha-m} \varphi_m(z, t) ,$$

where φ_m is continuous at $z=0$. It follows from (24) that

(25) $$\frac{\partial^m \log \Phi(z, t)}{\partial z^m} = z^{\alpha-m} \psi_m(z, t) ,$$

where ψ_m is continuous at $z=0$. Let $G(z, t)=G(\xi, \eta; x, y)$ be Green's function for Δu in R. Since

$$G(z, t) = -(2\pi)^{-1} \log |f(z, t)| ,$$

it follows from (25) that for $m=0, 1, 2, \cdots$ and $z \in R$,

(26) $$\frac{\partial^m G(z, t)}{\partial x^\mu \partial y^\nu} = r^{\alpha-m} \Psi_m(z, t) \qquad (\mu+\nu=m) ,$$

where Ψ_m is continuous at $z=0$.

Now the function u has the integral representation [1, pp. 182–183]

$$u(x, y) = \lambda \iint_R G(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta .$$

Hence

(27) $$\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}$$

[1] The author wishes to thank Professor Lewy for this reference.

[2] Lichtenstein actually asserts that (24) is without question true for all α, but that his proof is valid only for irrational α. Warschawski [**13**] has found a simple proof of (24), valid for all α in the range $\frac{1}{2} \leq \alpha < \infty$.

Added in April 1954: For asymptotic expansions of Φ at a corner, see R. Sherman Lehmann, "Development of the mapping function at an analytic corner," Technical Report No. 21, Applied Mathematics and Statistics Laboratory, Stanford University, California, March 31, 1954, 17 pp.

$$=\lambda\iint_R \frac{G(x+\Delta x,\, y;\, \xi,\, \eta)-G(x,\, y;\, \xi,\, \eta)}{\Delta x}\, u(\xi,\, \eta)d\xi d\eta$$

$$=\lambda\iint_R \frac{\partial G}{\partial x}(x+\theta\Delta x,\, y;\, \xi,\, \eta)u(\xi,\, \eta)d\xi d\eta\ ,$$

where $0<\theta=\theta(x, y, \Delta x)<1$. Since $G(z, t)=G(t, z)$, it is clear that $\partial G/\partial x=\partial G/\partial\xi$ and, as a function of t, $\partial G/\partial x$ behaves like $|t-t_0|^{\alpha-1}$ at any corner t_0 of R, uniformly in z for z bounded away from C. Hence $(\partial G/\partial x)u(\xi, \eta)$ in (27) is dominated by an integrable function of ξ, η, uniformly with respect to Δx. By Lebesgue's convergence theorem, letting $\Delta x\to 0$ in (27) proves that

$$\frac{\partial u}{\partial x}=\lambda\iint_R \frac{\partial G}{\partial x}(x,\, y;\, \xi,\, \eta)u(\xi,\, \eta)d\xi d\eta\ . \tag{27$'$}$$

Setting the expression (26) for $m=\mu=1$ into the last equation proves the case $m=\mu=1$ of (23).

In a similar way one can prove all the cases $m=0, 1, 2, 3, 4$ of (23), and the lemma is established.

LEMMA 2. *The functions* u_{xx}^2, u_xu_{xxx}, uu_{xxxx}, u_{yy}^2, u_yu_{yyy}, *and* uu_{yyyy} *are Lebesgue integrable in* R. *The Lebesgue integrals* $\int_C u_xu_{xx}dy$ *and* $\int_C u_yu_{yy}dx$ *exist.*

Proof. By Lemma 1 the functions $u_{xx}^2, \cdots, uu_{yyyy}$ are continuous in $R\cup C$ except possibly at the corners, where they are $O(r^{2\alpha-4})$. Since $0<\alpha$, the first sentence follows. The second sentence is proved analogously.

REMARK. The proof of Lemma 2 breaks down for corners of angle $\pi\ (\alpha-1)$, as r^{-2} is not integrable.

LEMMA 3. *At any node* (x, y) *of* R_h *whose neighbors are denoted as in* (2), *one has*

$$\Delta^{(h)}u=\Delta u+\tfrac{2}{3}h[\theta_xu'_{xxx}+\theta_yu''_{yyy}]\ ,$$

where $-1<\theta_x<1$, $-1<\theta_y<1$, *and where*

$$\begin{cases} u'_{xxx}=u_{xxx}(x',\, y)\ , & x-h_1<x'<x+h_2\ ,\\ u''_{yyy}=u_{yyy}(x,\, y')\ , & y-h_3<y'<y+h_4\ . \end{cases} \tag{28}$$

Proof. By Lemma 1, u_{xxx} is continuous in the open line segment from $(x-h_1, y)$ to $(x+h_2, y)$, but may become infinite if the endpoint is a corner of C. Since u is continuous in $R\cup C$, it nevertheless follows from Taylor's formula as stated in [6, p. 357] that, if we fix y and set $\phi(x)=u(x, y)$,

$$\frac{\phi(x+h_2)-\phi(x)}{h_2}=\phi'(x)+\frac{h_2}{2}\phi''(x)+\frac{h_2^2}{6}\phi'''(x+\theta_2 h_2) ,$$

where $0<\theta_2<1$.

Writing a similar formula for h_1 and subtracting, we find in the notation of (3) that

$$D_x^{(h)}\phi(x)=\phi''(x)+\left[\frac{h_2^2}{3}\phi'''(x+\theta_2 h_2)-\frac{h_1^2}{3}\phi'''(x-\theta_1 h_1)\right](h_1+h_2)^{-1}.$$

If one writes $k=\max(h_1, h_2)\leq h$, the last term can be bounded in absolute value by

$$\frac{2k^2}{3k}\max\left[|\phi'''(x+\theta_2 h_2)|, |\phi'''(x-\theta_1 h_1)|\right] ,$$

and hence can be written in the form $(2h/3)\theta_x u'_{xxx}$. Addition of a similar expression for $D_y^{(h)}u(x, y)$ proves the lemma.

Lemma 4. *For each node* (x, y) *of* R_h^2 *defined in* (15) *use the notation of* (28). *Then, as* $h\to 0$, *one has*

$$h\sum_{R_h^2} u(|u'_{xxx}|+|u''_{yyy}|)=o(1) \qquad (h\to 0) . \tag{29}$$

Proof. The lemma is proved much like Lemma 6 of [4]. The functions $u|u_{xxx}|$ and $u|u_{yyy}|$ are continuous in $R\cup C$, except at a corner of interior angle $\pi\alpha$, where Lemma 1 states that they behave like $r^{2\alpha-3}$ with $2\alpha-3>-1$. The sum (29) can be majorized by the Lebesgue integral of a step function over a polygonal arc in R which converges in length to C as $h\to 0$. The integrability of $r^{2\alpha-3}$ in $(0, 1)$ permits the application of Lebesgue's convergence theorem as $h\to 0$. Since $u=0$ on C, (29) follows. Details are omitted.

Lemma 5. *At each node in* R_h^3, *defined in* (15), *one has*

$$\Delta^{(h)}u=\Delta u+\frac{1}{12}h^2(u'_{xxxx}+u''_{yyyy}) ,$$

where

$$\begin{cases} u'_{xxxx}=u_{xxxx}(x+\theta' h, y) , & -1<\theta'<1 , \\ u''_{yyyy}=u_{yyyy}(x, y+\theta'' h) , & -1<\theta''<1 . \end{cases} \tag{30}$$

Proof. In [4]; the points of R_h^3 all have four neighbors in R_h^3, each at a distance h.

LEMMA 6. *At each node of* R_h^3, *defined in* (15), *use the notation of* (30). *Then, as* $h \to 0$, *one has*

$$h^2 \sum_{R_h^3} u(u'_{xxxx} + u''_{yyyy}) = \iint_R u(u_{xxxx} + u_{yyyy})dxdy + o(1) \qquad (h \to 0).$$

Proof. In [4].

LEMMA 7. *Define* u_n *and* τ *as in Theorem* 1. *One then has*

$$\iint_R u(u_{xxxx} + u_{yyyy})dxdy = \iint_R (u_{xx}^2 + u_{yy}^2)dxdy + \int_C u_n^2 \sin^2 2\tau d\tau ,$$

where the latter is a Riemann-Stieltjes integral.

Proof. The proof repeats that of Lemma 7 in [4] down to (29) of that paper. It then remains only to prove for smooth convex curves C that

$$\int_C u_{yy}(u_y dx + u_x dy) = \int_C u_n^2 \sin^2 2\tau d\tau . \tag{31}$$

Let s denote arclength on C, and let primes denote d/ds. Differentiating the relations $u_x = -u_n \sin \tau$, $u_y = u_n \cos \tau$, we find that, on C,

$$\begin{cases} u_x' = -u_n' \sin \tau - u_n \tau' \cos \tau = u_{xy} \sin \tau + u_{xx} \cos \tau , \\ u_y' = u_n' \cos \tau - u_n \tau' \sin \tau = u_{xy} \cos \tau + u_{yy} \sin \tau . \end{cases} \tag{32}$$

Changing u_{xx} to $-u_{yy}$ by (1), we can solve (32) for u_{yy} on C:

$$u_{yy} = u_n' \sin 2\tau + u_n \tau' \cos 2\tau .$$

Since $dx = ds \cos \tau$ and $dy = ds \sin \tau$, we obtain

$$\begin{aligned} \int_C u_{yy}(u_y dx + u_x dy) &= \int_C (u_n' \sin 2\tau + u_n \tau' \cos 2\tau)(u_n \cos 2\tau) ds \\ &= \int_C u_n^2 \tau' \cos^2 2\tau \, ds + \int_C u_n u_n' \cos 2\tau \sin 2\tau \, ds . \end{aligned} \tag{33}$$

By partial integration, we have

$$\begin{aligned} \int_C u_n u_n' \cos 2\tau \sin 2\tau \, ds &= \tfrac{1}{4} \int_C (u_n^2)' \sin 4\tau \, ds \\ &= \tfrac{1}{4}[u_n^2 \sin 4\tau]_C - \int_C u_n^2 \tau' \cos 4\tau \, ds . \end{aligned} \tag{34}$$

Since $\cos^2 2\tau - \cos 4\tau \equiv \sin^2 2\tau$, substitution of (34) into (33) shows that

$$\int_C u_{yy}(u_y dx + u_x dy) = \int_C u_n^2 \tau' \sin^2 2\tau \, ds \, .$$

Since $\tau' ds = d\tau$, the identity (31) is proved, and with it, the lemma.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.) No. 10 1-chome Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

UNIVERSITY OF CALIFORNIA PRESS · BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics

Vol. 5, No. 5 BadMonth, 1955