APPLICATIONS OF THE RAYLEIGH RITZ METHOD TO VARIATIONAL PROBLEM

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Introduction. Let R be a bounded either simply or multiply connected plane region with boundary Γ, consisting of a finite number of non-intersecting simply closed regular arcs of class c^k. A plane curve is a regular arc if the defining functions $x(t)$, $y(t)$, $a \leq t \leq b$ have continuous derivatives with $x'(t)^2 + y'(t)^2 \neq 0$ on $a \leq t \leq b$. A regular arc is of class c^k if the defining functions $x(s)$, $y(s)$, s being arc length, have continuous derivatives of order k. We shall say a function $h(x, y)$ defined on $\bar{R} = R + \Gamma$ is of class c^k if the partial derivatives of h of order r, $0 \leq r \leq k$ exist in R and have limits on Γ so as to define functions continuous on $\bar{R}$. Let $g(x, y)$ be a given function of class c^k on $\bar{R}$. The main problem considered is that of finding the function ψ_0 which yields minimum value to the functional

$$I[\psi] = \iint_R (a\psi_x^2 + b\psi_y^2 + c\psi^2 + 2f\psi)dx \, dy$$

defined over the admissible class of functions ψ which are of class c^k on $\bar{R}$ and assume the values of g on Γ.

We shall assume $a(x, y) > 0$, $b(x, y) > 0$, $c(x, y) \geq 0$ on $\bar{R}$; a, b, c bounded and integrable in $\bar{R}$; $f(x, y)$ integrable in $\bar{R}$. In the sequel, unless otherwise specified, integrations will be taken over R and the symbol R omitted.

Let $G(x, y)$ be of class c^k on $\bar{R}$, vanishing on Γ, positive in R, with normal derivative $\partial G/\partial \nu$ on Γ different from 0. We show that, if $k \geq 3$, every admissible function ψ has a uniformly convergent expansion on $\bar{R}$

$$\psi = g + \sum_{i=1}^{\infty} b_i f_i(x, y)$$

where f_i are obtained by a Gram-Schmidt process from the functions $\{G x^i y^j\}$ $i, j = 0, 1, 2, \cdots$ and b_i are generalized Fourier coefficients connected with the quadratic functional

$$D[\psi] = \iint (a\psi_x^2 + b\psi_y^2 + c\psi^2)dx \, dy \, .$$

In fact, $b_i = D[\psi - g, f_i]$ where

$$D[\xi, \eta] = \iint (a\,\xi_x\,\eta_x + b\,\xi_y\,\eta_y + c\,\xi\eta)dx\,dy\,.$$

An estimate of the error obtained by using for ψ only the first n terms of the expansion is given in terms of n and k. Sufficient conditions are obtained for the convergence of

$$\nabla\left[g + \sum_{i=1}^{n} b_i f_i\right]$$

to $\nabla\psi$ and an estimate is given for the rate of convergence.

In particular, if ψ_0 is an admissible function minimizing $I[\psi]$, then the expansion

$$\psi_0 = g + \sum_{i=1}^{\infty} a_i f_i$$

yields an explicit solution for ψ_0, since the coefficients a_i are given, in this case, by

$$a_i = -\iint f f_i\,dx\,dy - D[g, f_i]$$

which are independent of ψ_0.

The problem of minimizing the functional $I[\psi]$, with $g \equiv 0$, has been studied by Kryloff and Bogoliubov [4] and by Kantorovitch [2], both obtaining estimates for convergence to ψ_0 of functions obtainable by the Rayleigh Ritz method. The first paper deals with convex regions R, the second with regions R bounded by $x=0$, $x=1$, $y=g(x)$, $y=h(x)$; $h > g$ on $0 \leq x \leq 1$. Neither obtains an explicit solution for ψ_0 nor studies the convergence of the derivatives.

In the final section of this paper, we assume the existence of a function ψ_0 yielding minimum value, for $p \geq 1$, to

$$D^p[\psi] = \iint_R (a\psi_x^2 + b\psi_y^2 + c\psi^2)^p\,dx\,dy\,, \qquad \psi = g \quad \text{on} \quad \Gamma$$

and obtain an estimate for the rate of convergence to ψ_0 of functions obtained by the Rayleigh Ritz method.

§ 1. **Preliminary Considerations.** A variation v shall mean a function of class c^k on $\overline{R}$ vanishing on Γ. Form the Hilbert space H by completing the linear manifold V of variations v using the positive definite quadratic form $D[v]$ as the square of the norm of a variation. If $h \in H$, we represent the norm of h by $\|h\|$. If ξ and η are variations, the inner product will be

$$(\xi, \eta) = D[\xi, \eta]\,.$$

Let f_i be any complete orthonormal set of variations in H. If ψ is admissible, then $\psi - g$ is a variation and thus expressible in H as

$$\psi - g = \sum_{i=1}^{\infty} b_i f_i$$

with $b_i = D[\psi - g,\ f_i]$.

If ψ_0 is an admissible function yielding a minimum value to $I[\psi]$, if λ is real, and v is a variation, then $\psi_0 + \lambda v$ is admissible, and

$$I[\psi_0] \leq I[\psi_0 + \lambda v] = I[\psi_0] + \lambda(2D[\psi_0,\ v] + \iint 2fv\,dx\,dy) + \lambda^2 D[v].$$

This implies that the coefficient of λ must vanish so that

$$D[\psi_0,\ v] = -\iint fv\,dx\,dy$$

and

$$I[\psi_0 + \lambda v] = I[\psi_0] + \lambda^2 D[v]$$

for every variation v.

The first relation shows that the Fourier coefficients of $\psi_0 - g$,

$$a_i = D[\psi_0 - g,\ f_i] = D[\psi_0,\ f_i] - D[g,\ f_i] = -\iint f f_i\,dx\,dy - D[g,\ f_i]$$

are independent of ψ_0.

The second relation implies that if ψ is admissible,

$$I[\psi] = I[\psi_0 + \psi - \psi_0] = I[\psi_0] + D[\psi - \psi_0].$$

Thus if

$$\phi_n = g + \sum_{i=1}^{n} a_i f_i,$$

then

$$0 = \lim_{n \to \infty} \left\| \psi_0 - \left(g + \sum_{i=1}^{n} a_i f_i \right) \right\|^2 = \lim_{n \to \infty} D[\psi_0 - \phi_n] = \lim_{n \to \infty} I[\phi_n] - I[\psi_0]$$

so that ϕ_n is a minimizing sequence.

Moreover,

$$D\left[\psi_0 - g - \sum_{i=1}^{n} c_i f_i \right]$$

is a minimum when $c_i = a_i$ implying that ϕ_n are chosen to yield minimum value to $I[\phi_n] - I[\psi_0]$ and hence to $I[\phi_n]$ in the class of functions

$$\psi_n = g + \sum_{i=1}^{n} c_i f_i.$$

Thus ϕ_n may be obtained by the Rayleigh Ritz process applied to the functional $I[\psi]$.

We will prove, in Theorem 1, that the class of functions $\{G\,P\}$ where P is a polynomial in x and y, is dense in H. This class is the linear manifold determined by the set $\{G\,x^i\,y^j\}$, a set linearly independent in H. For, if

$$v = \sum_{i=1}^{m}\sum_{i=1}^{n}\alpha_{ij}\,G\,x^i\,y^j,$$

then $D[v]=0$ implies $\alpha_{ij}=0$.

It follows that we can obtain an orthonormal set f_i complete in H by orthonormalizing the set $\{G\,x^i\,y^j\}$. Let

$$v_1 = G\,x^0\,y^0$$

$$v_2 = G\,x^1\,y^0\,, \quad v_3 = G\,x^0\,y^1$$

$$\vdots$$

$$v_{\frac{k(k+1)}{2}+1} = G\,x^k\,y^0, \cdots, \quad v_{\frac{k(k+1)}{2}+k+1} = G\,x^0\,y^k\,.$$

Then

$$f_n = \frac{v_n - \sum_{j=1}^{n-1}f_j(v_n,\,f_j)}{\left\| v_n - \sum_{j=1}^{n-1}f_j(v_n,f_j) \right\|}$$

$$= \begin{vmatrix} (v_1,\,v_1)\cdots(v_1,\,v_{n-1}) & v_1 \\ \vdots \qquad\quad \vdots & \vdots \\ (v_n,\,v_1)\cdots(v_n,\,v_{n-1}) & v_n \end{vmatrix} \cdot \begin{vmatrix} (v_1,\,v_1) & \\ & \ddots \\ & (v_{n-1},\,v_{n-1}) \end{vmatrix}^{-1/2} \begin{vmatrix} (v_1,\,v_1) & \\ & \ddots \\ & (v_n,\,v_n) \end{vmatrix}^{-1/2}.$$

The function f_n is of the form $G\,P_n$, where the degree of the polynomial P_n is that of v_n/G. If $v_n = G\,x^r\,y^s$ with $r+s=k$, then

$$\frac{k(k+1)}{2}+1 \leq n \leq \frac{k(k+1)}{2}+k+1$$

so that $k^2 < k(k+1) < 2n-2$ and the degree k of P_n is less than $\sqrt{2n-2}$. Similarly k is greater than $\sqrt{2n-2}$.

§ 2. **The Minimizing sequence.** We shall use certain approximation theorems which can be derived by methods used by Mickelson [5][1]. To simplify the notation, let

$$x = (x_1,\cdots x_s)\,,$$

$$x^{(1)} = (x_1^{(1)},\cdots x_s^{(1)})\,,$$

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[1] For detailed proofs of Lemmas 1, 2 see J. Indritz "Applications of the Rayleigh Ritz method to the solutions of partial differential equations" Ph. D. Thesis, U. of Minnesota, 1953.

$$f(x) = f(x_1, \cdots, x_s) ,$$

$$f_r(x) = \frac{\partial^{r_1 + \cdots + r_s}}{\partial x_1^{r_1} \cdots \partial x_s^{r_s}} [f(x_1, \cdots, x_s)] ,$$

$$\| x^{(1)} - x^{(2)} \| = \sqrt{\sum_{i=1}^{s} (x_i^{(1)} - x_i^{(2)})^2} .$$

The modulus of continuity for a function f defined over a closed set A: $-1 \leq x_i \leq 1$ $(i=1, \cdots, s)$ is

$$\Omega(\delta, f) = \sup |f(x^{(1)}) - f(x^{(2)})|$$

for all points $x^{(1)}$, $x^{(2)}$ in A with $\| x^{(1)} - x^{(2)} \| \leq \delta$. The uniform modulus of continuity of a finite number of functions $f_1, \cdots, f_N$ is the largest of the moduli of each f_i for each δ.

LEMMA 1. *Let $F(\theta)$ be a continuous periodic function of period 2π in each θ_i and of class c^k. Let $\omega(\delta)$ be the uniform modulus of continuity of the partial derivatives of F of order 1 to k for $\delta \leq \pi \sqrt{s}$. Let $j \leq k$. Then, corresponding to every set $m_1, \cdots, m_s$ of positive integers, there is a trigonometric sum T^m of order at most m_i in θ_i such that*

$$|F_r(\theta) - T_r^m(\theta)| \leq K_1 \Big(\sum_{i=1}^{s} \frac{1}{m_i} \Big)^{k-j} \sum_{i=1}^{s} \omega \Big(\frac{1}{m_i} \Big) \ for \quad 0 \leq r_1 + \cdots + r_s \leq j$$

where K_1 is a constant independent of F, s, m_i.

If the partial derivatives of order 1 to k satisfy

$$|F_r(\theta^{(1)}) - F_r(\theta^{(2)})| \leq L \Big(\sum_{i=1}^{s} |\theta_i^{(1)} - \theta_i^{(2)}| \Big)$$

then

$$|F_r(\theta) - T_r^m| \leq L \ K_2 \Big(\sum_{i=1}^{s} \frac{1}{m_i} \Big)^{k-j+1} \quad 0 \leq r_1 + \cdots + r_s \leq j$$

where K_2 is also a constant independent of F, s, m_i.

If F is even in each θ_i separately, T contains only cosine terms.

LEMMA 2. *Let $f(x)$ be of class c^k in the set A: $-1 \leq x_i \leq 1$ $(i=1, \cdots, s)$. Let M be the maximum of the absolute values of the derivatives of order 1 to k, and $\Omega(\delta)$ the uniform modulus of continuity of the derivatives of order k. Let B denote a closed set interior to A. Let $j \leq k$. Then, for every set of positive integers $m_1, \cdots, m_s$ with $m_i \geq k$ there is a polynomial P^m of order at most m_i in x_i such that*

$$|f_r(x) - P_r^{\,m}(x)| \leq K_3 \Big(\sum_{i=1}^{s} \frac{1}{m_i}\Big)^{k-j} \sum_{i=1}^{s} \Omega\Big(\frac{1}{m_i}\Big)$$

*for x in B and $0 \leq r_1 + \cdots + r_s \leq j$.   Here K_3 is a constant independent of
f and m_i .*

   *If also, the k-th partial derivatives of $f(x)$ satisfy a Lipschitz con-
dition with parameter λ, then, for x in B,*

$$|f_r(x) - P_r^{\,m}(x)| \leq K_4 \Big(\sum_{i=1}^{s} \frac{1}{m_i}\Big)^{k-j+1} \quad for \quad 0 \leq r_1 + \cdots + r_s \leq j ,$$

and where K_4 is a constant independent of f and m_i .

   To apply the lemmas to a function defined over the region R, we
shall extend the domain of definition of the function.   The ques-
tion arises whether the differentiability properties of the function are
maintained under the extension.   The answer depends upon the pro-
perties of the boundary Γ of R.   For example, Hirschfeld [1] has
shown that even a cusp in the complementary region may prevent c^1
extension of a function of class c^∞ on a closed set through a continu-
ous boundary arc.   Whitney [6] has given a different definition for a
function to be of class c^k in a closed set A.   If f is of Whitney class
c^k in A, then there exists an extension F to the whole plane E_2 which
is of class c^k in the ordinary sense on E_2 and is analytic in $E_2 - A$.
The derivatives of F of order $\leq k$ coincide with those of f at any
interior point of A.   Moreover Whitney [7] has shown the following:
Let (a) f be of class c^k on $R + \Gamma$, where R is a region, Γ its bound-
ary, in the sense we have defined in the introduction, and (b) R have
the property "P", that any two points P_1, P_2 in R, whose linear
distance apart may be represented by $|P_1 - P_2|$, can be joined by a
rectifiable curve in R of length L, with $L/|P_1 - P_2|$ bounded uniformly
with respect to P_1 and P_2; then f is also of Whitney class c^k and thus
can be extended to E_2 to be of class c^k on E_2 .

   For our purposes we assume R to be a bounded region with bound-
ary Γ consisting of a finite number of non-intersecting simply closed
regular arcs Γ_i and we will show R has property "P".

   Choose, for each Γ_i, a $\delta > 0$ such that no two tangents to Γ_i on
any portion of arc length $< \delta$ make with each other an angle greater
than 5°.   We may choose δ independent of i and smaller than one-
fourth the distance between any two Γ_i.   Now fix i, and let P_1, P_2
be points on Γ_i on a subarc of length $< \delta$.   There is a point Q on that
subarc between P_1 and P_2 such that the tangent line at Q is parallel
to the chord $P_1 P_2$.   Set up an (x, y) coordinate system at Q, using the
tangent line as x-axis, the normal as y-axis, and note that the subarc

considered has an equation $y = y(x)$ of class c^1 in view of the implicit function theorems. Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $\|P_1 - P_2\| = $ distance between P_1 and P_2, $\|\widehat{P_1 P_2}\| = $ length of the subarc joining P_1 to P_2. Then $\|P_1 - P_2\| = |x_1 - x_2|$ and $|y'(x)| \leq 1$ so that

$$(1) \qquad \|P_1 - P_2\| \leq \|\widehat{P_1 P_2}\| = \left| \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \right|$$

$$\leq \sqrt{2}\, |x_1 - x_2| = \sqrt{2}\, \|P_1 - P_2\| .$$

Moreover, since $\tan 5° < 1/10$, the mean value theorem shows that $\sup |y(x)| \leq \|P_1 - P_2\|/10$.

We shall also use the well known property that if Γ_i is a regular arc, there is an $\omega_1 > 0$ such that for any subarc joining points P_3, P_4 on Γ_i, we have $\|\widehat{P_3 P_4}\|/\|P_3 - P_4\| \leq \omega_1$. ω_1 can be chosen independent of i.

Now suppose S_1, S_2 are any two points interior to the region R. If the segment $S_1 S_2$ is interior to R, we of course have $\|\widehat{S_1 S_2}\|/\|S_1 - S_2\| = 1$ by using the segment as the arc. Otherwise, let Q_1 be the first intersection of the directed line $S_1 S_2$ with the boundary, say with Γ_1. Let Q_1^1 be a point on $S_1 Q_1$ in R. Let Q_2 be the first point of intersection of the directed line $S_2 S_1$ with Γ_1 and Q_2^1 a point in R on $S_2 Q_2$ such that the open interval $Q_2 Q_2^1$ is also in R. Note that Q_1 and Q_2 may coincide. If $Q_2^1 S_2$ is not in R, let Q_3 be the first point of intersection of the directed line $Q_2^1 S_2$ with the boundary, say with Γ_2 and Q_3^1 in R and on $Q_2^1 Q_3$. Let Q_4 be the first point of intersection of the directed line $S_2 Q_2^1$ with Γ_2 and Q_4^1 a point in R, on $Q_4 S_2$, with the open interval $Q_4 Q_4^1$ in R. Continuing in this way, after at most n steps, we form a finite sequence of points $Q_0^1 = S_1$, Q_1^1, $Q_2^1, \cdots, Q_{2m}^1$, $Q_{2m+1}^1 = S_2$ such that Q_{2k-1} and Q_{2k} are on the same regular arc, and the lines joining Q_{2k}^1 to Q_{2k+1}^1, $k = 0, \cdots, m$ are in R. If we can show there is an $\omega > 0$, independent of the points, and arcs λ_i in R joining consecutive points Q_j^1 to Q_{j+1}^1 such that $\|\widehat{Q_j^1 Q_{j+1}^1}\| \leq \omega \|Q_j^1 - Q_{j+1}^1\|$, then we can attain the desired results by addition. It suffices to show that Q_1^1 and Q_2^1 and an arc λ joining Q_1^1 to Q_2^1 and in R may be chosen so that $\|\widehat{Q_1^1 Q_2^1}\| \leq \omega \|Q_1^1 - Q_2^1\|$. Suppose first that Q_1 and Q_2 coincide. A sufficiently small circle with Q_1 as center will have one of the arcs cut off by $S_1 S_2$ entirely in R and we may choose Q_1^1 and Q_2^1 as the intersections of $S_1 S_2$ with this circle. In this case

$$\|\widehat{Q_1^1 Q_2^1}\| = \frac{\pi}{2} \|Q_1^1 - Q_2^1\| .$$

Otherwise, let L be the length of an arc on Γ_1 joining Q_1 to Q_2,

Divide this arc into N equal segments of length $\beta = L/N$ where N is sufficiently large so that $\beta < \delta$. Draw circles of radius $r = \beta/\sqrt{2}$ about each of the division points and the end points. We first show that consecutive circles intersect. If R_1 and R_2 are two consecutive centers, (1) implies

$$\| R_1 - R_2 \| \leq \beta \leq \sqrt{2} \, \| R_1 - R_2 \|$$

so that

$$\frac{\| R_1 - R_2 \|}{2} \leq \frac{\| R_1 - R_2 \|}{\sqrt{2}} \leq \frac{\beta}{\sqrt{2}} \leq \| R_1 - R_2 \| \, ,$$

and the circles must intersect.

Moreover, since $r \geq \| R_1 - R_2 \|/\sqrt{2}$, the semi-length τ of the common chord is

$$\tau = \sqrt{r^2 - \frac{\| R_1 - R_2 \|^2}{4}} \geq \frac{\| R_1 - R_2 \|}{2} \, ,$$

whereas the arc joining R_1 to R_2 has distance $< \| R_1 - R_2 \|/10$ from the chord. Hence the arc lies entirely within the circles.

Now let Q_1^1 be an intersection of $S_1 S_2$ with the circle whose center is Q_1 and Q_2^1 an intersection of $S_1 S_2$ with the circle whose center is Q_2, the points being chosen to lie in R and have the desired properties. Starting from Q_1^1 we may proceed to Q_2^1 via the circumferences of the circles. The total length of the curve thus formed will be less than

$$(N+1) 2\pi \frac{\beta}{\sqrt{2}} = \frac{N+1}{N} \frac{2\pi}{\sqrt{2}} L \leq \frac{4\pi}{\sqrt{2}} L$$

and

$$\frac{\| \widehat{Q_1^1 Q_2^1} \|}{\| Q_1^1 - Q_2^1 \|} \leq \frac{4\pi}{\sqrt{2}} \frac{L}{\| Q_1^1 - Q_2^1 \|} \leq \frac{4\pi}{\sqrt{2}} \frac{L}{\| Q_1 - Q_2 \|} \leq \frac{4\pi}{\sqrt{2}} \omega_1 \, .$$

This concludes the proof that R has property "P".

We will be particularly interested in extending a function of the form $v(x, y)/G(x, y)$ where $G(x, y) > 0$ in R, $\partial G/\partial \nu > 0$ on Γ, $G = v = 0$ on Γ and we seek differentiability conditions on v and G which insure that v/G is of class c^k on $R + \Gamma$. Here again the nature of the boundary is of importance. The next two lemmas deal with this problem. The letter P will refer to a point in R and Q to a point on Γ, the boundary of R. By a neighborhood $N(Q)$ in $R + \Gamma$ we will mean a set of points S in $R + \Gamma$ such that for some sufficiently small circle with center at Q, every point of the circle which lies in $R + \Gamma$ also lies in S.

LEMMA 3. a) *Let R be a region bounded by Γ, a finite number of closed Jordan curves, no two having a point in common. Let γ be a regular subarc of Γ, and Q_0 an interior point of γ. Let N be the normal to γ at Q_0. Then there is a neighborhood $N(Q_0)$ in $R + \Gamma$ such that through each point P in $RN(Q_0)$, the line parallel to N cuts $\gamma_0 = \gamma N(Q_0)$ in one and only one point Q, PQ lies in $N(Q_0)$, and Q ranges over γ_0.*

b) *Let $\psi(x, y)$ be of class c^1 in $RN(Q_0)$ and suppose ψ, ψ_x, ψ_y have continuous limits on γ_0. Define $(\partial\psi/\partial s)(P)$ to be the derivative at $P \in RN(Q_0)$ in the direction of the tangent at the corresponding point Q on γ_0. The derivative $(\partial\psi/\partial s)(P)$ has continuous limits on γ_0 which we will denote by $(\partial\psi/\partial s)(Q)$.*

If $\psi = 0$ on γ_0, then $(\partial\psi/\partial s)(Q) = 0$ for Q on γ_0.

Proof. Let γ be given by $x(t)$, $y(t)$ and Q_0 defined by the parameter value t_0. Let (ξ, η) be rectangular axes along the tangent and normal at Q_0. In a suitable neighborhood of t_0, $t_1 < t < t_2$, defining an arc λ_0 containing Q_0, γ admits a representation $\eta = \eta(\xi)$. We may assume λ_0 so small that no two tangents to it make with each other an angle greater than $5°$. There is a positive distance d between $\Gamma - \lambda_0$ and the arc λ_1 defined by the parameter range $(t_1 + t_0)/2 < t < (t_0 + t_2)/2$. Take

$$\delta < \min[d, \ |\xi(t_0) - \xi((t_0 + t_2)/2)|, \ |\xi(t_0) - \xi((t_0 + t_1)/2)|]$$

and draw a square T of side δ with sides parallel to the (ξ, η) axes and center at Q_0. Let $\gamma_0 = \gamma T$, the projection of RT on γ by lines parallel to N, and let γ_h be the arcs formed by displacing γ_0 a distance h parallel to itself into R along N. For $h < h_1$ sufficiently small, $\gamma_h \subset T$. The regular arc γ_0 may be given a representation $x = x(s)$, $y = y(s)$, $0 \leq s \leq L$, in terms of arc length s, where L is the length of γ_0. Then γ_h is given by

$$x = x(s) + h \cos \alpha, \qquad y = y(s) + h \cos \beta,$$

where $\cos \alpha$, $\cos \beta$ are the direction cosines of the line N directed inward into R. The neighborhood $N(Q_0)$ may be chosen as given by these equations with $0 < s < L$. $0 \leq h < h_1$.

It is clear that

$$\frac{\partial\psi}{\partial s} = \frac{\partial\psi}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial\psi}{\partial y} \frac{dy}{ds}$$

has continuous limits on γ_0. Write

$$\frac{\partial\psi}{\partial s}(P) = \frac{\partial\psi}{\partial s}(x(s) + h \cos \alpha, \ y(s) + h \cos \beta) = F(s, \ h).$$

If λ is any closed subarc of γ_0, we have

$$\lim_{h \to 0} F(s, h) = \frac{\partial \psi}{\partial s}(Q)$$

uniformly in s.

Along γ_h we have

$$\psi(P_2) - \psi(P_1) = \int_{s_1}^{s_2} F(s, h) \, ds$$

where P_1 and P_2 are points on γ_h corresponding to points Q_1 and Q_2 on λ with parameter values s_1 and s_2. As h approaches 0, the limits on the integral remain fixed. Since $\psi = 0$ on λ, we find, by letting $h \to 0$,

$$0 = \int_{s_1}^{s_2} \frac{\partial \psi}{\partial s}(Q) \, ds$$

for arbitrary s_1, s_2. Thus $(\partial \psi / \partial s)(Q) = 0$ on λ and hence on γ_0.

LEMMA 4. *Let R, γ, Q_0, $N(Q_0)$, N, γ_0 be defined as in Lemma 3. Let $v(x, y)$ and $G(x, y)$ be of class c^0 on $N(Q_0)$ and of class c^1 on $N(Q_0)[R + Q_0]$. Let $v = G = 0$ on γ_0, $G > 0$ in $RN(Q_0)$, $(\partial G / \partial \nu)(Q_0) \neq 0$. Then there exists $\lim_{P \to Q_0} v(P)/G(P)$ for $P \in R$.*

If γ is of class c^{k+1} on $N(Q_0)$ and v, G are of class c^k in $N(Q_0)$ and of class c^{k+1} on $N(Q_0)[R + Q_0]$, then v/G is of class c^k on $N(Q_0)[R + Q_0]$.

Proof. Denote differentiation along a line parallel to N by $\partial / \partial h$. By the mean value theorem one finds that $(\partial G / \partial \nu)(Q_0)$ is the limiting value of $(\partial G / \partial h)(P)$ as $P \in RN(Q_0)$ approaches Q_0 along the normal at Q_0, and hence $(\partial G / \partial \nu)(Q_0)$ is the limiting value of $(\partial G / \partial h)(P)$ as P approaches Q_0 by any approach in $RN(Q_0)$. A similar statement is true for $(\partial v / \partial \nu)(Q_0)$.

Let P_n be any sequence of points in $RN(Q_0)$ converging to Q_0 and let Q_n be the points on γ_0 associated, by projection along N, with P_n. By the generalized mean value theorem,

$$\frac{v(P_n)}{G(P_n)} = \frac{v(P_n) - v(Q_n)}{G(P_n) - G(Q_n)} = \frac{(\partial v / \partial h)(P_n')}{(\partial G / \partial h)(P_n')}$$

where P_n' is interior to the line segment $P_n Q_n$.

Thus

$$\lim_{P_n \to Q_0} \frac{v(P_n)}{G(P_n)} = \frac{(\partial v / \partial \nu)(Q_0)}{(\partial G / \partial \nu)(Q_0)} \ .$$

It is clear from the construction of $N(Q_0)$ that the equations

$$x = X(s, h) = x(s) + h \cos \alpha, \quad y = Y(s, h) = y(s) + h \cos \beta$$

yield a one to one transformation of $N(Q_0)$ into $N^*(Q_0)$: $0 \leq h < h_1$, $0 < s < L$ and γ_0 into γ_0^*: $h = 0$, $0 < s < L$ and Q_0 into Q_0^*: $h = 0$, $s = s_0$. In fact, in view of the restriction on the slope of the tangent to γ_0, the Jacobian of the transformation is

$$J = x'(s) \cos \beta - y'(s) \cos \alpha \neq 0.$$

If $x(s)$, $y(s)$ are of class c^{k+1} on $0 < s < L$ then so are $X(s, h)$, $Y(s, h)$ in $N^*(Q_0) - \gamma_0^*$. Any partial derivative of $X(s, h)$, $Y(s, h)$ of order $r \leq k+1$ converges, as $h \to 0$, uniformly on any closed subinterval of γ_0^* and thus this derivative has a continuous limit on γ_0^*. By the implicit function theorems, the inverse functions $s = S(x, y)$, $h = H(x, y)$ are of class c^{k+1} in $RN(Q_0)$. Moreover, the partial derivatives of S, H of order $r \leq k+1$ have continuous limits on γ_0, for the relationships

$$1 = \frac{\partial X}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial X}{\partial h}\frac{\partial h}{\partial x} = x'(s)\frac{\partial s}{\partial x} + \cos \alpha \frac{\partial h}{\partial x}$$

$$0 = \frac{\partial Y}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial Y}{\partial h}\frac{\partial h}{\partial x} = y'(s)\frac{\partial s}{\partial x} + \cos \beta \frac{\partial h}{\partial x}$$

can be solved for $\partial s/\partial x$, $\partial h/\partial x$, $\partial s/\partial y$, $\partial h/\partial y$ and the resulting equations indicate that these derivatives and their derivatives of order $\leq k$ have continuous limits on γ_0.

Under this transformation $v(x, y)$ becomes $v^*(s, h)$ and $G(x, y)$ becomes $G^*(s, h)$. It is sufficient to show v^*/G^* is of class c^k at Q_0^* since any partial derivative of order $r \leq k$ of $v(x, y)/G(x, y)$ is a polynomial in the derivatives of v^*/G^* and in the derivatives of s and h with respect to x and y of order $\leq r$.

By the hypothesis and comments above, $v^*(s, h)$ and $G^*(s, h)$ are of class c^k on $N^*(Q)$ and of class c^{k+1} on $(N^*(Q_0) - \gamma_0^*) + Q_0^*$. In view of the continuity of $\partial G/\partial h$ at Q_0. there is a neighborhood of Q_0 where $(\partial G/\partial h)(P) > \delta > 0$, It is no loss of generality to assume $(\partial G/\partial h) > \delta > 0$ in $N(Q_0)$ and we shall do so. By Lemma 3, $\partial v/\partial s$ and $\partial G/\partial s$ vanish on γ_0. By repeated application of Lemma 3, $\partial^r v/\partial s^r$ and $\partial^r G/\partial s^r$ $(0 \leq r \leq k)$ vanish on γ_0.

The proof is greatly facilitated by an auxiliary transformation. Let $t = s$, $z = G^*(s, h)$ carrying Q_0^* into Q_0^{**}, γ_0^* into γ_0^{**}, $N^*(Q_0)$ into $N^{**}(Q_0)$. For each s, z is a monotone increasing function of h and the inverse function $h = H^*(t, z)$ is a monotone increasing function of z for each t. As above, we see that $v^*(s, h) = v^{**}(t, z)$ is of class c^k on

$N^{**}(Q_0)$ and of class c^{k+1} on $(N^{**}(Q_0) - \gamma_0^{**}) + (Q_0^{**})$. Moreover, it suffices to prove that $v^{**}(t, z)/z$ is of class c^k at Q_0^{**}. For notational simplicity, let $w(t, z) = v^{**}(t, z)$. Note that $N^{**}(Q_0)$ is the set $0 \leq z < G(t, h_1)$, $0 < t < L$.

By induction, we verify

$$\frac{\partial^r}{\partial z^r}\left(\frac{w}{z}\right) = \frac{r!(-1)^r}{z^{r+1}}\left(w - z\frac{\partial w}{\partial z} + \frac{z^2}{2!}\frac{\partial^2 w}{\partial z^2} + \cdots + (-1)^r\frac{z^r}{r!}\frac{\partial^r w}{\partial z^r}\right)$$

for $0 \leq r \leq k$ when $z > 0$.

For t fixed, $w(t, z)$ has a Taylor expansion of the form

$$w(t, \zeta) = w(t, z) + \frac{\partial w}{\partial z}(t, z)(\zeta - z) + \cdots + \frac{\partial^r w}{\partial z^r}(t, z)\frac{(\zeta - z)^r}{r!}$$

$$+ \frac{\partial^{r+1} w}{\partial z^{r+1}}(t, \xi)\frac{(\zeta - z)^{r+1}}{(r+1)!}$$

for $0 \leq r \leq k$, where $0 \leq \zeta < \xi(t, z, \zeta, r) < z$ so that, when $\zeta = 0$,

$$0 = w(t, 0) = w(t, z) - z\frac{\partial w}{\partial z}(t, z) + \cdots + \frac{(-1)^r}{r!}z^r\frac{\partial^r w}{\partial z^r}(t, z)$$

$$+ \frac{(-1)^{r+1}z^{r+1}}{(r+1)!}\frac{\partial^{r+1} w}{\partial z^{r+1}}(t, \xi).$$

Hence

$$\frac{\partial^r}{\partial z^r}\left(\frac{w}{z}\right) = \frac{1}{r+1}\frac{\partial^{r+1}}{\partial z^{r+1}}w(t, \xi),$$

which has a limit as the point (t, z) approaches Q_0^{**}.

We have thus shown that the partial derivatives of w/z, with respect to z alone, of order $\leq k$ have limits at Q_0^{**}.

We next show that the partial derivatives of w/z with respect to t alone have limits at Q_0^{**}. First note that the derivatives of w with respect to t alone vanish at $z = 0$. For, $w(t, z) = v^*(s, h)$ so that

$$\frac{\partial v}{\partial s} = \frac{\partial v^*}{\partial s} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial G}{\partial s}$$

and, as we have seen, $\partial v/\partial s$ and $\partial G/\partial s$ vanish at $z = 0$. Thus $\partial w/\partial t = 0$ at $z = 0$. Similarly, successive differentiation shows $\partial^r w/\partial t^r = 0$ on γ_0^{**}, $0 \leq r \leq k$.

We apply Taylor's theorem to obtain

$$\frac{\partial^r}{\partial t^r}\left(\frac{w}{z}\right) = \frac{1}{z}\frac{\partial^r w(t, z)}{\partial t^r} = \frac{1}{z}\left\{z\frac{\partial}{\partial z}\left[\frac{\partial^r w(t, \xi)}{\partial t^r}\right]\right\} = \frac{\partial}{\partial z}\frac{\partial^r w(t, \xi)}{\partial t^r},$$

$$0 < \xi(z, r) < z$$

and conclude that $(\partial^r/\partial t^r)(w/z)$ has a limit at Q_0^{**} for $0 \leq r \leq k$,

Finally, any mixed derivative may be written as

$$\frac{\partial^{n+m}}{\partial z^n \partial t^m}\left(\frac{w}{z}\right), \qquad n+m=r \leq k$$

and this may be written as $\dfrac{\partial^u}{\partial z^n}\left\{\dfrac{1}{z}\dfrac{\partial^m w}{\partial t^m}\right\}$

where $\partial^m w/\partial t^m$ vanishes on γ_0^{**} and is of class c^{k-m} on $N^{**}(Q_0)$ and of class c^{k-m+1} on $(N^{**}(Q_0)-\gamma_0^{**})+Q_0^{**}$. By the first results for derivatives with respect to z, the mixed derivatives have the desired property.

THEOREM 1. *Let R be a bounded region whose boundary Γ consists of a finite number of non-intersecting simply closed regular arcs of class c^k, $(k \geq 2)$. Let $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial \nu \geq \delta > 0$ on Γ.*

Let H be the Hilbert space formed by completing the linear vector space V of variations—functions of class c^k on $\overline{R}$ and vanishing on Γ—, using the functional

$$D[\xi] = \iint (a\xi_x^2 + b\xi_y^2 + c\xi^2)dx\,dy$$

for $\xi \in V$ as the square of the norm, where a, b, c are bounded and integrable, $a > 0$, $b > 0$, $c \geq 0$ in $R+\Gamma$.

Then the set of functions $G\tau$, where τ is a polynomial in x and y, is dense in H. The set $\{f_i\}$ obtained by orthonormalizing the set $\{Gx^i y^j\}$ is complete in H.

If $g(x, y)$ is a function of class c^k on $\overline{R}$ and Ψ is the set of functions ψ of class c^k on $\overline{R}$, assuming the values of $g(x, y)$ on Γ, and if for any $\psi \in \Psi$ we define $b_i = D[\psi-g, f_i]$, then

$$\left\| \psi - g - \sum_{i=1}^n b_i f_i \right\|^2 = D\left[\psi - g - \sum_{i=1}^n b_i f_i\right] \leq \frac{\theta(n)}{n^{k-2}}$$

where $\lim_{n \to \infty} \theta(n) = 0$, θ depending on $\psi - g$.

In particular, if f is integrable,

$$I[\psi] = \iint (a\psi_x^2 + b\psi_y^2 + c\psi^2 + 2f\psi)dx\,dy,$$

and there exists an admissible function ψ_0 which minimizes $I[\psi]$ for $\psi \in \Psi$, and we define

$$a_i = -\iint f f_i \, dx \, dy - D[g, f_i], \quad \phi_n = g + \sum_{i=1}^{n} a_i f_i \,,$$

then

$$\| \psi_0 - \phi_n \|^2 = D[\psi_0 - \phi_n] \leq \frac{\theta(n)}{n^{k-2}}$$

where $\lim_{n \to \infty} \theta(n) = 0$.

Proof. If v is a variation, we show there is a sequence Q_j of polynomials such that

$$\lim_{j \to \infty} \| v - G \, Q_j \|^2 = \lim_{j \to \infty} \iint [a(v - G \, Q_j)_x^2 + b(v - G \, Q_j)_y^2 + c(v - G \, Q_j)^2] \, dx \, dy = 0.$$

In view of Lemma 4, v/G is of class c^{k-1} on $\overline{R}$ and it is thus possible to extend the definition of v/G over the entire plane so that it is of class c^{k-1} over the entire plane. Let $\Omega(\delta)$ be the uniform modulus of continuity of the $(k-1)$st partial derivatives of v/G over a rectangle with sides parallel to the axes containing R in its interior.

By Lemma 2, with $s=2$, $j=1$, $m_1 = m_2 = j$ there is a sequence Q_j of polynomials of degree $2j$ in x and y such that, for (x, y) in $\overline{R}$,

$$\left| \frac{v}{G} - Q_j \right|, \quad \left| \left(\frac{v}{G} \right)_x - Q_{j_x} \right|, \quad \text{and} \quad \left| \left(\frac{v}{G} \right)_y - Q_{j_y} \right| \text{ are all } O\left(\frac{1}{j^{k-2}} \Omega\left(\frac{1}{j} \right) \right).$$

Hence

$$(v - G \, Q_j)_x^2 = \left[\left(\frac{v}{G} - Q_j \right) G \right]_x^2 = \left[\left(\frac{v}{G} - Q_j \right)_x G + \left(\frac{v}{G} - Q_j \right) G_x \right]^2$$

$$\leq \left(\frac{v}{G} - Q_j \right)_x^2 G^2 + G_x^2 \left(\frac{v}{G} - Q_j \right)^2 + 2G|G_x| \left| \left(\frac{v}{G} - Q_j \right)_x \right| \left| \frac{v}{G} - Q_j \right|$$

$$= O\left(\frac{1}{j^{2(k-2)}} \left[\Omega\left(\frac{1}{j} \right) \right]^2 \right).$$

A similar result is true for $(v - G \, Q_j)_y^2$ and $(v - G \, Q_j)^2$. Thus $\lim_{j \to \infty} D[v - G \, Q_j] = 0$ for $k \geq 2$.

It has thus been proved that the linear manifold formed by $\{Gx^iy^j\}$ is dense in V and thus in H. By the previous discussion the set $\{f_i\}$ is complete in H.

Now let v in the above be the particular variation $\psi - g$ and let $[N]$ represent the largest integer $\leq N$. For fixed n, let $j = [(\sqrt{n}/2) - 1]$ and $\tau_n(x, y) = Q_j(x, y)$. Thus there is a sequence τ_n of degree at most

$$2j \leq \left[2\left(\frac{\sqrt{n}}{2} - 1 \right) \right] = [\sqrt{2n} - 2]$$

such that

$$D[\psi - g - G\,\tau_n] = O\left(\frac{1}{n^{k-2}}\,\Omega^2\left(\frac{8}{\sqrt{n}}\right)\right).$$

Now $\sum_{i=1}^{n} b_i f_i = G\mu_n$ where μ_n is a polynomial of degree greater than $\sqrt{2n} - 2$, and it is known that

$$\left|\psi - g - \sum_{i=1}^{n} c_i f_i\right|$$

is a minimum when $c_i = (\psi - g,\ f_i) = b_i$. Thus

$$D\left[\psi - g - \sum_{i=1}^{n} b_i f_i\right] = O\left(\frac{1}{n^{k-2}}\,\theta\right), \quad \lim_{n\to\infty} \theta = 0.$$

In particular, if ψ_0 minimizes $I[\psi]$, then we have seen that

$$D[\psi_0 - g,\ f_i] = -\iint f f_i\,dx\,dy - D[g,\ f_i].$$

Thus, in this case, the Fourier coefficients depend only on known quantities.

COROLLARY $$b_n = O\left(\sqrt{\frac{\theta(n)}{n^{k-2}}}\right).$$

Proof. $b_n^2 = D[b_n f_n] = D\left[\left(\psi - g - \sum_{i=1}^{n-1} b_i f_i\right) - \left(\psi - g - \sum_{i=1}^{n} b_i f_i\right)\right]$

$$\leq D\left[\psi - g - \sum_{i=1}^{n-1} b_i f_i\right] + 2\left(D\left[\psi - g - \sum_{i=1}^{n-1} b_i f_i\right]\right.$$

$$\left.\cdot D\left[\psi - g - \sum_{i=1}^{n} b_i f_i\right]\right)^{1/2} + D\left[\psi - g - \sum_{i=1}^{n} b_i f_i\right] = O\left(\frac{\theta(n)}{n^{k-2}}\right).$$

§ 3. **Expansion Theorems.** We use the notations in Theorem 1 and seek conditions which insure that convergence in H yields uniform convergence in $\bar{R}$.

THEOREM 2. *Let R be a bounded region with boundary Γ. Let ψ, ψ_n be continuous on $\bar{R}$, absolutely continuous on each line in $\bar{R}$ and all taking on the same values on Γ. Let $D[\psi] < \infty$, $D[\psi_n] < \infty$. If $\lim_{n\to\infty} D[\psi - \psi_n] = 0$, then a necessary and sufficient condition that $\lim_{n\to\infty} \psi_n = \psi$ uniformly on $\bar{R}$ is that ψ_n be equicontinuous on $\bar{R}$. If $\lim_{n,\,m\to\infty} D[\psi_n - \psi_m] = 0$ then a necessary and sufficient condition that $\lim_{n\to\infty} \psi_n$ exists uniformly on $\bar{R}$ is that ψ_n be equicontinuous on $\bar{R}$.*

Proof. The necessity is clear since a sequence of continuous functions

which converge uniformly are equicontinuous.

Let $u(x, y)$ be a function with the continuity properties of $\psi(x, y)$ and vanishing on Γ. Let P_0 be a point interior to R. Place polar coordinates at P_0. If a ray from P_0 meets the circle S_ρ of radius $\rho \leq d$, d being the diameter of R, with P_0 as center, before it meets Γ, label P_1 the first intersection point with S_ρ and Q the first intersection with Γ. Otherwise both P_1 and Q will refer to the first intersection point of the ray and Γ.

$$\frac{1}{2\pi}\left\{\int_0^{2\pi} |u(P_1)|\, d\theta\right\}^2 \leq \int_0^{2\pi} u^2(P_1)\, d\theta = \int_0^{2\pi}\left[\int_{P_1}^{Q} \frac{\partial u}{\partial r}\, dr\right]^2 d\theta$$

$$= \int_0^{2\pi}\left[\int_{P_1}^{Q} \frac{1}{\sqrt{r}}\sqrt{r}\,\frac{\partial u}{\partial r}\, dr\right]^2 d\theta \leq \int_0^{2\pi} \log\frac{d}{\rho} \int_{P_1}^{Q} r\left(\frac{\partial u}{\partial r}\right)^2 dr\, d\theta$$

$$\leq \log\frac{d}{\rho} \iint (u_x^2 + u_y^2)\, dx\, dy \leq \alpha \log\frac{d}{\rho} D[u]$$

where $\alpha = 1/\min_{R+\Gamma}(a, b)$, since

$$\iint (au_x^2 + bu_y^2 + cu^2)\, dx\, dy \geq \min(a, b)\iint (u_x^2 + u_y^2)\, dx\, dy .$$

Apply this result to the functions $u_n = \psi - \psi_n$ (or to $u_{nm} = \psi_n - \psi_m$) which are equicontinuous on $R + \Gamma$ and thus have a uniform modulus of continuity $\omega(\delta)$, which approaches 0 with δ.

Since P_1 is on or interior to the circle of radius ρ, we have $|u_n(P_1) - u_n(P_0)| \leq \omega(\rho)$, whence $|u_n(P_1)| \geq |u_n(P_0)| - \omega(\rho)$ and

$$2\pi[|u_n(P_0)| - \omega(\rho)] \leq \sqrt{2\pi\alpha \log d/\rho\, D[u_n]} .$$

Thus

$$|u_n(P_0)| \leq \sqrt{\frac{\alpha}{2\pi} D[u_n] \log\frac{d}{\rho}} + \omega(\rho) ,$$

which is true even if P_0 is on Γ.

Now, for $\varepsilon > 0$, choose $\rho = \rho_1$ so small that $\omega(\rho_1) < \varepsilon/2$ and then choose N so large that

$$\frac{\alpha}{2\pi} D[u_n] \log\frac{d}{\rho_1} < \frac{\varepsilon^2}{4}$$

for $n > N$. Hence

$$\varepsilon > 0 \cdot \supset \cdot \exists\, N(\varepsilon) \ni n > N \cdot \supset \cdot |\psi(P_0) - \psi_n(P_0)| < \varepsilon .$$

LEMMA 5. *Let R be a bounded region with boundary Γ and diameter d.*

Let $u(x, y)$ be continuous on $R+\Gamma$, absolutely continuous on each line in $R+\Gamma$, and vanish on Γ, and let $0<D[u]<\infty$. Let $\alpha = 1/\min_{R+\Gamma}(a, b)$. Let $P_0 \in R+\Gamma$. If there exists $\delta>0$, $K\geq0$ and

$$|u(P) - u(P_0)| \leq K \| P - P_0 \|^\delta$$

for all points P such that the ray P_0P is in $R+\Gamma$, then

$$|u(P_0)| \leq \sqrt{\frac{\alpha D[u]}{2\pi\delta} \log^+ \frac{d^\delta K}{\Delta D[u]}} + \Delta D[u]$$

where Δ is any number >0, and

$$\log^+ x = \begin{cases} \log x & if \quad x>1 \\ 0 & if \quad x\leq1 . \end{cases}$$

Proof. If P_0 is interior to R, and $\rho\leq d$, then as in Theorem 2

$$\frac{1}{2\pi} \left\{ \int_0^{2\pi} |u(P_1)| \, d\theta \right\}^2 \leq \alpha \log d/\rho \, D[u],$$

where P_1 is a point which is the first intersection of a ray from P_0 with either Γ or the circle of radius $\rho\leq d$ about P_0 as center.

Since P_1 is on or interior to the circle of radius ρ, we have

$$|u(P_1) - u(P_0)| \leq K\rho^\delta, \quad |u(P_1)| \geq |u(P_0)| - K\rho^\delta,$$

$$2\pi[u(P_0) - K\rho^\delta] \leq \sqrt{2\pi\alpha \log d/\rho \, D[u]},$$

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi} D[u] \log \frac{d}{\rho}} + K\rho^\delta,$$

which holds even if P_0 is on Γ.

Let $\Delta>0$. If

$$\left(\frac{\Delta D[u]}{K}\right)^{1/\delta} < d,$$

choose

$$\rho = \left(\frac{\Delta D[u]}{K}\right)^{1/\delta}$$

to obtain

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi\delta} D[u] \log \frac{d^\delta K}{\Delta D[u]}} + \Delta D[u] .$$

Otherwise,

$$\left(\frac{\Delta D[u]}{K}\right)^{1/\delta} \geq d, \qquad K \leq \frac{\Delta D[u]}{d^\delta}$$

and we may replace K to obtain

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi} D[u]} \log \frac{d}{\rho} + \frac{\Delta D[u]}{d^\delta} \rho^\delta .$$

Choose $\rho = d$ to obtain $|u(P_0)| \leq \Delta D[u]$.

COROLLARY 1. *A sufficient condition that a sequence u_n, absolutely continuous on each line in $\overline{R}$, vanishing on Γ, continuous on $\overline{R}$, and having* $\lim_{n\to\infty} D[u_n] = 0$, *converge to 0 at P_0 is that $\exists \delta > 0$ and a sequence K_n, with* $\lim_{n\to\infty} D[u_n] \log K_n = 0$ *such that*

$$|u_n(p) - u_n(P_0)| \leq K_n \| P - P_0 \|^\delta$$

for all P with ray $P_0 P$ in $\overline{R}$. If δ, K_n are independent of P_0, the convergence is uniform. In any case,

$$|u_n(P_0)| \leq \sqrt{\frac{\alpha}{2\pi\delta} D[u_n]} \log^+ \frac{d^\delta K_n}{\Delta_n D[u_n]} + \Delta_n D[u_n]$$

for any $\Delta_n > 0$.

LEMMA 6. *Let R be a bounded domain with boundary Γ. Let $P_0 \in \overline{R}$ and suppose there is a circle of radius ε lying in $\overline{R}$ and containing P_0. Place polar coordinates (r, θ) at P_0. Let $u(x, y)$ be of class c^1 in $\overline{R}$ and suppose that there exist $\lambda > 0$, $\sigma \geq 0$ such that*

$$|u_r(P) - u_r(P_0)| \leq \sigma \| P - P_0 \|^\lambda$$

for all points P such that the ray $P_0 P$ is in $\overline{R}$.
   Then

$$|\nabla u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)} \left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)} (\lambda+1) + \frac{8\alpha D[u]}{\pi\varepsilon^2} \left(\frac{\lambda+1}{\lambda}\right) .$$

Proof. $|u_r(P_0)| \leq |u_r(P)| + \sigma r^\lambda$

Integrating over a circle S_ρ of radius $\rho \leq \varepsilon$ which contains P_0, $S_\rho \subset S_\varepsilon$, we obtain

$$\iint_{S_\rho} |u_r(P_0)|^2 r \, dr \, d\theta \leq 2\iint_{S_\rho} u_r(P)^2 r \, dr \, d\theta + 2\iint_{S_\rho} \sigma^2 r^{2\lambda} r \, dr \, d\theta .$$

We may assume that the polar axis lies in the direction of $\nabla u(P_0)$. Hence $u_r(P_0) = |\nabla u(P_0)| \cos \theta$ and

$$\iint_{S_\rho} |\nabla u(P_0)|^2 (\cos^2 \theta) r \, dr \, d\theta < 2\alpha D[u] + 2\sigma^2 (2\rho)^{2\lambda} \cdot \pi \rho^2 .$$

We will show that the minimum value of $\iint_{S_\rho}(\cos^2\theta)r\,dr\,d\theta$ is $\pi\rho^2/4$.
Suppose first that the pole O is interior to S_ρ. Let $r(\theta)$ be the equation of the circle relative to the pole O. Let Q be the point $(r(\theta),\,\theta)$ and Q' the point $(r(\theta+\pi),\,\theta+\pi)$. Q and Q' are thus the intersections of a ray through O with the circle. Let O' be the center of the circle and suppose the co-ordinates of O' relative to O are $(c,\,\phi)$. Then the angle between OQ and OO' is $\phi-\theta$. Drop a perpendicular from O' to QOQ' hitting the latter at T, the length of OT being $|c\cos(\phi-\theta)|$. Thus one of the lengths $\|OQ\|$, $\|OQ'\|$ is $m+|c\cos(\phi-\theta)|$ and the other is $m-|c\cos(\phi-\theta)|$ where $2m$ is the length of QQ', and the product $\|OQ'\|\cdot\|OQ\|=m^2-c^2\cos^2(\phi-\theta)$. Also, if OO' meets the circle in points A, A' it is easily seen that $\|OA'\|\,\|OA\|=\|OQ\|\,\|OQ'\|$ so that $(\rho+c)(\rho-c)=m^2-c^2\cos^2(\phi-\theta)$ and $m^2=\rho^2-c^2+c^2\cos^2(\phi-\theta)$. Hence
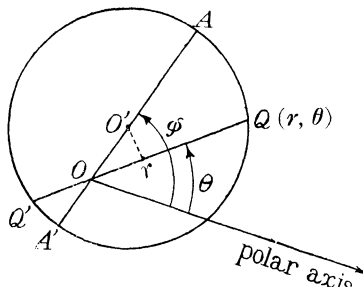
$$\|OQ\|^2+\|OQ'\|^2=[m+|c\cos(\phi-\theta)|]^2+[m-|c\cos(\phi-\theta)|]^2$$

$$=2m^2+2c^2\cos^2(\phi-\theta)=2\rho^2-2c^2+4c^2\cos^2(\phi-\theta).$$

We note that

$$\iint_{S_\rho}(\cos^2\theta)r\,dr\,d\theta=\frac{1}{2}\int_0^{2\pi}r^2(\theta)\cos^2\theta\,d\theta=\frac{1}{2}\int_0^\pi(\|OQ\|^2+\|OQ'\|^2)\cos^2\theta\,d\theta$$

$$=\frac{1}{2}\int_0^\pi[2\rho^2-2c^2+4c^2\cos^2(\phi-\theta)]\cos^2\theta\,d\theta\;.$$

Moreover this formula holds even if O is a point on the circumference for in this case

$$\iint_{S_\rho}(\cos^2\theta)\,r\,dr\,d\theta=1/2\int_\gamma^{\gamma+\pi}r^2(\theta)\cos^2\theta\,d\theta$$

where γ is the angle between the polar axis and the tangent to the circle at O in that direction which has the area to the left of the tangent line. Here $r^2=[2\rho\cos(\phi-\theta)]^2$ and since the square of the cosine has period π, the integral reduces to

$$\frac{1}{2}\int_0^\pi 4\rho^2\cos^2(\phi-\theta)\cos^2\theta\,d\theta\;.$$

Thus, in any case,

$$\iint_{S_\rho} (\cos^2 \theta) r \, dr \, d\theta = \frac{1}{2} \int_0^\pi [2\rho^2 - 2c^2 + 4c^2 \cos^2(\phi - \theta)] \cos^2 \theta \, d\theta$$

$$= \frac{1}{2} \pi \rho^2 - \frac{1}{2} \pi c^2 + \frac{\pi c^2}{4} [1 + 2 \cos^2 \phi] \, .$$

For fixed c, the minimum is obtained when $\phi = \dfrac{\pi}{2}$, and is $\dfrac{\pi \rho^2}{2} - \dfrac{1}{4} \pi c^2$. The absolute minimum is obtained when $c = \rho$ and is $\pi \rho^2 / 4$.

It follows from this result, that

$$\frac{\pi \rho^2}{4} |\nabla u(P_0)|^2 \leq 2\alpha D[u] + 2\sigma^2 (2\rho)^{2\lambda} \pi \rho^2 \, ,$$

$$|\nabla u(P_0)|^2 < \frac{8\alpha}{\pi \rho^2} D[u] + 2^{2\lambda+3} \sigma^2 \rho^{2\lambda} \, .$$

Consider the function $y = A/\rho^2 + B\rho^{2\lambda}$ where $A = 8\alpha D[u]/\pi$, $B = 2^{2\lambda+3}\sigma^2$. The minimum value is

$$y_{\min} = A^{\lambda/(\lambda+1)} B^{1/(\lambda+1)} (\lambda+1) \lambda^{-\lambda/(\lambda+1)}$$
$$= (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)} \left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)} (\lambda+1) 2^{(5\lambda+3)/(\lambda+1)}$$

obtained when

$$\rho = \left(\frac{A}{B\lambda}\right)^{1/(2\lambda+2)} = \left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)} \, .$$

If

$$\left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)} < \varepsilon \, .$$

choose

$$\rho = \left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)}$$

and have

$$|\nabla u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)} \left(\frac{\alpha}{\pi\lambda}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)} (\lambda+1).$$

However, if

$$\left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)} > \varepsilon \, ,$$

we have

$$\sigma^2 \leq \frac{\alpha D[u]}{2^{2\lambda}\lambda\pi\varepsilon^{2\lambda+2}}$$

and integrating over S_ε, as in the beginning of this proof, we find that

$$\iint_{S_\varepsilon} |u_r(P_0)|^2 r\, dr\, d\theta \leq 2 \iint_{S_\varepsilon} u_r(P)^2 r\, dr\, d\theta + 2\, \frac{\alpha\, D[u]}{2^{2\lambda}\lambda\pi\varepsilon^{2\lambda+2}}(2\varepsilon)^{2\lambda}\pi\varepsilon^2,$$

$$\frac{\pi\varepsilon^2}{4}\, |\triangledown u(P_0)|^2 \leq 2\alpha D[u] + \frac{2\alpha\, D[u]}{\pi\lambda\varepsilon^2}\cdot\pi\varepsilon^2,$$

$$|\triangledown u(P_0)|^2 \leq \frac{8\alpha\, D[u]}{\pi\varepsilon^2} + \frac{8\alpha\, D[u]}{\lambda\pi\varepsilon^2} = \left(1 + \frac{1}{\lambda}\right)\frac{8\alpha\, D[u]}{\pi\varepsilon^2}.$$

Thus, in any case,

$$|\triangledown u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)}\left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)}(\lambda+1) + \frac{8\alpha\, D[u]}{\pi\varepsilon^2}\left(\frac{\lambda+1}{\lambda}\right).$$

LEMMA 7. *Let R be a bounded region with boundary Γ and diameter d and let R have the property that there exists an $\varepsilon>0$ such that every point of $R+\Gamma$ is within some circle of radius ε lying in $R+\Gamma$.*

Let $u(x,\ y)=G\tau+H$ where τ is a polynomial of degree m, G and H are of class c^1 on $R+\Gamma$ and vanish on Γ, $G>0$ in R, $|\triangledown G|\geq\delta>0$ on Γ. Let $|G|\leq G_1$, $|H|\leq H_1$, $|\triangledown G|\leq G_2$, $|\triangledown H|\leq H_2$ for constants G_1, G_2, H_1, H_2.

Suppose also that

$$|G_x(P)-G_x(P_0)|\leq G_0\,\|\,P-P_0\,\|\,, \quad |G_y(P)-G_y(P_0)|\leq G_0\,\|\,P-P_0\,\|\,,$$

$|H_x(P)-H_x(P_0)|\leq H_0\,\|\,P-P_0\,\|$, $\quad |H_y(P)-H_y(P_0)|\leq H_0\,\|\,P-P_0\,\|$ *for constants G_0, H_0, whenever P, P_0 are points in $\bar{R}$ such that the line P_0P is in $\bar{R}$. Let A be an upper bound for $D[u]$ and $D[u]\log m$.*

Then there exists a constant B, depending only on α, A, G_0, G_1, G_2, H_0, H_1, H_2, δ, ε, d, G but not on m or τ, such that for $P_0 \in \bar{R}$.

$$|u(P_0)|\leq\sqrt{\frac{2\alpha}{\pi}D[u]}\,\log^+\frac{m}{\varDelta\, D[u]}+B(\varDelta\, D[u])^{\natural}(D[u])^{1/4}$$

for any $\varDelta>0$. (m to be replaced by 1 if it is 0).

Proof. We may assume $D[u]>0$ for otherwise $u=0$ in $\bar{R}$.

Let $L=\max_{\bar{R}}|\tau|$. By a theorem of Kellogg [3], $|\triangledown\tau(P)|\leq Lm^2/\varepsilon$ for $P \in \bar{R}$.

If P and P_0 are on a straight line in $\bar{R}$, then

$$|\tau(P)-\tau(P_0)| = \left|\int_{P_0}^{P}\frac{\partial\tau}{\partial r}dr\right|\leq\frac{Lm^2}{\varepsilon}\,\|\,P-P_0\,\|.$$

$$|H(P)-H(P_0)|\leq H_2\,\|\,P-P_0\,\|\,, \quad |G(P)-G(P_0)|\leq G_2\,\|\,P-P_0\,\|\,,$$

$$|u(P)-u(P_0)|\leq|G(P)\tau(P)-G(P)\tau(P_0)|+|G(P)\tau(P_0)-G(P_0)\tau(P_0)|+|H(P)-H(P_0)|$$

$$\leq\Big(G_1\frac{Lm^2}{\varepsilon}+LG_2+H_2\Big)\|P-P_0\|=K\|P-P_0\|\,.$$

By Lemma 5, with $\varDelta=D[u]^{-1/2}$,

(2)
$$|u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+\frac{dK}{\sqrt{D[u]}}}+\sqrt{D[u]}\,.$$

Also, τ_x, τ_y are polynomials of degree m and absolute value less than or equal to Lm^2/ε, so that $|\nabla\tau_x|\leq(Lm^2/\varepsilon)(m^2/\varepsilon,)$ and

$$|\tau_x(P)-\tau_x(P_0)|\leq\int_{P_0}^{P}|\nabla\tau_x|dr\leq\frac{Lm^4}{\varepsilon^2}\|P-P_0\|\,.$$

Thus

$$|\nabla\tau(P)-\nabla\tau(P_0)|\leq\frac{2Lm^4}{\varepsilon^2}\|P-P_0\|\,.$$

Then

$$|\nabla u(P)-\nabla u(P_0)|$$

$$\leq|G(P)\nabla\tau(P)-G(P_0)\nabla\tau(P_0)|+|\tau(P)\nabla G(P)-\tau(P_0)\nabla G(P_0)|+|\nabla H(P)-\nabla H(P_0)|$$

$$\leq|G(P)\nabla\tau(P)-G(P)\nabla\tau(P_0)|+|G(P)\nabla\tau(P_0)-G(P_0)\nabla\tau(P_0)|$$

$$+|\tau(P)\nabla G(P)-\tau(P)\nabla G(P_0)|+|\tau(P)\nabla G(P_0)-\tau(P_0)\nabla G(P_0)|$$

$$+|H_x(P)-H_x(P_0)|+|H_y(P)-H_y(P_0)|$$

$$\leq\Big(G_1\frac{2Lm^4}{\varepsilon^2}+\frac{Lm^2}{\varepsilon}G_2+L\,2G_0+G_2\frac{Lm^2}{\varepsilon}+2H_0\Big)\|P-P_0\|=\sigma\|P-P_0\|\,.$$

Whence Lemma 6 yields

(3)
$$|\nabla u(P_0)|\leq\sqrt{(\sigma^2D[u])^{1/2}\Big(\frac{\alpha}{\pi}\Big)^{1/2}32+\frac{16\alpha}{\pi\varepsilon^2}D[u]}\,.$$

By use of inequalities (2) and (3) we now find a bound for L.

Either $L\leq1$ or else there exist constants c_1, c_2 such that $K<c_1Lm^2$, $\sigma<c_2Lm^4$ where the factor m is to be omitted if it is zero, and c_1, c_2 depend only on ε, G_1, G_2, H_2, H_0, G_0.

Assume $L>1$. Since $|\nabla G|\neq0$ on $\varGamma$, there exists a continuous curve (or curves) γ dividing $\bar{R}$ into two closed sets $\bar{R}_1$ and $\bar{R}_2$, such that $\bar{R}_1\bar{R}_2=\gamma$, $\bar{R}_1$ being a boundary set where $|\nabla G|\geq\delta/2>0$, and $\bar{R}_2$ the set separated from $\varGamma$ by γ. There is a constant c_3 such that $G(P)\geq c_3>0$ for $P\in\bar{R}_2$.

Suppose first that $|\tau|$ assumes its maximum L at a point $P_0\in\bar{R}_2$. Then, by (2),

$$|G(P_0)\tau(P_0)+H(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+\frac{d\,c_1Lm^2}{\sqrt{D[u]}}}+\sqrt{D[u]}$$

or,

$$(4) \qquad L \leq \frac{1}{c_3}\left[H_1 + \sqrt{\frac{\alpha}{2\pi}D[u]}\, \log^+ \frac{d\, c_1\, Lm^2}{\sqrt{D[u]}} + \sqrt{D[u]} \right].$$

Since $D[u]\log m$ and $D[u]$ are bounded by A, equation (4) implies the existence of a constant c_4 depending on c_3, c_1, d, A, α, H_1 such that $L < c_4$.

On the other hand, if $|\tau|$ assumes its maximum L at a point $P_0 \in \bar{R}_1$, write

$$\nabla u = G\nabla\tau + \tau\nabla G + \nabla H \qquad \tau^2\nabla G = \tau\nabla u - u\nabla\tau + H\nabla\tau - \tau\nabla H,$$

$$\tau^2|\nabla G| \leq |\tau|\,|\nabla u| + |u|\,|\nabla\tau| + |H|\,|\nabla\tau| + |\tau|\,|\nabla H|$$

$$\leq L\sqrt{(c_2^2 L^2 m^8 D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2}32 + \frac{16\alpha\, D[u]}{\pi\varepsilon^2}}$$

$$+ \frac{Lm^2}{\varepsilon}\left(\sqrt{\frac{\alpha}{2\pi}D[u]}\,\log^+ \frac{d\, c_1\, Lm^2}{\sqrt{D[u]}} + \sqrt{D[u]}\right) + H_1\frac{Lm^2}{\varepsilon} + LH_2.$$

Therefore,

$$(5) \quad L \leq \frac{2}{\partial}\left\{\sqrt{32(c_2^2 L^2 m^8 D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} + \frac{16\alpha\, D[u]}{\pi\varepsilon^2}}\right.$$

$$\left. + \frac{m^2}{\varepsilon}\left(\sqrt{\frac{\alpha\, D[u]}{2\pi}}\,\log^+ \frac{d\, c_1\, Lm^2}{\sqrt{D[u]}} + \sqrt{D[u]}\right) + \frac{H_1 m^2}{\varepsilon} + H_2\right\}.$$

This inequality, which is of the form

$$L \leq K_1 + K_2 m^2 + K_3 m^2\sqrt{\log L} + K_4 m^2\sqrt{L},$$

shows that

$$\sqrt{L} \leq \frac{K_1}{\sqrt{L}} + \frac{K_2 m^2}{\sqrt{L}} + K_3 m^2\sqrt{\frac{\log L}{L}} + K_4 m^2 \leq K_1 + K_2 m^2 + K_3 m^2 + K_4 m^2,$$

since $L > 1$, whence $L \leq c_5 m^4$ for some constant c_5.

Thus, in any case, there is a constant c_6 such that $L < c_6 m^4$, where the factor m is to be omitted if it is zero. From this one can conclude that $K < c_1 c_6 m^6$. However, we may obtain a better estimate by noticing that K merely serves as a number such that $|u(P) - u(P_0)| \leq K\|P - P_0\|$ whenever P and P_0 are on a straight line in $\bar{R}$. Hence K may be replaced by $\sup_{\bar{R}} |\nabla u|$.

The inequality $\sigma < c_2 Lm^4 < c_2 c_6 m^8$ and formula (3) yield

$$|\nabla u(P_0)| \leq \sqrt{c_2 c_6 m^8 D[u]^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2}\cdot 32 + \frac{16\alpha D[u]^{1/2}D[u]^{1/2}}{\pi\varepsilon^2}} = c_7 D[u]^{1/4} m^4,$$

since $D[u] \leq A$. Thus we may replace K by $c_7 D[u]^{1/4} m^4$ and substitute in Lemma 5 to obtain

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi}} D[u] \log^+ \frac{dc_7 \overline{D[u]^{1/4} m^4}}{4D[u]} + 4D[u] \ .$$

Let $4 = dc_7 4_1^4 D[u]^{13/4}$ and $B = dc_7$ to obtain the conclusion.

LEMMA 8. *Let R, G, have the properties in Lemma 7 and let $u = G\tau$ where τ is a polynomial of degree m.*

Then $D[u] \geq c_{12} |u(P_0)|^2 / \log m$ where $c_{12} > 0$ is a constant depending only on G_0, G_1, G_2, d, ε, α, δ, G. The factor $\log m$ is to be omitted if $m = 0$ or 1.

Proof. Whether $L \leq 1$ or not, the formulas for K, σ show that $K < c_1 L m^2$, $\sigma < c_2 L m^4$. Moreover, either formula (4) or (5) holds, with $H_1 = 0$, $H_2 = 0$. If (4) holds, we have

$$\frac{L}{\sqrt{D[u]}} \leq \frac{1}{c_3} \left[\sqrt{\frac{\alpha}{2\pi}} \log^+ \frac{dc_1 L m^2}{\sqrt{D[u]}} + 1 \right] \ .$$

Let $w = L / \sqrt{D[u]}$. The above inequality is then of the form $w \leq K_1 \sqrt{\log w m^2} + K_2$ whence $L / \sqrt{D[u]} \leq c_8 \log m$ for some constant c_8, depending on α, c_3, d, c_1. Here the factor $\log m$ is to be omitted if $m = 0$ or 1. On the other hand, if (5) holds, we have

$$\frac{L}{\sqrt{D[u]}} \leq \frac{2}{\delta} \left\{ \sqrt{c_2 m^4} \frac{L}{\sqrt{D[u]}} \left(\frac{\alpha}{\pi} \right)^{1/2} \cdot 32 + \sqrt{\frac{16\alpha}{\pi \varepsilon^2}} \right.$$

$$\left. + \frac{m^2}{\varepsilon} \left(\sqrt{\frac{\alpha}{2\pi}} \log^+ dc_1 m^2 \frac{L}{\sqrt{D[u]}} + 1 \right) \right\} \ ,$$

from which we conclude $L / \sqrt{D[u]} \leq c_9 m^4 \log m$ (m and $\log m$ to be omitted if $m = 0$ or 1).

Thus, in any case, there is a constant c_{10} such that $L / \sqrt{D[u]} \leq c_{10} m^5$. Therefore

$$\frac{K}{\sqrt{D[u]}} \leq \frac{c_1 L m^2}{\sqrt{D[u]}} \leq c_1 c_{10} m^7 \ .$$

Substituting in equation (2), we have

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi}} D[u] \log^+ dc_1 c_{10} m^7 + \sqrt{D[u]} \leq c_{11} \sqrt{\log m} \sqrt{D[u]} \ ,$$

m to be omitted if it is 0 or 1.

THEOREM 3. *Let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k \geq 3$. Let $G(x, y)$ be a function of class c^k on $R + \Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial \nu \geq \delta > 0$ on Γ. Let f_i be the set obtained by orthonormalizing the set $\{Gx^i y^j\}$ using the functional*

$$D[\xi] = \iint_R (a\xi_x^2 + b\xi_y^2 + c\xi^2)dx\,dy$$

as the square of the norm, where a, b, c are bounded and integrable, $a > 0$, $b > 0$, $c \geq 0$ on $R + \Gamma$. Let $g(x, y)$ be any function of class c^k on $R + \Gamma$. Let $\psi(x, y)$ be any function of class c^k on $R + \Gamma$ assuming the values of $g(x, y)$ on Γ. Define $b_i = D[\psi - g, f_i]$.
   Then

$$\left| \psi - g - \sum_{i=1}^{n} b_i f_i \right| = O\left(\sqrt{\frac{\sqrt{\theta(n)}}{n^{k-2}}} \log \frac{n}{(\log n)^N} \right),$$

where

$$D\left[\psi - g - \sum_{i=1}^{n} b_i f_i \right] \leq \frac{\theta(n)}{n^{k-2}},$$

with $\lim_{n \to \infty} \theta(n) = 0$, θ depending on $\psi - g$, and where N is any fixed constant > 0. Moreover, if $k \geq 10$, then

$$\left| \nabla\psi - \nabla\left(g + \sum_{i=1}^{n} b_i f_i \right) \right| = O\left(\left[\frac{\theta(n)}{n^{k-10}} \right]^{1/4} \right).$$

Finally, if S is any closed domain in R, $k \geq 7$, then for points P in S,

$$\left| \nabla\psi - \nabla\left(g + \sum_{i=1}^{n} b_i f_i \right) \right| = O\left(\left[\frac{\theta(n) \log n}{n^{k-6}} \right]^{1/4} \right).$$

Proof. Let $u_n = \psi - g - \sum_{i=1}^{n} b_i f_i$. Then u_n is of the form $G\tau_n + H$ where the degree m_n of τ_n is less than $\sqrt{2n} - 2$ and greater than $\sqrt{2n} - 2$. By Theorem 1, $D[u_n] \leq \theta(n)/n^{k-2}$, $k \geq 3$, where $\lim_{n \to \infty} \theta(n) = 0$ so that $D[u_n] \log m_n \leq A$ for some constant A independent of n. By Lemma 7,

$$|u_n(P_0)| \leq \sqrt{\frac{2\alpha}{\pi}} \, D[u_n] \log^+ \frac{m_n}{\Delta_n D[u_n]} + B(\Delta_n D[u_n])^4 D[u_n]^{1/4}$$

for any $\Delta_n > 0$.

$$|u_n| \leq \sqrt{\frac{2\alpha}{\pi}} \, D[u_n] \log^+ \frac{\sqrt{2n}}{\Delta_n D[u_n]} + B(\Delta_n D[u_n])^4 D[u_n]^{1/4}.$$

There is a constant E, depending on N, such that $1/n < E/(\log n)^N e$, $n \geq 3$. Then

$$D[u_n] \leq \frac{\theta(n)}{n^{k-2}} \leq \frac{\theta(n) E^{k-2}}{e^{k-2}(\log n)^{N(k-2)}} = \frac{\sqrt{2n}}{e^{k-2} \varDelta_n} \leq \frac{\sqrt{2n}}{e \varDelta_n}$$

if

$$\varDelta_n = \frac{\sqrt{2n}(\log n)^{N(k-2)}}{\theta(n) E^{k-2}}.$$

The function $x \log(\sqrt{2n}/\varDelta x)$ is monotone increasing for $0 \leq x \leq \sqrt{2n}/e\varDelta$ so that we may replace $D[u_n]$ by $\theta(n)/n^{k-2}$ to obtain

$$|u_n| \leq \sqrt{\frac{2\alpha}{\pi} \frac{\theta(n)}{n^{k-2}} \log^+ \frac{\theta(n) E^{k-2}}{(\log n)^{N(k-2)} \theta(n)/n^{k-2}}}$$

$$+ B\left(\frac{\sqrt{2n}(\log n)^{N(k-2)}}{\theta(n) E^{k-2}} \frac{\theta(n)}{n^{k-2}}\right)^4 \left(\frac{\theta(n)}{n^{k-2}}\right)^{1/4}$$

$$= O\left(\sqrt{\frac{\sqrt{\theta(n)}}{n^{k-2}} \log \frac{n}{(\log n)^N}}\right).$$

In the proof of Lemma 7, we saw that $L < c_6 m^4$ and $\sigma < c_2 L M^4 < c_2 c_6 m^8$. Hence by equation (3) of Lemma 7,

$$|\nabla u_n| \leq \sqrt{c_2 c_6 m_n^8 \left(\frac{\theta(n)}{n^{k-2}}\right)^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32} + \frac{16\alpha}{\pi \varepsilon^2} \frac{\theta(n)}{n^{k-2}}.$$

Since $m_n < \sqrt{2n}$, we obtain the statement of the theorem regarding uniform convergence in $\bar{R}$ of $|\nabla u_n|$ for $k \geq 10$.

Next, let S be any closed domain in R. We may suppose the boundary $\varGamma''$ of S is sufficiently smooth so that a circle of radius ε may be rolled around $\varGamma''$ while lying in S. Let $L_n' = \sup_S |\tau_n|$ and $P_0^{(n)}$ be the point in S where $L_n' = |\tau_n(P_0^{(n)})|$. As in the proof of Lemma 7,

$$|\nabla u_n(P_0)| \leq \sqrt{(\bar{\sigma}_n^2 D[u_n])^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32} + \frac{16\alpha D[u_n]}{\pi \varepsilon^2} \qquad \text{for } P_0 \in S$$

where

$$\bar{\sigma}_n = G_1 \frac{2 L_n' m_n^4}{\varepsilon^2} + \frac{L_n' m_n^2}{\varepsilon} G_2 + L_n' 2 G_0 + \frac{G_2 L_n' m_n^2}{\varepsilon} + 2 H_0.$$

Using $G\tau_n$ as the function u of Lemma 8 defined over $\bar{R}$ and remembering that $G\tau_n = -\sum_{i=1}^{n} b_i f_i$, we obtain

$$|G(P_0^{(n)})L_n'| \leq \sqrt{D\left[\sum_{i=1}^{n} b_i f_i\right]} \log m_n \Big/ c_{12} \ .$$

In S, $|G(P)| \geq c_{13} > 0$. Also

$$D\left[\sum_{i=1}^{n} b_i f_i\right] = \sum_{i=1}^{n} b_i^2 \leq \sum_{i=1}^{\infty} b_i^2 = D[\psi - g] \ .$$

Therefore $L_n' < c_{14} \sqrt{\log n}$, $\bar{\sigma}_n < c_{15} n^2 \sqrt{\log n}$, and

$$|\nabla u_n(P_0)| \leq \sqrt{c_{15} n^2 \sqrt{\log n}} \left(\frac{\theta(n)}{n^{k-2}}\right)^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha\theta(n)}{\pi\varepsilon^2 n^{k-2}}$$

$$= O\left(\left[\frac{\theta(n) \log n}{n^{k-6}}\right]^{1/4}\right) \ .$$

THEOREM 4. *Let R, Γ, G, f_i be defined as in Theorem 3. Then there is a constant c_{17} such that whenever $P_0 \in \bar{R}$, then*

$$\sum_{K=1}^{n} f_K^2(P_0) \leq c_{17} \log n \ .$$

The theorem is true if P_0 is a point where $f_1, \cdots, f_n$ all vanish, in particular on Γ. Let P_0 be a point in R where not all f_K, $K = 1, \cdots, n$ vanish. Consider the problem of minimizing $D[u]$, where u is of the form $u = \sum_{K=1}^{n} c_K f_K$, under the condition $u(P_0) = T \neq 0$. Now

$$D[u] = D\left[\sum_{K=1}^{n} c_K f_K\right] = \sum_{K=1}^{n} c_K^2 \ ,$$

so that we must minimize $\sum_{K=1}^{n} c_K^2$ under the condition $\sum_{K=1}^{n} c_K f_K(P_0) = T$. By Lagrange multipliers we find a necessary condition for a minimum to be

$$c_K = \bar{c}_K = \frac{T f_K(P_0)}{\sum\limits_{j=1}^{n} f_j^2(P_0)} \ ,$$

and the function $\bar{u} = \sum_{K=1}^{n} \bar{c}_K f_K$ satisfies

$$D[\bar{u}] = T^2 \Big/ \sum_{K=1}^{n} f_K^2(P_0) \ .$$

This is actually a minimum value, for, if $u = \sum_{K=1}^{n} c_K f_K$, then

$$T^2 = [\sum_{K=1}^{n} c_K f_K(P_0)]^2 \leq \sum_{K=1}^{n} c_K^2 \sum_{K=1}^{n} f_K^2(P_0)$$

so

$$\frac{T^2}{\sum\limits_{K=1}^{n} f_K^2(P_0)} \leq \sum\limits_{K=1}^{n} c_K^2 = D[u] \ .$$

Now $\bar{u}$ is of the form $G\,\tau_n$ where τ_n has the degree of f_n and this degree is less than $\sqrt{2n-2}$.

By Lemma 8, we have $D[\bar{u}] \geq c_{12} T^2 / \log \sqrt{2n-2}$.
Hence

$$\frac{T^2}{\sum\limits_{K=1}^{n} f_K^2(P_0)} = D[\bar{u}] \geq \frac{c_{12} T^2}{\log \sqrt{2n-2}} \ ,$$

$$\sum\limits_{K=1}^{n} f_K^2(P_0) \leq \frac{1}{c_{12}} \log \sqrt{2n-2} \ .$$

4. **An Associated Problem.** As in the previous sections, let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k \geq 3$; $G(x, y)$ be a function of class c^k on $R + \Gamma$, vanishing on Γ, positive in R, with $\partial G / \partial \nu \geq \delta > 0$ on Γ; $g(x, y)$ be any function of class c^k on $R + \Gamma$; a variation be a function of class c^k on $R + \Gamma$ vanishing on Γ.

Let

$$D^p[\xi] = \iint_R (a \xi_x^2 + b \xi_y^2 + c \xi^2)^p dx \, dy \ ,$$

where $a > 0$, $b > 0$, $c \geq 0$ on $R + \Gamma$; a, b, and c are bounded and integrable on $\bar{R}$; p is a real number greater than or equal to 1.

Assuming the existence of a function ψ_0, yielding minimum value to $D^p[\psi]$ in the set of admissible functions of class c^k on $R + \Gamma$, which take the value of g on Γ, can we obtain ψ_0 by the Rayleigh Ritz method? This question is answered in the affirmative and an estimate is obtained for the rate of convergence.

Let $\| \xi \| = (D^p[\xi])^{1/2p}$, for ξ in the set of functions of class c^k on $R + \Gamma$. This functional has the properties $\| \xi \| \geq 0$, $\| a \xi \| = |a| \, \| \xi \|$ for real a, $\| \xi + \eta \| \leq \| \xi \| + \| \eta \|$.

The functional $\| \xi \|$ is a true norm in the linear space V of variations. Let H be the Banach space formed by completing V with respect to this norm. As in the proof of Theorem 1, we see that the set of functions $G\tau$, where τ is a polynomial in x and y, is dense in H. Moreover, if ψ is admissible, there exists a sequence of polynomials Q_j of degree at most j such that

$$\| \psi - g - GQ_j \| = D^p[\psi - g - GQ_j]^{1/2p} = O\left(\frac{\theta(j)}{j^{k-2}} \right),$$

where θ depends on $\psi - g$ and $\lim_{j \to \infty} \theta(j) = 0$.

There exists $\inf D^p[\psi] \geq 0$ for admissible ψ. Let τ_j be a polynominal of degree at most j which makes $D^p[g + G\tau_j] \leq D^p[g + GQ_j]$ for all polynomials Q_j of degree at most j.

That such a polynomial τ_j exists can be seen as follows. The class of all functions GQ_j where Q_j is a polynominal of degree at most j is also the linear manifold determined by $f_i = GT_i$, the orthonormal sequence of Theorem 1, whose polynomial factor T_i is of degree at most j. As stated in the introduction, $1 \leq i \leq j\binom{j+1}{2} + j + 1 = \sigma$ so that we may write $GQ_j = \sum_{i=1}^{\sigma} c_i f_i$. Now let Q'_j be any fixed Q_j. We may restrict ourselves to those Q_j such that $D^p[g + GQ_j] \leq D^p[G + GQ'_j]$. For such Q_j we have

$$\| g \| + \| g + GQ'_j \| \geq \| g \| + \| g + GQ_j \| \geq \| GQ_j \|.$$

Since $D[\xi] \leq D^p[\xi]^{1/p} |R|^{1/q}$ where $(1/p) + (1/q) = 1$, $|R| = $ area of R, we find that

$$|R|^{1/q}[\| g \| + \| g + GQ'_j \|]^2 \geq D[GQ_j] = D\left[\sum_{i=1}^{\sigma} c_i f_i \right] = \sum_{i=1}^{\sigma} c_i^2.$$

Thus the permissible c_i lie in a bounded closed set S in σ-dimensional space. Since

$$D^p[g + GQ_j] = D^p\left[g + \sum_{i=1}^{\sigma} c_i f_i \right]$$

is a continuous function of c_i in S, it attains its minimum in S.

Since $D^p[g + G\tau_j]$ is a decreasing function of j, we have

$$\lim_{j \to \infty} \| g + G\tau_j \| \leq \liminf_{j \to \infty} \| g + GQ_j \|.$$

Let ψ be admissible and choose Q_j so that $\lim_{j \to \infty} D^p[\psi - g - GQ_j] = 0$. Then $\| g + GQ_j \| \leq \| \psi \| + \| \psi - g - GQ_j \|$ implies that $\liminf_{j \to \infty} \| g + GQ_j \| \leq \| \psi \|$. It follows that $\lim_{j \to \infty} \| g + G\tau_j \| \leq \| \psi \|$ for every admissible ψ and thus $g + G\tau_j$ is a minimizing sequence.

If $c > 0$ in a set of positive measure in R, the functional $\| \xi \|$ is a true norm in the linear space (c^k) of functions of class c^k on $R + \Gamma$. If $c = 0$, a.e. in R, this is still true provided we identify functions differing by a constant. In either case we will complete the space (c^k) to form a Banach space B.

A set S in a normed linear space is uniformly convex if there exists a continuous monotone increasing function $g(\varepsilon)$, $0 \leq \varepsilon < 1$, with $\lim_{\varepsilon \to \infty} g(\varepsilon) = 0$, such that whenever ξ, η are in S and $\|\xi\| = \|\eta\| = 1$, $\|(\xi + \eta)/2\| \geq 1 - \varepsilon$, then $\|\xi - \eta\| \leq g(\varepsilon)$.

We shall show (c^k) is uniformly convex. It is easily verified that if α, β are ≥ 0, and $p \geq 1$ then

$$3\alpha^p + \beta^p \leq 2|\alpha + \beta|^p + |\alpha - \beta|^p.$$

Apply the inequality to the integrand below, where we assume ϕ and ψ are in (c^k), $\|\phi\| = \|\psi\| = 1$, $\|(\phi + \psi)/2\| \geq 1 - \varepsilon$.

$$\iint 3 \left[a\left(\frac{\phi + \psi}{2}\right)_x^2 + b\left(\frac{\phi + \psi}{2}\right)_y^2 + c\left(\frac{\phi + \psi}{2}\right)^2 \right]^p$$

$$+ \left[a\left(\frac{\phi - \psi}{2}\right)_x^2 + b\left(\frac{\phi - \psi}{2}\right)_y^2 + c\left(\frac{\phi - \psi}{2}\right)^2 \right]^p dx\, dy$$

$$\leq \frac{2}{2^p} \iint [(a\phi_x^2 + b\phi_y^2 + c\phi^2) + (a\psi_x^2 + b\psi_y^2 + c\psi^2)]^p dx\, dy$$

$$+ \iint |a\phi_x\psi_x + b_y\phi_y\psi_y + c\phi\psi|^p dx\, dy$$

$$\leq \frac{2}{2^p} \left\{ \left(\iint (a\phi_x^2 + b\phi_y^2 + c\phi^2)^p dx\, dy \right)^{1/p} + \left(\iint_R (a\psi_x^2 + b\psi_y^2 + \psi c^2)^p dx\, dy \right)^{1/p} \right\}^p$$

$$+ \iint (\sqrt{a\phi_x^2 + b\phi_y^2 + c\phi^2} \sqrt{a\psi_x^2 + b\psi_y^2 + c\psi^2})^p dx\, dy$$

$$\leq 2 + \sqrt{\iint (a\phi_x^2 + b\phi_y^2 + c\phi^2)^p dx\, dy} \sqrt{\iint (a\psi_x^2 + b\psi_y^2 + c\psi^2)^p dx\, dy} = 3$$

Hence

$$\iint \left(a\left(\frac{\phi - \psi}{2}\right)_x^2 + b\left(\frac{\phi - \psi}{2}\right)_y^2 + c\left(\frac{\phi - \psi}{2}\right)^2 \right)^p dx\, dy \leq 3 - 3(1 - \varepsilon)^{2p},$$

and

$$\|\phi - \psi\| = \left[\iint \left(a(\phi - \psi)_x^2 + b(\phi - \psi)_y + c(\phi - \psi)^2 \right)^p dx\, dy \right]^{1/2p}$$

$$\leq 2(3[1 - (1 - \varepsilon)^{2p}])^{1/2p} \leq 2(3^{1/2p})(2p\varepsilon)^{1/2p} = g(\varepsilon)$$

for $\varepsilon < 1$, since the function $y = [1 - (1 - x)^{2p}] - 2px$ vanishes at 0 and is a decreasing function of x for $0 \leq x < 1$.

LEMMA 9. *Let B be a Banach space, Y a set in B with the property that if y_1, y_2 are in Y, then so is $(y_1 + y_2)/2$. Let the linear manifold spanned by Y be a uniformly convex set in B. Let*

$$\rho = \inf_{y \in Y} \| y \| > 0,$$

let y_n be a sequence in Y with

$$\lim_{n \to \infty} \| y_n \| = \rho, \quad \rho_n = \| y_n \|.$$

Then there exists a unique x in B such that $\| x \| = \rho$ and we have

$$\| x - y_n \| \leq \rho g\left(\frac{\rho_n - \rho}{2\rho}\right) + \rho_n - \rho \, ,$$

where $g(\varepsilon)$ is the function in the definition of uniform convexity. If $\rho = \inf_{y \in Y} \| y \| = 0$, and $\lim_{n \to \infty} \| y_n \| = \rho$, then there is a unique x in B such that $\| x \| = \rho$, and we have $\| x - y_n \| = \rho_n - \rho$.

Proof. Let $z_n = y_n / \rho_n$ so that $\| z_n \| = 1$. Write

$$\frac{z_n + z_m}{2} - \frac{1}{\rho}\left(\frac{y_n + y_m}{2}\right) = \frac{y_n}{2}\left(\frac{1}{\rho_n} - \frac{1}{\rho}\right) + \frac{y_m}{2}\left(\frac{1}{\rho_m} - \frac{1}{\rho}\right).$$

$$\left\| \frac{z_n + z_m}{2} \right\| \geq \frac{1}{\rho}\left\| \frac{y_n + y_m}{2} \right\| - \frac{\| y_n \|}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_n}\right) - \frac{\| y_m \|}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_m}\right)$$

$$\geq \frac{1}{\rho} \cdot \rho - \frac{\rho_n}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_n}\right) - \frac{\rho_m}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_m}\right) = 1 - \frac{(\rho_n - \rho) + (\rho_m - \rho)}{2\rho} \, .$$

Hence

$$\| z_n - z_m \| \leq g\left(\frac{(\rho_n - \rho) + (\rho_m - \rho)}{2\rho}\right) \quad \text{for} \quad \frac{\rho_n - \rho + \rho_m - \rho}{2\rho} < 1 .$$

Thus there exists $z = \lim_{n \to \infty} z_n$ in B. Let $x = \rho z = \lim_{n \to \infty} \rho z_n = \lim_{n \to \infty} \rho_n z_n = \lim_{n \to \infty} y_n$. Then $\| x \| = \lim_{n \to \infty} \| y_n \| = \rho$. Also $\| z_n - z \| \leq g((\rho_n - \rho)/2\rho)$ implies

$$\| x - y_n \| = \| \rho z - \rho_n z_n \| \leq \| \rho z - \rho z_n \| + \| \rho z_n - \rho_n z_n \| \leq \rho g\left(\frac{\rho_n - \rho}{2\rho}\right) + \rho_n - \rho \, .$$

To show x is unique, suppose also $y'_n \in Y$, $\lim_{n \to \infty} \| y'_n \| = \rho$, $x' \in B$, $\| x' \| = \rho$, $x' = \lim_{n \to \infty} y'_n$. Then form the sequence $\{y''_n\} = y_1, y'_1, y_2, y'_2,$ etc. of elements of Y with $\lim_{n \to \infty} y''_n = \rho$. As above, $\exists x'' \in B$ with $x'' = \lim_{n \to \infty} y''_n = \lim_{n \to \infty} y'_n = \lim_{n \to \infty} y_n$. The last part of the lemma is obvious, since only $\| 0 \| = 0$.

To apply the lemma, let B be the completion of (c^k), Y the set of admissible functions,

$$y_n = g + G\tau_n, \quad \rho = \inf D^p[\psi]^{1/2p} \, ,$$

for admissible ψ. By the lemma, there is a unique x such that $\| x \| =$

ρ. Assuming that $x = \psi_0$ is in Y, we can choose polynomials Q_j of degree at most j such that

$$\| \psi_0 - g - GQ_j \| = O\left(\frac{\theta(j)}{j^{k-2}}\right) .$$

Then

$$\rho_j - \rho = \| g + G\tau_j \| - \| \psi_0 \| \leq \| g + GQ_j \| - \| \psi_0 \|$$

$$= \| g + GQ_j - \psi_0 + \psi_0 \| - \| \psi_0 \| \leq \| \psi_0 - g - GQ_j \| = O\left(\frac{\theta(j)}{j^{k-2}}\right) .$$

By the lemma,

$$\| \psi_0 - g - G\tau_j \| \leq 2\rho(6p)^{1/2p}\left(\frac{\rho_j - \rho}{2\rho}\right)^{1/2p} + \rho_j - \rho = O\left(\frac{\theta(j)}{j^{k-2}}\right)^{1/2p} ,$$

a better result, $O(\theta(j)/j^{k-2})$, is obtained in the case $\rho = 0$.

Since

$$D[u] < (D^p[u])^{1/p} |R|^{1/q} ,$$

where $|R|$ is the area of R and $(1/p) + (1/q) = 1$, we find

$$D[u_j] \leq \left(\frac{\theta(j)}{j^{k-2}}\right)^{1/p} ,$$

where $\lim_{j \to \infty} \theta(j) = 0$, when we take $u_j = \psi_0 - g - G\tau_j$. A proof similar to that of Theorem 3 can now be constructed for the following result.

THEOREM 5. *Let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k \geq 3$. Let $G(x, y)$ be a function of class c^k on $R + \Gamma$, vanishing on Γ, positive in R, with $\partial G / \partial \nu \geq \delta > 0$ on Γ. Let a, b, c be bounded and integrable on $\overline{R}$, and $a > 0$, $b > 0$, $c \geq 0$ on $\overline{R}$. Let $g(x, y)$ be any function of class c^k on $R + \Gamma$. Choose polynomials τ_j minimizing $D^p[g + GQ_j]$ in the set of all polynomials Q_j of degree at most j. Then, if ψ_0 yields minimum value to $D^p[\psi]$ for ψ in the set of functions of class c^k on $R + \Gamma$ assuming the values of g on Γ, we have*

$$|\psi_0 - g - G\tau_j| = O\left(\sqrt{\left(\frac{\sqrt{\theta(j)}}{j^{k-2}}\right)^{1/p}} \log \frac{j}{(\log j)^N}\right) ,$$

where N is any fixed positive constant, $\theta(j)$ depends on $\psi_0 - g$ and $\lim_{j \to \infty} \theta(j) = 0$.

If $k \geq 16p + 2$, *then*

$$|\nabla \psi_0 - \nabla(g + G\tau_j)| = O\left(\left[\frac{\theta(j)}{j^{k-2-16p}}\right]^{1/4p}\right).$$

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