ON A THEOREM OF L. LICHTENSTEIN

S. E. WARSCHAWSKI
ON A THEOREM OF L. LICHTENSTEIN

S. E. WARSCHAWSKI

1. Introduction. The object of this note is the proof of the following:

**Theorem.** Let $C$ be a closed Jordan curve in the $z$-plane which possesses a corner of opening $\pi \alpha$, $0 < \alpha \leq 2$ at $z = 0$. Suppose that this corner is formed by two regular analytic arcs $\gamma_a$ and $\gamma_b$:

$$\gamma_a : z = A(t) = \sum_{\nu = 1}^\infty a_{\nu} t^\nu; \quad \gamma_b : z = B(t) = \sum_{\nu = 1}^\infty b_{\nu} t^\nu, \quad 0 \leq t \leq 1, \quad a_{\nu} \neq 0, \quad b_{\nu} \neq 0.$$

If $\zeta = f(z)$ maps the interior $\Delta$ of $C$ conformally onto the half-plane $\Im [\zeta] > 0$ so that $f(0) = 0$, then, for every integer $n$,

$$\lim_{z \to 0} \left\{ z^{n-1/\alpha} \frac{d^n f(z)}{dz^n} \right\} = c \frac{1}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \cdots \left( \frac{1}{\alpha} - n + 1 \right),$$

for unrestricted approach, where $c = \lim_{z \to 0} [f(z) z^{-1/\alpha}]$.

This theorem was stated by L. Lichtenstein [2] and [3], but proved only for the case that $\alpha$ is irrational. He remarks, however, that it is most likely true for all $\alpha$, $0 < \alpha \leq 2$, but that his proof does not yield this result. In the following a simple proof based on a different approach is given for the complete theorem.

2. Lemmas. In the proof of theorem we shall make use of the following two lemmas.

**Lemma 1.** Suppose $\Gamma$ is a closed Jordan curve with a corner at $z = 0$ of opening $\pi \alpha$, $0 < \alpha \leq 2$, and that each of the two arcs forming the corner has bounded curvature in the neighborhood of $z = 0$. If $w = g(z)$ maps the interior $D$ of $\Gamma$ conformally onto the angle $0 < \arg w < \pi \alpha$, so that $g(0) = 0$, then for non-tangential approach,

$$\lim_{z \to 0} \frac{g(z)}{z} = \mu \quad \text{exists and} \quad \mu \neq 0.$$

This is just a weaker statement of a well known result [4, 5]; (2) holds under more general assumptions on the arcs which form the corner.

---

Received July 29, 1954. Prepared under contract Nonr 396 (00) (NR 044 004) between the Office of Naval Research and the University of Minnesota.

This note is the result of an inquiry from Dr. George Forsythe of the Institute of Numerical Analysis regarding the validity of Lichtenstein's theorem for all $\alpha$. Dr. Forsythe applies this result in his preceding paper on "Asymptotic lower bounds for the fundamental frequency of convex membranes."
and for unrestricted approach [5, p. 427]. However, for the sake of completeness we give an elementary proof of this lemma in § 4.

**Lemma 2.** Suppose that $F(w)$ is analytic in an angle $A$: $\alpha < \arg w < \beta$, $\beta - \alpha \leq 2\pi$, and that in every sub-angle $B$ of $A$ with the vertex at $w=0$, $\lim_{w \to 0} F(w) = \mu$. Then for any integer $n \geq 1$, as $w \to 0$ in any sub-angle $B$ of $A$

\[ \lim_{w \to 0} [w^{n-1}F^{(n)}(w)] = \begin{cases} \mu & \text{when } n=1 \\ 0 & \text{when } n>1 \end{cases}. \]

**Proof.** Let $B$ be the angle $\alpha + \delta < \arg w < \beta - \delta$, $0 < 2\delta < \beta - \alpha$. About $w \in B$ we describe a circle $c$ of radius $r$ which is contained in and tangent to a side of the angle $\alpha + \frac{\delta}{2} \leq \arg w \leq \arg \beta - \frac{\delta}{2}$. Clearly,

\[ \frac{r}{|w|} \geq \frac{\sin \frac{\delta}{2}}{2}. \]

We set $G(w) = F(w) - \mu w$. Then

\[ w^{n-1}G^{(n)}(w) = \frac{n!}{2\pi i} \int_{c} G(t)w^{n-1} \frac{dt}{t-w} = \frac{n!}{2\pi i} \int_{c} G(t) \frac{tw^{n-1}}{(t-w)^{n+1}} dt. \]

Since $|t| \leq |t-w| + |w|$ and $|t-w| = r$ for $t$ on $c$, we have

\[ |w^{n-1}G^{(n)}(w)| \leq \frac{n!}{2\pi} \int_{c} \left| \frac{G(t)}{t} \frac{|w|^{n-1}(r+|w|)}{r^{n+1}} \right| dt \]

and the last expression approaches 0 as $w \to 0$ in $B$. This proves (3).

3. **Proof of the theorem.** (i) We may and shall assume in the following that $C$ consists of two regular analytic arcs $\hat{OA}$ and $\hat{OB}$ and a circular arc $\gamma$ about $O$ through $A$ and $B$. (The size of the radius $r$ of this arc will be restricted below). For, if $D$ is a subregion of $\Delta$ bounded by the just described curves, and if $f(z)$ maps $D$ onto the upper half plane such that $f(0)=0$, then $f(z)=h[f_1(z)]$ for $z \in D$, where $h(\zeta)$ is an analytic function in a neighborhood of $\zeta=0$ and $h'(0)=\infty$. The result (1) on $f^{(n)}(z)$ follows then from that on $f_1^{(n)}(z)$.

The theorem will be proved by the following statement: if $w=g(z)$ maps $\Delta$ onto the angle $0 < \arg w < \pi \alpha$ such that $z=0$ corresponds to $w=0$, then, for unrestricted approach,

\[ \lim_{z \to 0} g'(z)=\lambda, \ 0<|\lambda|<\infty, \text{ and } \lim_{z \to 0} [z^{n-1}g^{(n)}(z)]=0, \text{ for } n>1. \]

The result (1) of the theorem is then obtained from (4) by use of the
fact that $f(z) = [g(z)]^{1/\alpha}$.

For the proof of (4) we may presuppose that $0 < \alpha < 1$; for if $1 < \alpha \leq 2$ we apply first the auxiliary transformation $z' = z^{1/\alpha}$. For $|t| \leq \delta$, where $\delta > 0$ is sufficiently small, $\widehat{OA}$ and $\widehat{OB}$ are transformed into regular analytic arcs in $\tau = t^{1/\alpha}$. We assume $r$ so small that $\widehat{OA}$ and $\widehat{OB}$ are obtained for values of the parameter $t \leq \delta$.

We now impose a further restriction on $\delta$ and thus on $r$. There exists a $\rho > 0$ such that $z = A(t)$ and $z = B(t)$ have analytic and univalent inverse functions $t = a(z)$ and $t = b(z)$ in $|z| \leq \rho$. We take $\delta$ so small that $\widehat{OA}$ and $\widehat{OB}$ are contained in $|z| < \rho$. Thus, $r < \rho$.

(ii) Consider the maps of $\Delta$ by means of $t = a(z)$: $\widehat{OA}$ is transformed into a segment $\widehat{O_1A_1}$ of the real $t$-axis and $\widehat{OB}$ into an arc $\widehat{O_2B_1}$ which makes an angle of opening $\pi \alpha$ with $O_1A_1$. The circular arc $\gamma: \widehat{AB}$ is mapped onto an arc $\widehat{A_2B_2}$. If $r$ is sufficiently small, the arcs $\widehat{O_2B_1}$ and $\widehat{A_2B_2}$ will lie in the upper half of the $t$-plane. We assume that $r$ has been so chosen (third and final restriction on $r$). Let $\Delta$ denote the image of $\Delta$ in the $t$-plane.

Suppose that $w = \phi(t)$ maps $\Delta$ onto the angle $0 < \arg w < \pi \alpha$ such that $t = 0$ corresponds to $w = 0$ and $A_1$ to $w = \infty$. The segment $O_1A_1$ is then transformed into the positive real axis of the $w$-plane. We reflect the arc $\widehat{O_2B_1}$ with respect to the positive real axis and denote the image of $B_1$ by $B'_1$. By Schwarz's reflection principle the function $w = \phi(t)$ maps the region bounded by the Jordan curve $\Gamma: O_1B_1A_2B'_1O_1$ conformally onto the angle $-\pi \alpha < \arg w < \pi \alpha$.

We apply now Lemma 1 to the curve $\Gamma$, which has a corner of opening $2\alpha \pi$, $0 < 2\alpha < 2$, at $t = 0$, formed by the regular analytic arcs $\widehat{O_1B_1}$ and $\widehat{O_2B'_1}$. Hence, for non-tangential approach,

$$\lim_{t \to 0} \frac{\phi(t)}{t} = \mu,$$

exists and $0 < |\mu| < \infty$. Next, observing that the mapping $w = \phi(t)$ preserves angles at $t = 0$ and applying Lemma 2 to the inverse $F(w)$ of $\phi(t)$ we find that in any angle $-\pi \alpha + \varepsilon < \arg w < \pi \alpha - \varepsilon$ ($0 < \varepsilon < \pi \alpha$):

$$\lim_{w \to 0} F'(w) = \mu, \quad \lim_{w \to 0} [w^{n-1}F^{(n-1)}(w)] = 0, \quad \text{for } n > 1.$$

Hence, in any sector $|\arg t| \leq \pi \beta$, $|t| \leq \eta$, where $0 < \beta < \alpha$ and $\eta$ is sufficiently small,

$$\lim_{t \to 0} \phi'(t) = \frac{1}{\mu}, \quad \lim_{t \to 0} [t^{n-1}\phi^{(n)}(t)] = 0, \quad \text{for } n > 1.$$

We assume here that $O, A, B$ follow in counter-clockwise order along $C$. 


Since $\phi[a(z)]=g(z)$, it follows from (5) that, for $\mu = \frac{a'(0)}{\mu_1}$,

\[
\lim_{z \to 0} g'(z) = \lambda \quad \text{and} \quad \lim_{z \to 0} [z^n - g^{(n-1)}(z)] = 0, \quad \text{for } n > 1,
\]

in any curvilinear angle in $C + \Delta$ formed by $OA$ and any Jordan arc $j$ in $\Delta$ which has a tangent at $O$ making the angle $\pi \beta$ with the tangent to $OA$ at $O$.

(iii) By applying the same argument in which the arc $\widehat{OB}$ takes the role of $\widehat{OA}$ we find that (6) holds in any curvilinear angle formed by $\widehat{OB}$ and any Jordan arc $j'$ in $\Delta$ which has a tangent at $O$ making an angle $\pi \beta$ with the tangent to $OB$ at $O$. Since $\beta$ may be taken so that the two curvilinear angles overlap, we obtain (4), and this completes the proof.

4. Proof of Lemma 1. We can construct a Jordan curve $I_i'$ contained in $D + I'$ and one $I_e'$ exterior to $D$, each consisting of two circular arcs intersecting at the angle $\pi \alpha$ at $z=0$ (and at another point). The interior $I(I_i')$ of $I_i'$ is in $D$, and we may assume that the exterior $E(I_e')$ contains $D$. If $h_i(z)$ and $h_e(z)$ are the bilinear transformations which map $I(I_i')$ and $E(I_e')$, respectively, onto the angle $0 < \arg \omega < \pi \alpha$, such that $h_i(0) = h_e(0) = 0$, then clearly

\[
\lim_{z \to 0} \frac{h_i(z)}{z} = \lambda_i \quad \text{and} \quad \lim_{z \to 0} \frac{h_e(z)}{z} = \lambda_e
\]

exist for unrestricted approach, $0 < |\lambda_i|, |\lambda_e| < \infty$. The function $\zeta = h_e^{-1}(z)$ maps $E(I_e')$ onto $\mathcal{S}[\zeta] > 0$, $I'$ and $I_e'$ onto closed curves $I_i^*$ and $I_e^*$, respectively, which lie in $\mathcal{S}[\zeta] > 0$ and are tangent to the real axis at $\zeta = 0$. Let $\phi(\zeta)$ and $\phi_i(\zeta)$ map the interiors of $I_i^*$ and $I_i^*$, respectively, onto the upper half plane, so that $\phi(0) = \phi_i(0) = 0$ and, for a point $\zeta_0$ interior to $I_i^*$, $\phi(\zeta_0) = \phi_i(\zeta_0)$. An application of the Wolff-Carathéodory-Landau-Valiron lemma [1, 5] shows that

\[
\lim_{\zeta \to 0} \frac{\phi(\zeta)}{\phi_i(\zeta)} = l, \quad 0 \leq l < \infty,
\]

exists for non-tangential approach. Since

\[
\phi_i(\zeta) = h_i^{-1}[h_e^{-1}(\zeta^*)],
\]

where $h_e^{-1}$ denotes the inverse of $h_e$, it follows that

\[
\phi(\zeta) = h_i^{-1}[h_e^{-1}(\zeta^*)] \left\{ h_e^{-1}(\zeta^*) \right\}^{-1/\alpha} \rightarrow \left\{ \lambda_i \right\}^{1/\alpha}
\]

for unrestricted approach. Hence, $l \geq \left\{ \lambda_i \lambda_e^{-1} \right\}^{1/\alpha} > 0$. 
Finally, we note that
\[ g(z) = \{ \phi[h^{1/\gamma}(z)] \}^* \]
and hence
\[ \lim_{z \to 0} g(z) = l^* \lambda \]
for non-tangential approach. This proves the lemma\(^3\).

REFERENCES


University of Minnesota

\(^3\)Another proof of Lichtenstein's theorem may be obtained from an asymptotic expansion due to R. S. Lehman, *Development of the mapping function at an analytic corner*, Stanford, Applied Math. Tech. Rep. No. 21, 1954. This would seem, however, more complicated than the proof given here. The author became aware of Lehman's work only after the present note was submitted for publication.
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, Alfred Horn at the University of California Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50; back numbers (Volumes 1, 2, 3) are available at $2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.) No. 10 1-chome Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

UNIVERSITY OF CALIFORNIA PRESS . BERKELEY AND LOS ANGELES
Henry A. Antosiewicz, *A theorem on alternatives for pairs of matrices* 641
F. V. Atkinson, *On second-order non-linear oscillation* 643
Frank Herbert Brownell, III, *Fourier analysis and differentiation over real separable Hilbert space* 649
Richard Eliot Chamberlin, *Remark on the averages of real functions* 663
Philip J. Davis, *On a problem in the theory of mechanical quadrature* 669
Douglas Derry, *On closed differentiable curves of order n in n-space* 675
Edwin E. Floyd, *Boolean algebras with pathological order topology* 687
George E. Forsythe, *Asymptotic lower bounds for the fundamental frequency of convex membrane* 691
Israel Halperin, *On the Darboux property* 703
Theodore Edward Harris, *On chains of infinite order* 707
Peter K. Henrici, *On certain series expansions involving Whittaker functions and Jacobi polynomial* 725
John G. Herriot, *The solution of Cauchy's problem for a third-order linear hyperbolic differential equation by means of Riesz integral* 745
Jack Indritz, *Applications of the Rayleigh Ritz method to variational problem* 765
E. E. Jones, *The flexure of a non-uniform beam* 799
Hukukane Nikaidô and Kazuo Isoda, *Note on non-cooperative convex game* 807
Raymond Moos Redheffer and W. Wasow, *On the convergence of asymptotic solutions of linear differential equation* 817
S. E. Warschawski, *On a theorem of L. Lichtenstei* 835
Philip Wolfe, *The strict determinateness of certain infinite game* 841