

Pacific Journal of Mathematics

**THE STRICT DETERMINATENESS OF CERTAIN INFINITE
GAME**

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1. **Introduction.** Gale and Stewart [1] have discussed an infinite two-person game in extensive form which is the generalization of a game as defined by Kuhn [3] obtained by deleting the requirement of finiteness of the game tree and regarding as plays all unicursal paths of maximal length originating in the distinguished vertex x_0 . In a *win-lose* game the set S of all plays is divided into two sets S_I and S_{II} such that player I wins the play s if $s \in S_I$ and player II wins it if $s \in S_{II}$. Gale and Stewart have shown that a two-person infinite win-lose game of perfect information with no chance moves (called a GS game here) is strictly determined if S_I belongs to the smallest Boolean algebra containing the open sets of a certain topology for S . Here we answer affirmatively the question posed by them: Is a GS game strictly determined if S_I is a G_δ (or, equivalently, an F_σ)? The notation and results of [1] are used throughout, as well as the partial ordering of X given by: $x > y$ if $f^n(x) = y$ for some $n \geq 1$.

2. **Alternative description of S_I .** Let I' be the game $(x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$, where

$$S_I = \bigcap_{n=1}^{\infty} E_n,$$

$E_1 \supseteq E_2 \supseteq \dots$, and E_n is open. Following [3], let the rank $rk(x)$, for $x \in X$, be the unique k such that $f^k(x) = x_0$. As in [1], $\mathfrak{U}(x)$ is the set of all plays passing through x (the topology for S is that in which $\mathfrak{U}(x)$ is a neighborhood of each play in it). Then for each n ,

$$E_n = \bigcup \{ \mathfrak{U}(y) : \mathfrak{U}(y) \subseteq E_n \};$$

and since for any $y \in X$ we have

$$\mathfrak{U}(y) = \bigcup \{ \mathfrak{U}(z) : f(z) = y \},$$

with

$$rk(z) = 1 + rk(y),$$

Received October 3, 1953. The work in this paper was done during the author's tenure of an Atomic Energy Commission Predoctoral Fellowship.

there exists for each n a subset Y_n of X such that $rk(y) > n$ for all $y \in Y_n$ and

$$E_n = \bigcup \{U(y) : y \in Y_n\} .$$

Furthermore, since of any two neighborhoods having a non-void intersection, one is contained in the other, each Y_n may be chosen so that $U(y), U(y')$ are disjoint for different y, y' in Y_n .

Since $s \in S_i$ if and only if $s \in E_n$ for an infinite number of values of n , we have: $s \in S_i$ if and only if for infinitely many n there exists i (dependent on n) such that $s(i) \in Y_n$. Thus, since on the one hand $i = rk(s(i)) > n$, and on the other for any n there is at most one i such that $s(i) \in Y_n$, letting

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

we have: $s \in S_i$ if and only if $s(i) \in Y$ for infinitely many i .

3. Lemmas.

LEMMA 1. *If Γ is a GS game with*

$$\sum_{II}^W(\Gamma) = A$$

and

$$T = S - \bigcup \{U(x) : \sum_{II}^W(\Gamma_x) \neq A\} ,$$

then

$$\Gamma_T = (x_0, X_T^I, X_{II}^T, X^T, f^T, T, S_T^I, S_{II}^T)$$

is a subgame of Γ ,

$$\sum_{II}^W(\Gamma_T) = A$$

implies

$$\sum_{II}^W(\Gamma) = A ,$$

and

$$\sum_{II}^W((\Gamma_T)_x) = A$$

for all $x \in X^T$.

Proof. Since T is a closed nonempty subset of S , Γ_T is a subgame of Γ by Theorem 5 of [1]. The second statement follows from assertion B [1, p. 260]. Finally suppose that

$$\sum_{II}^W((\Gamma_T)_x) \neq A$$

for some $x \in X^T$. Letting, in assertion A [1, p. 260],

$$F = U(x) \cap T ,$$

and noting that F is closed and nonempty and that

$$(I'_T)_x = (I'_x)_F,$$

we have

$$\sum_{II}^W(I'_x) \neq A,$$

which is impossible in view of the construction of T .

We assume hereafter that I' is a GS game with S_I described in terms of $Y \subseteq X$ as in § 2, and that

$$\sum_{II}^W(I') = A,$$

whence

$$\sum_{II}^W(I'_T) = A$$

by Lemma 1. The strict determinateness of I' will follow from Lemma 1 and the fact that

$$\sum_{II}^W(I'_T) \neq A,$$

proved in § 4.

LEMMA 2. For $x \in X^T$, we have

$$s \in S_I^{Tx}$$

if and only if

$$s \in S^{Tx} \text{ and } s(i) \in Y$$

for infinitely many i .

LEMMA 3. For $x \in X^T$ there exists

$$\sigma_x \in \sum_{II}((I'_T)_x)$$

such that for any

$$\tau \in \sum_{II}((I'_T)_x)$$

we have

$$\langle \sigma_x, \tau \rangle(i) \in Y$$

for some $i > rk(x)$.

Proof. Let Y_x be the set of all

$$y \in Y \cap X^T$$

such that $y > x$ and no members of Y fall between x and y . Let I' be the game

$$(x_0, X_I^{Tx}, X_{II}^{Tx}, X^{Tx}, f^{Tx}, S^{Tx}, S'_I, S'_{II}),$$

where

$$S'_I = S^{Tx} \cap \bigcup \{U(y) : y \in Y_x\}$$

and

$$S'_{II} = S^{Tx} - S'_I$$

(that is, the game in which I wins if the play passes through any member of Y following x). Noting that

$$S_I^{Tx} \subseteq S'_I,$$

we have

$$S'_{II} \subseteq S_{II}^{Tx}$$

and hence

$$\sum_{II}^W(I') = A.$$

But S'_I is open in S^{Tx} and so I' is strictly determined by Corollary 10 of [1], whence there exists

$$\sigma_x \in \sum_{II}^W(I'),$$

which satisfies the conclusion of the lemma.

4. Winning I' . Let

$$Y' = (Y \cap X^T) \cup \{x_0\}.$$

For each $x \in Y'$ let σ_x be as given by Lemma 3, and let σ'_x be the restriction of σ_x to the set of all z in X^T such that $x \leq z$ and that there exists no y in Y' with $x < y \leq z$. We show that the domains of the σ'_x cover X^T and are disjoint: First, if $x_0 \in X_I^T$, then x_0 belongs to the domain of σ_{x_0} . For

$$z \in X_I^T - \{x_0\},$$

let

$$x = \max\{z' : z' \in Y' \ \& \ z' < z\}.$$

Then $x \in Y'$ and z belongs to the domain of σ'_x ; thus the domains of the σ'_x cover X_I^T . Now suppose that $x_1, x_2 \in Y'$, $x_1 \neq x_2$, and that there exists x_3 common to the domains of σ'_{x_1} and σ'_{x_2} ; then $x_1 \leq x_3$ and $x_2 \leq x_3$, so that either $x_1 < x_2 \leq x_3$ or $x_2 < x_1 \leq x_3$, which is impossible in view of the restriction imposed upon σ_x in obtaining σ'_x .

Since the domains of the σ'_x cover X_I^T and are disjoint, they have

a common extension σ^* , which necessarily maps the elements of X_I^T on their immediate successors, and thus belongs to $\sum_I(I^T)$.

We show that σ^* wins I^T . Let

$$\tau \in \sum_{II}(I^T) .$$

For this τ and any x in Y' , let $i(x)$ be the least i such that $\langle \sigma_x, \tau \rangle(i) \in Y'$, whose existence is given by Lemma 3. Define $\{x_n\}$ inductively by

$$x_{n+1} = \langle \sigma^*, \tau \rangle(i(x_n)) \quad n=0, 1, \dots$$

(x_0 is the distinguished vertex). Since

$$rk(x_{n+1}) = i(x_n) > rk(x_n) ,$$

and x_n, x_{n+1} are on a common path, we have $x_{n+1} > x_n$ for all n , and so if $x_n \in Y'$ then

$$x_{n+1} = \langle \sigma^*, \tau \rangle(i(x_n)) = \langle \sigma_{x_n}, \tau_{x_n} \rangle(i(x_n)) \in Y' ,$$

where

$$\tau_{x_n} \in \sum_{II}((I^T)_{x_n})$$

is the restriction of τ to $X_{II}^{T_{x_n}}$. Thus by induction $x_n \in Y'$ for all n , and hence

$$\langle \sigma^*, \tau \rangle(i) \in Y$$

for infinitely many values of i , so that

$$\langle \sigma^*, \tau \rangle \in S_I^T .$$

Since τ is arbitrary,

$$\sigma^* \in \sum_I^W(I^T) ,$$

so that by Lemma 1, we have

$$\sum_I^W(I) \cong A .$$

As this is the consequence of the sole fact that

$$\sum_{II}^W(I) = A ,$$

I is strictly determined.

Reversing the roles of the players in the above gives the result that a GS game is strictly determined if S_I is an F_σ .

The strict determinateness of a two-person zero-sum game with G payoff having *chance moves* can be shown. The proof is more complicated, but uses the same ideas [4].

5. An application. Let

$$I = (x_0, X_I, X_{II}, X, f, S, \phi)$$

be a zero-sum two-person infinite game of perfect information with no chance moves having payoff ϕ such that there exists a real function h on X ($|h(x)| < K < \infty$) with

$$\phi(s) = \limsup_{i \rightarrow \infty} h(s(i)) \quad \text{for all } s \in S.$$

Γ is the result of an attempt to reduce the following situation to a game: The tree K of a GS game and a function h as above are given; the two players make choices in K in the belief that every play will terminate in some unknown, but distant, vertex x , at which time player I will receive the amount $h(x)$ from player II . A payoff function ϕ is sought such that $\phi(s)$ ($-\phi(s)$) expresses the utility to player I (II) of a play s in K .

The payoff ϕ defined above arises from ascription to players I and II respectively of "optimistic" and "pessimistic" behaviors in this way: Player I assumes that the play s will terminate in some "distant" vertex $s(i)$ at which h assumes nearly its supremum on all "distant" vertices of s ; he thus makes his choices so as to maximize the expression

$$\limsup_{i \rightarrow \infty} h(s(i)) = \phi(s);$$

and player II supposes that s will terminate in some "distant" vertex at which his gain $-h(s(i))$ assumes nearly its infimum for all such vertices, and thus seeks to maximize

$$\liminf_{i \rightarrow \infty} -h(s(i)) = -\phi(s),$$

that is, to minimize ϕ . The derived game is thus zero-sum. Ascription, however, of such "optimistic" or "pessimistic" payoffs to both players yields, in general, a non-zero sum game.

We show now that the game Γ of this section is strictly determined, using the method of Theorem 15 of [1] which asserts the strict determinateness of Γ for the more special case of continuous ϕ . (Gillette [2] has shown the strict determinateness of an infinite game of perfect information with chance moves which consists in repeated play from a finite set of finite games and has payoff

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n g_n(s),$$

where $g_n(s)$ is the gain from the n th game played.)

First, as a converse to the equivalence of § 2, let $Y \subseteq X$, and denote by Y_n the set of all members of Y having rank greater than n . Then

$$\begin{aligned} \{s : s(i) \in Y \text{ for infinitely many } i\} &= \bigcap_n \{s : s(i) \in Y_n \text{ for some } i\} \\ &= \bigcap_n \bigcup \{11(y) ; y \in Y_n\}, \end{aligned}$$

which is a G_δ .

Now in I' , for t real, let

$$S_t^i = \{s : h(s(i)) > t \text{ for infinitely many } i\},$$

and $S_{II}^t = S - S_t^i$. Then S_t^i is a G_δ , and thus the GS game

$$I_t = (x_0, X_I, X_{II}, X, f, S, S_t^i, S_{II}^t)$$

is strictly determined. Let

$$v = \sup \{t : \sum_I^W(I_t) \approx A\}.$$

Since $S_t^k = A$, $S_t^{-k} = S$, and S_t^i is a decreasing function of t , we have

$$-K \leq v \leq K, \quad \sum_I^W(I_t) \approx A \quad \text{if } t < v,$$

and

$$\sum_{II}^W(I_t) \approx A \quad \text{if } t > v.$$

Given $\varepsilon > 0$, choose

$$\sigma_0 \in \sum_I^W(I_{v-\varepsilon}) \quad \text{and} \quad \tau_0 \in \sum_{II}^W(I_{v+\varepsilon}).$$

Then for any

$$\sigma \in \sum_I(I), \quad \tau \in \sum_{II}(I),$$

we have

$$h(\langle \sigma, \tau \rangle(i)) > v - \varepsilon \quad \text{for infinitely many } i$$

and do not have

$$h(\langle \sigma, \tau_0 \rangle(i)) > v + \varepsilon \quad \text{for infinitely many } i;$$

so that

$$\Phi(\langle \sigma_0, \tau \rangle) \geq v - \varepsilon \quad \text{and} \quad \Phi(\langle \sigma, \tau_0 \rangle) < v + 2\varepsilon.$$

Hence

$$v - \varepsilon \leq \sup_\sigma \inf_\tau \Phi(\langle \sigma, \tau \rangle) \leq \inf_\tau \sup_\sigma \Phi(\langle \sigma, \tau \rangle) \leq v + 2\varepsilon;$$

thus I' is strictly determined, and has value v .

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50; back numbers (Volumes 1, 2, 3) are available at \$2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.) No. 10 1-chome Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

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