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THE STRICT DETERMINATENESS OF CERTAIN INFINITE GAME

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1. Introduction. Gale and Stewart [1] have discussed an infinite two-person game in extensive form which is the generalization of a game as defined by Kuhn [3] obtained by deleting the requirement of finiteness of the game tree and regarding as plays all unicursal paths of maximal length originating in the distinguished vertex x_0 . In a *win-lose* game the set S of all plays is divided into two sets S_I and S_{II} such that player I wins the play s if $s \in S_I$ and player II wins it if $s \in S_{II}$. Gale and Stewart have shown that a two-person infinite win-lose game of perfect information with no chance moves (called a GS game here) is strictly determined if S_I belongs to the smallest Boolean algebra containing the open sets of a certain topology for S . Here we answer affirmatively the question posed by them: Is a GS game strictly determined if S_I is a G_δ (or, equivalently, an F_σ)? The notation and results of [1] are used throughout, as well as the partial ordering of X given by: $x > y$ if $f^n(x) = y$ for some $n \geq 1$.

2. Alternative description of S_I . Let Γ be the game $(x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$, where

$$S_I = \bigcap_{n=1}^{\infty} E_n,$$

$E_1 \supseteq E_2 \supseteq \dots$, and E_n is open. Following [3], let the rank $rk(x)$, for $x \in X$, be the unique k such that $f^k(x) = x_0$. As in [1], $\mathfrak{U}(x)$ is the set of all plays passing through x (the topology for S is that in which $\mathfrak{U}(x)$ is a neighborhood of each play in it). Then for each n ,

$$E_n = \bigcup \{ \mathfrak{U}(y) : \mathfrak{U}(y) \subseteq E_n \};$$

and since for any $y \in X$ we have

$$\mathfrak{U}(y) = \bigcup \{ \mathfrak{U}(z) : f(z) = y \},$$

with

$$rk(z) = 1 + rk(y),$$

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there exists for each n a subset Y_n of X such that $rk(y) > n$ for all $y \in Y_n$ and

$$E_n = \bigcup \{U(y) : y \in Y_n\} .$$

Furthermore, since of any two neighborhoods having a non-void intersection, one is contained in the other, each Y_n may be chosen so that $U(y), U(y')$ are disjoint for different y, y' in Y_n .

Since $s \in S_i$ if and only if $s \in E_n$ for an infinite number of values of n , we have: $s \in S_i$ if and only if for infinitely many n there exists i (dependent on n) such that $s(i) \in Y_n$. Thus, since on the one hand $i = rk(s(i)) > n$, and on the other for any n there is at most one i such that $s(i) \in Y_n$, letting

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

we have: $s \in S_i$ if and only if $s(i) \in Y$ for infinitely many i .

3. Lemmas.

LEMMA 1. *If Γ is a GS game with*

$$\sum_{II}^W(\Gamma) = A$$

and

$$T = S - \bigcup \{U(x) : \sum_{II}^W(\Gamma_x) \neq A\} ,$$

then

$$\Gamma_T = (x_0, X_T^I, X_{II}^T, X^T, f^T, T, S_T^I, S_{II}^T)$$

is a subgame of Γ ,

$$\sum_{II}^W(\Gamma_T) \neq A$$

implies

$$\sum_{II}^W(\Gamma) \neq A ,$$

and

$$\sum_{II}^W((\Gamma_T)_x) = A$$

for all $x \in X^T$.

Proof. Since T is a closed nonempty subset of S , Γ_T is a subgame of Γ by Theorem 5 of [1]. The second statement follows from assertion B [1, p. 260]. Finally suppose that

$$\sum_{II}^W((\Gamma_T)_x) \neq A$$

for some $x \in X^T$. Letting, in assertion A [1, p. 260],

$$F = U(x) \cap T ,$$

and noting that F is closed and nonempty and that

$$(I'_{\tau})_x = (I'_x)_F,$$

we have

$$\sum_{II'}^W(I'_x) \neq A,$$

which is impossible in view of the construction of T .

We assume hereafter that Γ is a GS game with S_I described in terms of $Y \subseteq X$ as in § 2, and that

$$\sum_{II'}^W(\Gamma) = A,$$

whence

$$\sum_{II'}^W(\Gamma_{\tau}) = A$$

by Lemma 1. The strict determinateness of Γ will follow from Lemma 1 and the fact that

$$\sum_{II'}^W(\Gamma_{\tau}) \neq A,$$

proved in § 4.

LEMMA 2. For $x \in X^T$, we have

$$s \in S_I^{T^x}$$

if and only if

$$s \in S^{T^x} \text{ and } s(i) \in Y$$

for infinitely many i .

LEMMA 3. For $x \in X^T$ there exists

$$\sigma_x \in \sum_{II'}((\Gamma_T)_x)$$

such that for any

$$\tau \in \sum_{II'}((\Gamma_T)_x)$$

we have

$$\langle \sigma_x, \tau \rangle(i) \in Y$$

for some $i > rk(x)$.

Proof. Let Y_x be the set of all

$$y \in Y \cap X^T$$

such that $y > x$ and no members of Y fall between x and y . Let Γ' be the game

$$(x_0, X_I^{Tx}, X_{II}^{Tx}, X^{Tx}, f^{Tx}, S^{Tx}, S'_I, S'_{II}) ,$$

where

$$S'_I = S^{Tx} \cap \bigcup \{U(y) : y \in Y_x\}$$

and

$$S'_{II} = S^{Tx} - S'_I$$

(that is, the game in which I wins if the play passes through any member of Y following x). Noting that

$$S_I^{Tx} \subseteq S'_I ,$$

we have

$$S'_{II} \subseteq S_{II}^{Tx}$$

and hence

$$\sum_{II}^W(I'') = A .$$

But S'_I is open in S^{Tx} and so I'' is strictly determined by Corollary 10 of [1], whence there exists

$$\sigma_x \in \sum_I^W(I'') ,$$

which satisfies the conclusion of the lemma.

4. Winning I'_T . Let

$$Y' = (Y \cap X^T) \cup \{x_0\} .$$

For each $x \in Y'$ let σ_x be as given by Lemma 3, and let σ'_x be the restriction of σ_x to the set of all z in X^T such that $x \leq z$ and that there exists no y in Y' with $x < y \leq z$. We show that the domains of the σ'_x cover X^T and are disjoint: First, if $x_0 \in X_I^T$, then x_0 belongs to the domain of σ_{x_0} . For

$$z \in X_I^T - \{x_0\} ,$$

let

$$x = \max \{z' : z' \in Y' \ \& \ z' < z\} .$$

Then $x \in Y'$ and z belongs to the domain of σ'_x ; thus the domains of the σ'_x cover X_I^T . Now suppose that $x_1, x_2 \in Y'$, $x_1 \not\leq x_2$, and that there exists x_3 common to the domains of σ'_{x_1} and σ'_{x_2} ; then $x_1 \leq x_3$ and $x_2 \leq x_3$, so that either $x_1 < x_2 \leq x_3$ or $x_2 < x_1 \leq x_3$, which is impossible in view of the restriction imposed upon σ_x in obtaining σ'_x .

Since the domains of the σ'_x cover X_I^T and are disjoint, they have

a common extension σ^* , which necessarily maps the elements of X_I^r on their immediate successors, and thus belongs to $\sum_I(\Gamma_\tau)$.

We show that σ^* wins Γ_τ . Let

$$\tau \in \sum_{II}(\Gamma_\tau) .$$

For this τ and any x in Y' , let $i(x)$ be the least i such that $\langle \sigma_x, \tau \rangle(i) \in Y'$, whose existence is given by Lemma 3. Define $\{x_n\}$ inductively by

$$x_{n+1} = \langle \sigma^*, \tau \rangle(i(x_n)) \quad n=0, 1, \dots$$

(x_0 is the distinguished vertex). Since

$$rk(x_{n+1}) = i(x_n) > rk(x_n) ,$$

and x_n, x_{n+1} are on a common path, we have $x_{n+1} > x_n$ for all n , and so if $x_n \in Y'$ then

$$x_{n+1} = \langle \sigma^*, \tau \rangle(i(x_n)) = \langle \sigma_{x_n}, \tau_{x_n} \rangle(i(x_n)) \in Y' ,$$

where

$$\tau_{x_n} \in \sum_{II}((\Gamma_\tau)_{x_n})$$

is the restriction of τ to $X_{II}^{T_{x_n}}$. Thus by induction $x_n \in Y'$ for all n , and hence

$$\langle \sigma^*, \tau \rangle(i) \in Y$$

for infinitely many values of i , so that

$$\langle \sigma^*, \tau \rangle \in S_I^r .$$

Since τ is arbitrary,

$$\sigma^* \in \sum_I^W(\Gamma_\tau) ,$$

so that by Lemma 1, we have

$$\sum_I^W(\Gamma) \cong A .$$

As this is the consequence of the sole fact that

$$\sum_{II}^W(\Gamma) = A ,$$

Γ is strictly determined.

Reversing the roles of the players in the above gives the result that a GS game is strictly determined if S_I is an F_σ .

The strict determinateness of a two-person zero-sum game with G payoff having *chance moves* can be shown. The proof is more complicated, but uses the same ideas [4].

5. An application. Let

$$\Gamma = (x_0, X_I, X_{II}, X, f, S, \phi)$$

be a zero-sum two-person infinite game of perfect information with no chance moves having payoff ϕ such that there exists a real function h on X ($|h(x)| < K < \infty$) with

$$\phi(s) = \limsup_{i \rightarrow \infty} h(s(i)) \quad \text{for all } s \in S.$$

Γ is the result of an attempt to reduce the following situation to a game: The tree K of a GS game and a function h as above are given; the two players make choices in K in the belief that every play will terminate in some unknown, but distant, vertex x , at which time player I will receive the amount $h(x)$ from player II . A payoff function ϕ is sought such that $\phi(s)$ ($-\phi(s)$) expresses the utility to player I (II) of a play s in K .

The payoff ϕ defined above arises from ascription to players I and II respectively of "optimistic" and "pessimistic" behaviors in this way: Player I assumes that the play s will terminate in some "distant" vertex $s(i)$ at which h assumes nearly its supremum on all "distant" vertices of s ; he thus makes his choices so as to maximize the expression

$$\limsup_{i \rightarrow \infty} h(s(i)) = \phi(s);$$

and player II supposes that s will terminate in some "distant" vertex at which his gain $-h(s(i))$ assumes nearly its infimum for all such vertices, and thus seeks to maximize

$$\liminf_{i \rightarrow \infty} -h(s(i)) = -\phi(s),$$

that is, to minimize ϕ . The derived game is thus zero-sum. Ascription, however, of such "optimistic" or "pessimistic" payoffs to both players yields, in general, a non-zero sum game.

We show now that the game Γ of this section is strictly determined, using the method of Theorem 15 of [1] which asserts the strict determinateness of Γ for the more special case of continuous ϕ . (Gillette [2] has shown the strict determinateness of an infinite game of perfect information with chance moves which consists in repeated play from a finite set of finite games and has payoff

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n g_n(s),$$

where $g_n(s)$ is the gain from the n th game played.)

First, as a converse to the equivalence of § 2, let $Y \subseteq X$, and denote by Y_n the set of all members of Y having rank greater than n . Then

$$\begin{aligned} \{s : s(i) \in Y \text{ for infinitely many } i\} &= \bigcap_n \{s : s(i) \in Y_n \text{ for some } i\} \\ &= \bigcap_n \bigcup \{11(y) ; y \in Y_n\}, \end{aligned}$$

which is a G_δ .

Now in I' , for t real, let

$$S_t^i = \{s : h(s(i)) > t \text{ for infinitely many } i\},$$

and $S_{II}^t = S - S_t^i$. Then S_t^i is a G_δ , and thus the GS game

$$I_t = (x_0, X_I, X_{II}, X, f, S, S_t^i, S_{II}^t)$$

is strictly determined. Let

$$v = \sup \{t : \sum_{II}^W(I_t) = A\}.$$

Since $S_I^K = A$, $S_I^{-K} = S$, and S_t^i is a decreasing function of t , we have

$$-K \leq v \leq K, \quad \sum_{II}^W(I_t) = A \quad \text{if } t < v,$$

and

$$\sum_{II}^W(I_t) = A \quad \text{if } t > v.$$

Given $\varepsilon > 0$, choose

$$\sigma_0 \in \sum_{II}^W(I_{v-\varepsilon}) \quad \text{and} \quad \tau_0 \in \sum_{II}^W(I_{v+\varepsilon}).$$

Then for any

$$\sigma \in \sum_I(I), \quad \tau \in \sum_{II}(I),$$

we have

$$h(\langle \sigma, \tau \rangle(i)) > v - \varepsilon \quad \text{for infinitely many } i$$

and do not have

$$h(\langle \sigma, \tau_0 \rangle(i)) > v + \varepsilon \quad \text{for infinitely many } i;$$

so that

$$\Phi(\langle \sigma, \tau \rangle) \geq v - \varepsilon \quad \text{and} \quad \Phi(\langle \sigma, \tau_0 \rangle) < v + 2\varepsilon.$$

Hence

$$v - \varepsilon \leq \sup_{\sigma} \inf_{\tau} \Phi(\langle \sigma, \tau \rangle) \leq \inf_{\tau} \sup_{\sigma} \Phi(\langle \sigma, \tau \rangle) \leq v + 2\varepsilon;$$

thus I' is strictly determined, and has value v .

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