

# Pacific Journal of Mathematics

**ON A THEOREM OF S. BERNSTEI**

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**1. Introduction and proof of the main theorem.** A result of S. Bernstein [4] is the following.

**THEOREM A.** *If  $p(z)$  is a polynomial of degree  $n$  such that*  
 $[\max |p(z)|, |z|=1]=1$ , *then*

$$(1) \quad [\max |p(z)|, |z|=R>1] \leq R^n,$$

*with equality only for  $p(z)=\lambda z^n$ , where  $|\lambda|=1$ .*

We propose to show here that if we restrict ourselves to polynomials of degree  $n$  having no zero within the unit circle the right hand member of (1) can be made smaller. In particular we have the following result.

**THEOREM 1.** *If  $p(z)$  is a polynomial of degree  $n$  such that*  
 $[\max |p(z)|, |z|=1]=1$ , *and  $p(z)$  has no zero within the unit circle, then*

$$[\max |p(z)|, |z|=R>1] \leq \frac{1+R^n}{2},$$

*with equality only for  $p(z)=(\lambda+\mu z^n)/2$ , where  $|\lambda|=|\mu|=1$ .*

In order to prove Theorem 1 we use a conjecture of Erdős first proved by Lax [2] (See also [1]).

**THEOREM B.** *If  $p(z)$  is a polynomial of degree  $n$  such that*  
 $[\max |p(z)|, |z|=1]=1$ , *and  $p(z)$  has no zero within the unit circle, then*

$$[\max |p'(z)|, |z|=1] \leq \frac{n}{2}.$$

Turning now to Theorem 1, let us assume that  $p(z)$  does not have the form  $(\lambda+\mu z^n)/2$ . In view of Theorem B

$$(2) \quad |p'(e^{i\varphi})| \leq \frac{n}{2}, \quad 0 \leq \varphi < 2\pi,$$

from which we may deduce that

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$$(3) \quad |p'(re^{i\varphi})| < \frac{n}{2} r^{n-1}, \quad 0 \leq \varphi < 2\pi, \quad r > 1,$$

by applying Theorem A to the polynomial  $p'(z)/(n/2)$  and observing that we have the strict inequality in (3) because  $p(z)$  does not have the form  $(\lambda + \mu z^n)/2$ . But for each  $\varphi$ ,  $0 \leq \varphi < 2\pi$ , we have

$$p(Re^{i\varphi}) - p(e^{i\varphi}) = \int_1^R e^{i\varphi} p'(re^{i\varphi}) dr.$$

Hence

$$|p(Re^{i\varphi}) - p(e^{i\varphi})| \leq \int_1^R |p'(re^{i\varphi})| dr < \frac{n}{2} \int_1^R r^{n-1} dr = \frac{R^n - 1}{2},$$

and

$$|p(Re^{i\varphi})| < \frac{R^n - 1}{2} + |p(e^{i\varphi})| \leq \frac{1 + R^n}{2}.$$

Finally, if  $p(z) = (\lambda + \mu z^n)/2$ ,  $|\lambda| = 1$ , then

$$[\max |p(z)|, |z| = R > 1] = \frac{1 + R^n}{2}.$$

As a corollary of Theorem 1 we may deduce

**THEOREM 2.** *If  $p(z)$  is a polynomial of degree  $n$  with real coefficients having all zeros of nonpositive real part and if for some  $R > 1$*

$$p(R) > p(1) \left( \frac{R^k + R^n}{2} \right),$$

*$k$  a nonnegative integer, then  $p(z)$  has at least  $(k+1)$  zeros in  $|z| < 1$ .*

*Proof.* Suppose  $p(z)$  has  $m$  zeros in  $|z| < 1$  and  $m \leq k$ . Let

$$p(z) = (z - z_1) \cdots (z - z_m)(z - z_{m+1}) \cdots (z - z_n),$$

and suppose  $|z_j| < 1$ , ( $j = 1, \dots, m$ ). Put

$$g(z) = (z - z_1) \cdots (z - z_m)$$

and

$$h(z) = (z - z_{m+1}) \cdots (z - z_n).$$

The polynomials  $p(z)$ ,  $g(z)$  and  $h(z)$  have positive coefficients, hence for all  $R > 1$

$$g(R) \leq g(1)R^m$$

and

$$h(R) \leq h(1) \left( \frac{1+R^{n-m}}{2} \right)$$

according to Theorems A and 1 respectively.

Thus

$$p(R) = h(R)g(R) \leq p(1) \left( \frac{R^m + R^n}{2} \right) \leq p(1) \left( \frac{R^k + R^n}{2} \right),$$

a contradiction, establishing Theorem 2.

**2. The converse problem.** The converse of Theorem 1 is false as the simple example  $p(z) = (z + \frac{1}{2})(z + 3)$  shows. However, the following result in the converse direction is valid.

**THEOREM 3.** *If  $p(z)$  is a polynomial of degree  $n$  such that*

$$p(1) = [\max |p(z)|, |z|=1] = 1$$

and

$$[\max |p(z)|, |z|=R > 1] \leq \frac{1+R^n}{2}$$

for  $0 < R - 1 < \delta$ , where  $\delta$  is any positive number, then  $p(z)$  does not have all its roots within the unit circle.

For the proof we need the following

**LEMMA.** *If*

$$q(z) = (z - z_1) \cdots (z - z_m)$$

where  $|z_j| < 1$ , ( $j=1, \dots, m$ ), then if  $|a|=1$  we have

$$\left| \frac{q'(a)}{q(a)} \right| > \frac{m}{2}.$$

*Proof.* According to Laguerre's Theorem [3, p. 38]

$$\frac{q'(a)}{q(a)} = \frac{m}{a-w},$$

where  $|w| < 1$ , hence  $|a-w| < 2$  and

$$\left| \frac{q'(a)}{q(a)} \right| > \frac{m}{2}.$$

We turn now to the proof of Theorem 3. Suppose  $p(z)$  has all its zeros in  $|z| < 1$ . Let

$$p(z) = a_0 + a_1z + \cdots + a_nz^n,$$

put

$$\bar{p}(z) = \bar{a}_0 + \bar{a}_1z + \cdots + \bar{a}_nz^n$$

and consider the polynomial  $g(z) = p(z)\bar{p}(z)$  of degree  $2n$ .  $g(z)$  is real for real  $z$ ,

$$[\max |g(z)|, |z|=1] = g(1) = 1,$$

$$|g(Re^{i\varphi})| \leq \left(\frac{1+R^n}{2}\right)^2 \leq \frac{1+R^{2n}}{2}$$

and  $g(z)$  has all its zeros in  $|z| < 1$ . Now  $g'(1)$  is not only real but positive. This is so since, given any  $\eta > 0$ , we have  $g(1-\eta) < g(1)$ . Hence

$$g'(1) = \lim_{\eta \rightarrow 0} \frac{g(1-\eta) - g(1)}{-\eta} \geq 0.$$

Now  $g'(1) \neq 0$ , as all of the roots of  $g(z) = 0$  are inside the unit circle, hence, by Lucas' Theorem all roots of  $g'(z) = 0$  are within the convex closure of the unit circle namely the unit circle itself.

Given any  $\varepsilon > 0$ , sufficiently small,

$$|g(1+\varepsilon) - g(1)| = g(1+\varepsilon) - g(1) \leq \frac{(1+\varepsilon)^{2n} + 1}{2} - 1 = \frac{(1+\varepsilon)^{2n} - 1}{2},$$

or

$$|g(1+\varepsilon) - g(1)| \leq n\varepsilon + O(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0$$

and  $g'(1) \leq n$ . Therefore  $g'(1)/g(1) \leq n$  contradicting the lemma. Theorem 3 is established.

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