THE USE OF FORMS IN VARIATIONAL CALCULATION

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Introduction. The purpose of this paper is to present a method of calculating the first and second variation which is suitable for spaces which have a Euclidean connection. I then use this method to calculate the first and second variations along a geodesic in a Finsler space in terms of differential invariants of the Finsler metric. In the special case of Riemannian geometry, this calculation has been carried out by Schoenberg in [4].

Indications as to how this calculation should be made are originally due to E. Cartan [1]. I wish to thank Prof. S. S. Chern for the privilege of seeing his calculations on this matter for Riemann spaces.

1. Algebraic Preliminaries. Let \( I = [0, 1] \) and \( 0 \leq \xi, \xi \leq 1 \). Let \( M^n \) be an \( n \)-dimensional \( C^\infty \) manifold. Assume we have a one parameter family of mappings of \( I \) into \( M^n \) which we will denote by \( f(\xi, \xi_2) \), where \( \xi_2 \) is taken as the parameter along \( I \) and \( \xi_1 \) parametrizes the family of mappings. Then we may define a mapping \( \gamma: I \times I \rightarrow M^n \) by the equation

\[
\gamma(\xi_1, \xi_2) = f(\xi_1, \xi_2).
\]

We require that \( \gamma \) shall also be a \( C^\infty \) mapping.

Let \( \gamma^* \) denote the mapping induced by \( \gamma \) on the tangent space to \( I \times I \) into the tangent space to \( M^n \). Let \( \gamma^* \) denote the dual mapping induced on the cotangent spaces. Then we define two vector fields \( X_1 \) and \( X_2 \) over \( \gamma(I \times I) \) by

\[
X_1 = \gamma^*(\partial/\partial \xi_2) \quad \text{and} \quad X_2 = \gamma^*(\partial/\partial \xi_1).
\]

Then if \( w \) is any form in \( M^n \) we may write

\[
\gamma^*(w) = w_5 d\xi_1 + w_6 d\xi_2,
\]

where \( w_5 \) and \( w_6 \) are defined by the equation.

**Lemma 1.1.** If \( \langle X, w \rangle \) denotes the value that \( X \) takes on the covector \( w \) at each point, then

\[
w_5 = \langle X_1, w \rangle
\]

and

\[
w_6 = \langle X_2, w \rangle.
\]

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Proof. \[ w_d = \langle \partial / \partial \xi_1, \gamma^*(w) \rangle = \langle \gamma^*(\partial / \partial \xi_1), w \rangle = \langle X_1, w \rangle. \]
The proof is analogous for \( w_d \).

Let \( \Omega \) be any two form and let \( X_1 \) and \( X_2 \) be any two vector fields. It is well known that \( A(V) \) and \( A(V^*) \) are dually paired. Let this pairing be denoted by

\[ \langle X_1 \wedge X_2, \Omega \rangle. \]

Then if \( \Omega \) can be decomposed as \( w_1 \wedge w_2 \), where \( w_1 \) and \( w_2 \) are one forms, we have that the pairing may be defined by the following expression:

\[ \langle X_1 \wedge X_2, w_1 \wedge w_2 \rangle = \langle X_1, w_1 \rangle \langle X_2, w_2 \rangle - \langle X_1, w_2 \rangle \langle X_2, w_1 \rangle. \]

**Theorem 1.1.** \[ \langle X_1 \wedge X_2, w_1 \wedge w_2 \rangle = w_{18} w_{26} - w_{16} w_{28}. \]

The proof of this theorem is straightforward.

We define the symbols \( \delta w_a \) and \( dw_b \) by the following equations:

\[
\begin{align*}
\delta w_a &= \partial / \partial \xi_1 \langle X_a, w \rangle, \\
dw_b &= \partial / \partial \xi_2 \langle X_b, w \rangle.
\end{align*}
\]

If \( f \) is any function of \( \xi_1 \) and \( \xi_2 \), we define

\[ d^* \delta f = \frac{\partial f}{\partial \xi_1 \partial \xi_2}, \]

where \( t = r + s \). Define \( \delta^* d^* f \) similarly.

**Theorem 1.2.** \[ \langle X_1 \wedge X_2, dw \rangle = \delta w_a - dw_b. \]

*Proof.* Now, in terms of a local coordinate system \((x_1, \cdots, x_n)\),

\[ \langle X_1 \wedge X_2, dw \rangle = \sum \left[ \frac{\partial}{\partial \xi_1} \left( a_i \frac{\partial x_i}{\partial \xi_2} \right) - \frac{\partial}{\partial \xi_2} \left( a_i \frac{\partial x_i}{\partial \xi_1} \right) \right] \]

since

\[ \sum a_i \frac{\partial^2 x_i}{\partial \xi_1 \partial \xi_2} = \sum a_i \frac{\partial^2 x_i}{\partial \xi_2 \partial \xi_1}. \]

This and the definition of \( \delta w_a \) and \( dw_b \) prove the theorem.

2. **The First Variation.** Consider the integral

\[ I = \int_a^b F(q_1, \cdots, q_n; q'_1, \cdots, q'_n; t) dt \]

in a space \( M \) of \( 2n + 1 \) dimensions. Then in the cotangent space to the manifold \( M \) define the form \( w \) by the equation
(2.2) \[ w = \sum \frac{\partial F}{\partial q_i} dq_i - \left( \sum q_i \frac{\partial F}{\partial q_i} - F \right) dt. \]

Now let \( C \) be a curve in \( M^{2n+1} \) expressed by the equations
\[ q_i = q_i(\xi), \quad q'_i = q'_i(\xi), \quad t = (b-a)\xi + a. \]
Assume further that \( dq_i/d\xi = q_i \) for all values of \( \xi \). Let \( X_2 \) be the image of \( \partial/\partial \xi \) under the mapping described above. Then
\[ X_2 = \sum q'_i \frac{\partial}{\partial q_i} + \sum \frac{\partial q_i}{\partial \xi} \frac{\partial}{\partial q'_i} + (b-a) \frac{\partial}{\partial t}, \]
and
\[ w_0 d\xi_2 = F(q, q', t) \frac{dt}{(b-a)}. \]
Hence
\[ I = \int_0^i w_0 d\xi_2 = \int_{\alpha}^b F(q_1(t), \ldots, q_n(t); q_1(t), \ldots, q_n(t); t) dt. \]

Now consider a one parameter family of curves \( f(\xi_1, \xi_2) \) each with the property described above. For each curve in the family we get a vector field which we will denote by \( X_2(\xi) \). We may consider the variational problem for this family of curves. The crucial fact is that the requirement that \( f(\xi_1, \xi_2) \) is a mapping of a fixed interval for each fixed value of \( \xi_1 \) enables us to treat the problem of variable end point without the necessity of differentiating limits of integration. We consider
\[ I(\xi_1) = \int_0^i \langle X_2(\xi_1), w \rangle d\xi_2 \]
and
\[ (2.5) \quad \delta I = \frac{\partial I(\xi_1)}{\partial \xi_1} = \int_0^i \delta w_0 d\xi_2. \]
If we add and subtract \( dw_s \) under the integral sign we get
\[ (2.6) \quad \delta I = [w_0]_0^i + \int_0^i (\delta w_s - dw_s) d\xi_2 \]
\[ (2.7) \quad = [w_0]_0^i + \int_0^i w'(\delta, d) d\xi_2, \]
where
\[ (2.8) \quad w'(\delta, d) = \langle X_1 \wedge X_2, dw \rangle, \]
and

\[ w'(d, \delta) = \langle X_2 \wedge X_1, dw \rangle. \]

It may be noted that \( w'(\delta, d) = -w'(d, \delta) \). The term \([w_\delta]_J\) is called the transversality term.

**Theorem 2.1.** Assume \([w_\delta]_J = 0\). Then a necessary and sufficient condition for \( \delta I = 0 \) for all variations is that \( dw = 0 \) along \( C \).

**Proof.** The condition is clearly sufficient. An equivalent form of the hypothesis is that

\[ \int_0^1 \langle X_1 \wedge X_2, dw \rangle d\xi_2 = 0 \]

for all vector fields \( X_1 \) along \( C \). Assume \( dw \) does not equal zero along \( C \). Then there exists an \( X_1 \) such that \( \langle X_1 \wedge X_2, dw \rangle > 0 \) for some open interval \( a < \xi_2 < b \). Then we may choose a new vector field \( X_1 \) such that:

\[
\begin{align*}
\overline{X}_1 &= X_1 \text{ for } a < \xi_2 < b \\
\overline{X}_1 &= 0 \text{ for } 0 \leq \xi_2 \leq a - \varepsilon \text{ or } b + \varepsilon \leq \xi_2 \leq 1,
\end{align*}
\]

where \( \varepsilon \) may be chosen arbitrarily small. Then

\[
\int_0^1 \langle \overline{X}_1 \wedge X_2, dw \rangle d\xi_2 = \int_0^b \langle X_1 \wedge X_2, dw \rangle d\xi_2 + \varepsilon',
\]

where \( \varepsilon' \) depends on \( \varepsilon \) and \( \lim_{\varepsilon \to 0} \varepsilon' = 0 \). Hence we may choose \( \varepsilon \) in such a way that

\[ \int_0^1 \langle \overline{X}_1 \wedge X_2, dw \rangle d\xi_2 > 0. \]

This contradiction proves the theorem.

Remark: This is essentially the usual argument for the derivation of Euler's equation.

3. **Application to Finsler Geometry.** If we assume that our integral is of the Finsler type then we may proceed to calculate the second variation. For treating this special case we assume that the reader has a familiarity with Euclidean connections and we will use the Euclidean connection for a Finsler space as calculated by E. Cartan in [2] and Chern [3].

Let \( M \) be an \( n \)-dimensional differentiable manifold and let \( G \) be the principal bundle over \( M \) with fiber and group the \( n \)-dimensional orthogonal groups, \( O(n) \). Then in \( G \), we have forms \( w_i, w_{ij} \), where \( w_{ij} + w_{ji} = 0 \) and \( i, j = 1, \ldots, n \). The equations of structure are
(3.1) \[ \frac{dw_i}{dt} = w_j \wedge w_{ji} + \gamma_{jia} w_j \wedge w_{ai} \]

(3.2) \[ \frac{dw_{ij}}{dt} = w_{ik} \wedge w_{kj} + \Omega_{ij} , \]

where \( \alpha = 1, \ldots, n - 1 \). (Henceforth we will assume that Greek indices run from 1 to \( n - 1 \) and Latin indices run from 1 to \( n \).) The \( \gamma_{ij\alpha} \) are symmetric in all indices and zero if any index is \( n \). Also

(3.3) \[ \Omega_{ij} = \frac{1}{2} \sum_{\alpha, \beta} Q_{ij\alpha\beta} w_{\alpha n} \wedge w_{\beta n} + \sum_{i, \alpha} P_{ij\alpha} w_i \wedge w_{\alpha n} + \frac{1}{2} \sum_{i, \alpha} R_{ijk} w_i \wedge w_k . \]

Let \( C \) be any path in \( M^n \). Choose any path in \( G \) with the property that if \( e_1, \ldots, e_n \) represents a righthanded frame, that is, an element of \( O \) \((n)\), then \( e_i \) is in the tangent direction to \( C \). Then arclength along a path \( C \) is

\[ I = \int_0^1 (w_n)_0 d\xi . \]

This follows from equation (2.4) and the definition of \( w_n \) (see [3]).

Now \( X_2 = e_n \) and \( X_1 = \sum k_i e_i \). Therefore \( (w_n)_0 = \langle X_1, w_n \rangle = k_n \). Hence if \( X_1 \) is perpendicular to the curve \( C \), then the transversality term is zero. From equation (3.1), we have

\[ \frac{dw_n}{dt} = \sum w_{\alpha} \wedge w_{\alpha n} . \]

Hence

(3.4) \[ \delta I = \delta (w_n)_0 + \int_0^1 \sum (w_{\alpha})_0 (w_{\alpha n})_0 - (w_{\alpha})_0 (w_{\alpha n})_0 d\xi , \]

where\[ (w_{\alpha})_0 = \langle w_{\alpha}, e_n \rangle = 0 . \]

It is clear from the last equation that the symbols \( \delta \) and \( d \) and our indices make the notation awkward. Hence a \( w_{\delta} \) will be written as \( w \) and a \( w_{\delta} \) will be written as \( \phi \). In this notation equation (3.4) becomes

(3.5) \[ I = [\phi_n]_0 + \int_0^1 \sum \phi_a w_{an} d\xi , \]

since \( w_a = 0 \) along the path \( C \).

From Theorem 2.1 we have the following theorem.

**Theorem 3.1.** The differential equations of a geodesic in Finsler geometry are

\[ w_a = 0 , \quad w_{an} = 0 , \quad \alpha = 1, \ldots, n - 1 . \]

We will now compute the second variation along a geodesic. We have
and $\delta^2 I$ is the second variation. Hence we have to compute $\delta^2(w_n)$ along a geodesic. Now

$$\delta^2(w_n) = \delta d(\phi_n) + \phi_\alpha \delta(w_\alpha)$$

since $w_\alpha = 0$ along the geodesic. We have

$$\delta(w_\alpha) - \delta(\phi_\alpha) = \langle X_1 \wedge X_2, d\omega_\alpha \rangle.$$  

From equation (3.2) we obtain

$$\langle X_1 \wedge X_2, d\omega_\alpha \rangle = \langle X_1 \wedge X_2, \omega_\alpha \wedge \omega_\beta \rangle + \langle X_1 \wedge X_2, \Omega_\alpha \rangle.$$  

By Theorem 1.1 and since $C$ is a geodesic, we have

$$\delta w_\alpha = d\phi_\alpha - \omega_\alpha \phi_\beta + \langle \Omega_\alpha, X_1 \wedge X_2 \rangle.$$  

Now by equation (3.2) and the facts that

$$R_{ijkl} = -R_{jikl}, \quad R_{ij, kl} = R_{kl, ij}$$

we have

$$\langle X_1 \wedge X_2, \Omega_\alpha \rangle = \sum P_n \omega_\alpha \phi_\beta n + \sum R_n \phi_\beta \phi_\alpha \omega_\alpha.$$  

Therefore, from equations (3.6), (3.8) and (3.9), we obtain

$$\delta^2(w_n) = \delta d\phi_n + \sum \phi_\alpha [d\phi_\alpha - \phi_\beta \omega_\alpha \phi_\beta + P_n \omega_\alpha \phi_\beta n + R_n \phi_\beta \phi_\alpha \omega_\alpha].$$

Now,

$$\delta d\phi_n = d\delta \phi_n \quad \text{and} \quad d(\phi_\alpha \phi_\alpha) = \phi_\alpha (d\phi_\alpha) + \phi_\alpha (d\phi_\alpha).$$

Hence

$$\delta^2(w_n) = d[\delta \phi_n + \phi_\alpha \phi_\alpha] - \phi_\alpha d\phi_\alpha$$

$$+ [- \phi_\alpha \phi_\beta \omega_\alpha \phi_\beta + P_n \omega_\alpha \phi_\beta n + R_n \phi_\beta \phi_\alpha \omega_\alpha] \omega_\alpha.$$  

But from equation (3.1) we have

$$d\phi_\alpha = \delta w_\alpha + w_\alpha \phi_\alpha - \phi_\beta w_\beta$$

since

$$\gamma_{jap} [\phi_j w_\beta - w_\beta \phi_\beta] = 0$$

along the geodesic. Also $\delta w_\alpha = 0$ along the geodesic, since $w_\alpha \geq 0$ and equals zero along the geodesic and hence $w_\alpha$ must attain a minimum along a geodesic.
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Hence

\[ \delta^2 w_n = d[\delta \phi_n + \sum \phi_a \phi_a n] + \sum (\phi_{an} \phi_{an} + P_{nan} \phi_a \phi_{\beta n} + R_n \phi_{\alpha n} \phi_{\beta n}) w_n. \]

Hence the integral form of the second variation becomes

\[ \delta^2 I = [\delta \phi_n + \sum \phi_a \phi_a n]_0 + \int_0^t \sum (\phi_{an} \phi_{an} + P_{nan} \phi_a \phi_{\beta n} + R_n \phi_{\alpha n} \phi_{\beta n}) w_n d\xi. \]

For Riemannian geometry we have \( P_{ijkl} = 0 \) and \( \sum \phi_a \phi_{an} \) represents the second fundamental form of the geodesic surface perpendicular to the geodesic at the point.

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