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Louis Auslander

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LOUIS AUSLANDER

Introduction. The purpose of this paper is to present a method of calculating the first and second variation which is suitable for spaces which have a Euclidean connection. I then use this method to calculate the first and second variations along a geodesic in a Finsler space in terms of differential invariants of the Finsler metric. In the special case of Riemannian geometry, this calculation has been carried out by Schoenberg in [4].

Indications as to how this calculation should be made are originally due to E. Cartan [1]. I wish to thank Prof. S. S. Chern for the privilege of seeing his calculations on this matter for Riemann spaces.

1. Algebraic Preliminaries.

Let $I=[0, 1]$ and $0\leq\xi_1, \xi_2\leq 1$. Let M^n be an n-dimensional C^∞ manifold. Assume we have a one parameter family of mappings of I into M^n which we will denote by $f(\xi_1, \xi_2)$, where ξ_2 is taken as the parameter along I and ξ_1 parametrizes the family of mappings. Then we may define a mapping η: $I\times I\rightarrow M^n$ by the equation

$$\eta(\xi_1, \xi_2)=f(\xi_1, \xi_2).$$

We require that η shall also be a C^∞ mapping.

Let η_* denote the mapping induced by η on the tangent space to $I\times I$ into the tangent space to M^n. Let η^* denote the dual mapping induced on the cotangent spaces. Then we define two vector fields X_1 and X_2 over $\eta(I\times I)$ by

$$X_2=\eta_*(\partial/\partial\xi_2) \quad \text{and} \quad X_1=\eta_*(\partial/\partial\xi_1).$$

Then if w is any form in M^n we may write

$$\eta^*(w)=w_\delta d\xi_1+w_d d\xi_2,$$

where w_δ and w_d are defined by the equation.

LEMMA 1.1. *If $\langle X, w\rangle$ denotes the value that X takes on the covector w at each point, then*

$$w_\delta=\langle X_1, w\rangle$$

and

$$w_d=\langle X_2, w\rangle.$$

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Proof. $w_\delta = \langle \partial/\partial\xi_1, \eta^*(w)\rangle = \langle \eta^*(\partial/\partial\xi_1), w\rangle = \langle X_1, w\rangle$. The proof is analogous for w_d.

Let Ω be any two form and let X_1 and X_2 be any two vector fields. It is well known that $\Lambda^2(V)$ and $\Lambda^2(V^*)$ are dually paired. Let this pairing be denoted by

$$\langle X_1 \wedge X_2, \Omega\rangle .$$

Then if Ω can be decomposed as $w_1 \wedge w_2$, where w_1 and w_2 are one forms, we have that the pairing may be defined by the following expression:

$$\langle X_1 \wedge X_2, w_1 \wedge w_2\rangle = \langle X_1, w_1\rangle\langle X_2, w_2\rangle - \langle X_1, w_2\rangle\langle X_2, w_1\rangle .$$

THEOREM 1.1. $\langle X_1 \wedge X_2, w_1 \wedge w_2\rangle = w_{1\delta}w_{2d} - w_{1d}w_{2\delta}$.

The proof of this theorem is straightforward.

We define the symbols δw_d and dw_δ by the following equations:

$$\delta w_d = \partial/\partial\xi_1 \langle X_2, w\rangle ,$$
$$dw_\delta = \partial/\partial\xi_2 \langle X_1, w\rangle .$$

If f is any function of ξ_1 and ξ_2, we define

$$d^r\delta^s f = \frac{\partial^t f}{\partial\xi_2^r \partial\xi_1^s} ,$$

where $t = r+s$. Define $\delta^r d^s f$ similarly.

THEOREM 1.2. $\langle X_1 \wedge X_2, dw\rangle = \delta w_d - dw_\delta$.

Proof. Now, in terms of a local coordinate system $(x_1, \cdots, x_n)$,

$$\langle X_1 \wedge X_2, dw\rangle = \sum\left[\frac{\partial}{\partial\xi_1}\left(a_i \frac{\partial x_i}{\partial\xi_2}\right) - \frac{\partial}{\partial\xi_2}\left(a_i \frac{\partial x_i}{\partial\xi_1}\right)\right]$$

since

$$\sum a_i \frac{\partial^2 x_i}{\partial\xi_1 \partial\xi_2} = \sum a_i \frac{\partial^2 x_i}{\partial\xi_2 \partial\xi_1} .$$

This and the definition of δw_d and dw_δ prove the theorem.

2. The First Variation.

Consider the integral

$$I = \int_a^b F(q_1, \cdots, q_n ; q_1', \cdots, q_n' ; t)dt \tag{2.1}$$

in a space M of $2n+1$ dimensions. Then in the cotangent space to the manifold M define the form w by the equation

(2.2) $$w=\sum \frac{\partial F}{\partial q_i'}dq - \left(\sum q_i' \frac{\partial F}{\partial q_i'} - F\right)dt .$$

Now let C be a curve in M^{2n+1} expressed by the equations

$$q_i=q_i(\xi_2) , \qquad q_i'=q_i'(\xi_2) , \qquad t=(b-a)\xi_2+a .$$

Assume further that $dq_i/d\xi_2=q_i'$ for all values of ξ_2. Let X_2 be the image of $\partial/\partial\xi_2$ under the mapping described above. Then

(2.3) $$X_2=\sum q_i' \frac{\partial}{\partial q_i} + \sum \frac{\partial q_i}{\partial \xi_2}\frac{\partial}{\partial q_i'} + (b-a)\frac{\partial}{\partial t} ,$$

and

$$w_d d\xi_2=F(q, q', t)\frac{dt}{(b-a)} .$$

Hence

(2.4) $$I=\int_0^1 w_d d\xi_2= \int_a^b F\big(q_1(t), \cdots, q_n(t) ; q_1'(t), \cdots, q_n'(t) ; t\big)dt .$$

Now consider a one parameter family of curves $f(\xi_1, \xi_2)$ each with the property described above. For each curve in the family we get a vector field which we will denote by $X_2(\xi_1)$. We may consider the variational problem for this family of curves. The crucial fact is that the requirement that $f(\xi_1, \xi_2)$ is a mapping of a *fixed* interval for each fixed value of ξ_1 enables us to treat the problem of variable end point without the necessity of differentiating limits of integration. We consider

$$I(\xi_1)=\int_0^1 \langle X_2(\xi_1), w\rangle d\xi_2$$

and

(2.5) $$\delta I=\frac{\partial I(\xi_1)}{\partial \xi_1}=\int_0^1 \delta w_d d\xi_2 .$$

If we add and subtract dw_δ under the integral sign we get

(2.6) $$\delta I=[w_\delta]_0^1+\int_0^1 (\delta w_d-dw_\delta)d\xi_2$$

(2.7) $$=[w_\delta]_0^1+\int_0^1 w'(\delta, d)d\xi_2 ,$$

where

(2.8) $$w'(\delta, d)=\langle X_1\wedge X_2, dw\rangle ,$$

and

$$w'(d, \delta)=\langle X_2 \wedge X_1, dw \rangle .$$

It may be noted that $w'(\delta, d)=-w'(d, \delta)$. The term $[w_\delta]_0^1$ is called the transversality term.

THEOREM 2.1. *Assume* $[w_\delta]_0^1=0$. *Then a necessary and sufficient condition for* $\delta I=0$ *for all variations is that* $dw=0$ *along* C.

Proof. The condition is clearly sufficient. An equivalent form of the hypothesis is that

$$\int_0^1 \langle X_1 \wedge X_2, dw \rangle d\xi_2=0$$

for all vector fields X_1 along C. Assume dw does not equal zero along C. Then there exists an X_1 such that $\langle X_1 \wedge X_2, dw \rangle > 0$ for some open interval $a<\xi_2<b$. Then we may choose a new vector field X_1 such that:

$$\begin{aligned} &\overline{X}_1=X_1 \quad \text{for} \quad a<\xi_2<b \\ &\overline{X}_1=0 \quad\ \ \text{for} \quad 0\leq\xi_2\leq a-\varepsilon \quad \text{or} \quad b+\varepsilon\leq\xi_2\leq 1 , \end{aligned}$$

where ε may be chosen arbitrarily small. Then

$$\int_0^1 \langle \overline{X}_1 \wedge X_2, dw \rangle d\xi_2=\int_a^b \langle X_1 \wedge X_2, dw \rangle d\xi_2+\varepsilon',$$

where ε' depends on ε and $\lim_{\varepsilon\to 0} \varepsilon'=0$. Hence we may choose ε in such a way that

$$\int_0^1 \langle \overline{X}_1 \wedge X_2, dw \rangle d\xi_2 >0 .$$

This contradiction proves the theorem.

Remark: This is essentially the usual argument for the derivation of Euler's equation.

3. Application to Finsler Geometry.

If we assume that our integral is of the Finsler type then we may proceed to calculate the second variation. For treating this special case we assume that the reader has a familiarity with Euclidean connections and we will use the Euclidean connection for a Finsler space as calculated by E. Cartan in [2] and Chern [3].

Let M be an n-dimensional differentiable manifold and let G be the principal bundle over M with fiber and group the n-dimensional orthogonal groups, $O_{(n)}$. Then in G, we have forms w_i, w_{ij}, where $w_{ij}+w_{ji}=0$ and $i, j=1, \cdots, n$. The equations of structure are

$$dw_i = w_j \wedge w_{ji} + \gamma_{ji\alpha} w_j \wedge w_{\alpha n} \tag{3.1}$$

$$dw_{ij} = w_{ik} \wedge w_{kj} + \Omega_{ij}\,, \tag{3.2}$$

where $\alpha = 1, \cdots, n-1$. (Henceforth we will assume that Greek indices run from 1 to $n-1$ and Latin indices run from 1 to n.) The $\gamma_{ij\alpha}$ are symmetric in all indices and zero if any index is n. Also

$$\Omega_{ij} = \frac{1}{2} \sum_{\alpha,\beta} Q_{ij\alpha\beta} w_{\alpha n} \wedge w_{\beta n} + \sum_{l,\alpha} P_{ijl\alpha} w_l \wedge w_{\alpha n} + \frac{1}{2} \sum_{l,k} R_{ijlk} w_l \wedge w_k\,. \tag{3.3}$$

Let C be any path in M^n. Choose any path in G with the property that if $e_1, \cdots, e_n$ represents a righthanded frame, that is, an element of $O_{(n)}$, then e_n is in the tangent direction to C. Then arc length along a path C is

$$I = \int_0^1 (w_n)_d d\xi_2\,.$$

This follows from equation (2.4) and the definition of w_n (see [3]).

Now $X_2 = e_n$ and $X_1 = \sum k_i e_i$. Therefore $(w_n)_\delta = \langle X_1, w_n \rangle = k_n$. Hence if X_1 is perpendicular to the curve C, then the transversality term is zero. From equation (3.1), we have

$$dw_n = \sum w_\alpha \wedge w_{\alpha n}\,.$$

Hence

$$\delta I = [\delta(w_n)]_0^1 + \int_0^1 \sum \{(w_\alpha)_\delta (w_{\alpha n})_d - (w_\alpha)_d (w_{\alpha n})_\delta\}\, d\xi_2\,, \tag{3.4}$$

where $$(w_\alpha)_d = \langle w_\alpha, e_n \rangle = 0\,.$$

It is clear from the last equation that the symbols δ and d and our indices make the notation awkward. Hence a w_d will be written as w and a w_δ will be written as ϕ. In this notation equation (3.4) becomes

$$I = [\phi_n]_0^1 + \int_0^1 \sum \phi_\alpha w_{\alpha n} d\xi_2\,, \tag{3.5}$$

since $w_\alpha = 0$ along the path C.

From Theorem 2.1 we have the following theorem.

THEOREM 3.1. *The differential equations of a geodesic in Finsler geometry are*

$$w_\alpha = 0\,, \qquad w_{\alpha n} = 0\,, \qquad \alpha = 1, \cdots, n-1.$$

We will now compute the second variation along a geodesic. We have

$$\delta I = \int_0^1 \delta w_n d\xi_2 ,$$

and $\delta^2 I$ is the second variation. Hence we have to compute $\delta^2(w_n)$ along a geodesic. Now

$$\delta^2(w_n) = \delta d(\phi_n) + \phi_\alpha \delta(w_{\alpha n}) \tag{3.6}$$

since $w_{\alpha n}=0$ along the geodesic. We have

$$\delta(w_{\alpha n}) - d(\phi_{\alpha n}) = \langle X_1 \wedge X_2,\ dw_{\alpha n} \rangle . \tag{3.7}$$

From equation (3.2) we obtain

$$\langle X_1 \wedge X_2,\ dw_{\alpha n} \rangle = \langle X_1 \wedge X_2,\ w_{\alpha\beta} \wedge w_{\beta n} \rangle + \langle X_1 \wedge X_2,\ \Omega_{\alpha n} \rangle .$$

By Theorem 1.1 and since C is a geodesic, we have

$$\delta w_{\alpha n} = d\phi_{\alpha n} - w_{\alpha\beta}\phi_{\beta n} + \langle \Omega_{\alpha n},\ X_1 \wedge X_2 \rangle . \tag{3.8}$$

Now by equation (3.2) and the facts that

$$R_{ijkl} = -R_{jikl} , \qquad R_{ij,kl} = R_{kl,ij}$$

we have

$$\langle X_1 \wedge X_2,\ \Omega_{\alpha n} \rangle = \sum P_{n\alpha n\beta} w_n \phi_{\beta n} + \sum R_{n\alpha n\beta} \phi_\beta w_n . \tag{3.9}$$

Therefore, from equations (3.6), (3.8) and (3.9), we obtain

$$\delta^2(w_n) = \delta d\phi_n + \sum \phi_\alpha [d\phi_{\alpha n} - \phi_{\beta n} w_{\alpha\beta} + P_{n\alpha n\beta} w_n \phi_{\beta n} + R_{n\alpha n\beta} \phi_\beta w_n] . \tag{3.10}$$

Now,

$$\delta d\phi_n = d\delta\phi_n \quad \text{and} \quad d(\phi_\alpha \phi_{\alpha n}) = \phi_{\alpha n}(d\phi_\alpha) + \phi_\alpha(d\phi_{\alpha n}) .$$

Hence

$$\begin{aligned} \delta^2(w_n) = {} & d[\delta\phi_n + \phi_\alpha \phi_{\alpha n}] - \phi_{\alpha n} d\phi_\alpha \\ & + [-\phi_\alpha \phi_{\beta n} w_{\alpha\beta} + P_{n\alpha n\beta} \phi_\alpha \phi_{\beta n} + R_{n\alpha n\beta} \phi_\alpha \phi_\beta] w_n . \end{aligned} \tag{3.11}$$

But from equation (3.1) we have

$$d\phi_\alpha = \delta w_\alpha + w_j \phi_{j\alpha} - \phi_j w_{j\alpha} \tag{3.12}$$

since

$$\gamma_{j\alpha\beta}[\phi_j w_{\beta n} - w_j \phi_{\beta n}] = 0$$

along the geodesic. Also $\delta w_\alpha = 0$ along the geodesic, since $w_\alpha \geq 0$ and equals zero along the geodesic and hence w_α must attain a minimum along a geodesic.

Hence

$$\delta^2 w_n = d[\delta\phi_n + \sum \phi_\alpha\phi_{\alpha n}] + \sum(\phi_{\alpha n}\phi_{\alpha n} + P_{n\alpha n\beta}\phi_\alpha\phi_{\beta n} + R_{n\alpha n\beta}\phi_\alpha\phi_\beta)w_n\,. \tag{3.13}$$

Hence the integral form of the second variation becomes

$$\delta^2 I = [\delta\phi_n + \sum \phi_\alpha\alpha_{\alpha n}]_0^1 + \int_0^1 \sum (\phi_{\alpha n}\phi_{\alpha n} + P_{n\alpha n\beta}\phi_\alpha\phi_{\beta n} + R_{n\alpha n\beta}\phi_\alpha\phi_\beta)w_n d\xi_2\,.$$

For Riemannian geometry we have $P_{ijkl}=0$ and $\sum \phi_\alpha\phi_{\alpha n}$ represents the second fundamental form of the geodesic surface perpendicular to the geodesic at the point.

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YALE UNIVERSITY

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