ABSTRACT RIEMANN SUM

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1. Introduction. A theorem of B. Jessen [5] asserts that for \( f(x) \) of period one and Lebesgue integrable on \([0, 1]\)

\[
\lim_{n \to \infty} 2^{-n} \sum_{k=0}^{2^n-1} f(x + k2^{-n}) = \int_{0}^{1} f(t) \, dt
\]

almost everywhere.

We show that the theorem of Jessen is a special case of a theorem analogous to the Birkhoff ergodic theorem [1] but dealing with sums of the form

\[
2^{-n} \sum_{k=0}^{2^n-1} f(T^{k/2^n} x).
\]

In this form \( T \) is an operator on a \( \sigma \)-finite measure space such that \( T^{n/2^n} \) exists as a one-to-one point transformation which is measure preserving for \( n = 0, 1, \ldots \), and \( f(x) \) is integrable with \( f(x) = f(Tx) \). We also obtain in § 3 the analogues for abstract Riemann sums of the ergodic theorems of Hurewicz [4] and of Hopf [3].

We might remark that there is no use, due to the examples of Marcinkiewicz and Zygmund [6] and Ursell [8], in considering sums of the form

\[
\sum_{k=0}^{n-1} f(T^{k/2^n} x)
\]

without further hypothesis on \( f(x) \). However we may replace \( 2^n \) throughout by \( m_1 m_2 \cdots m_n \) with \( m_j \) integral and \( m_j \geq 2 \) without altering any argument.

In § 4 necessary and sufficient conditions are obtained on a transformation \( T \) in order that the sums (2) have a limit as \( n \to \infty \) for almost all \( x \). These conditions are analogous to those of Ryll-Nardzewski [7] in the ergodic case. We use the necessary conditions to establish an analogue of a form of the Hurewicz ergodic theorem for two operators [2].

2. Notation. Let \((S, \Omega, \mu)\) be a fixed \( \sigma \)-finite measure space. We consider throughout point transformations \( T \) which have measurable square roots of all orders, that is,

\[
\sum_{k=0}^{n-1} f(T^{k/2^n} x)
\]

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\[ T_0 = T; \quad T_n = T_{n-1} \quad n = 1, 2, \cdots. \]

(3.2) If \( X \in \Omega \), then \( T_n X \in \Omega \) and \( T_{n-1} X \in \Omega \), \( n = 0, 1, \cdots. \)

No requirement is made of the uniqueness of the sequence \( T_n \). For example in the theorem of Jessen, \( T \) is the identity transformation while \( T_n x = x + 2^{-n} \) (mod 1). We also suppose throughout that \( T \) is measure preserving

(3.3) \( \mu(TX) = \mu(X) \) for \( X \in \Omega. \)

3. Limit theorems. Let \( \Phi \) be a finite valued set function defined on \( \Omega \) and absolutely continuous with respect to \( \mu \). Form the sums

(4) \( \Phi_n(X) = \sum_{k=0}^{2^n-1} \Phi(T_k^n X) \quad n = 0, 1, \cdots, \)

and

(5) \( \mu_n(X) = \sum_{k=0}^{2^n-1} \mu(T_k^n X) \quad n = 0, 1, \cdots. \)

Then \( \Phi_n \) is absolutely continuous with respect to \( \mu_n \) and there exists an averaging sequence of point functions \( f_n(x) \) so that

(2) \( \Phi_n(X) = \int_X f_n(x) \mu_n(dx), \quad n = 0, 1, \cdots. \)

**Theorem 1.** Let \( T \) be a transformation such that (3.1), (3.2) and (3.3) are satisfied. Let \( \Phi \) be a finite valued set function defined on \( \Omega \), absolutely continuous with respect to \( \mu \) and such that \( \Phi(TX) = \Phi(X) \). Then for almost all \( x \in [\mu] \) the averaging sequence of point functions defined by (4), (5) and (6) has a limit as \( n \to \infty \). The limit function \( F(x) \) has the following properties:

(i) \( F(T_n x) = F(x) \) almost everywhere \([\mu]\), \( n = 0, 1, \cdots. \)

(ii) \( F(x) \) is integrable over \( S. \)

(iii) For any set \( X \) with \( T_n X = X, n = 0, 1, \cdots \) and \( \mu(X) < \infty \)

\[ \int_X F(x) \mu(dx) = \int_X f(x) \mu(dx). \]

**Proof.** Note first that since \( \Phi(TX) = \Phi(X) \),

(7) \( \Phi_n(T_n X) = \sum_{k=0}^{2^n-1} \Phi(T_k^{n+1} X) = \Phi(X). \)

Likewise
(8) \( \mu_n(T_nX) = \mu_n(X) \).

Therefore for all \( X \)
\[
\int_X f_n(T_nx) \mu_n(dx) = \int_{T_nX} f_n(x) \mu_n(dx) = \int_X f_n(x) \mu_n(dx)
\]
and consequently
\[
(9) \quad f_n(T_nx) = f_n(x) \quad \text{almost everywhere} \ [\mu_n].
\]

Relation (3.1) then implies
\[
\begin{cases}
\lim_{n \to \infty} f_n(T_nx) = \lim_{n \to \infty} f_n(x) \\
\lim_{n \to \infty} f_n(T_n^m x) = \lim_{n \to \infty} f_n(x) \quad \text{almost everywhere} \ [\mu] \quad j=1, \ldots, 2^m-1 \\
m=1, 2, \ldots
\end{cases}
\]

Let
\[
A = \{x | \sup_{a \leq b} f_n(x) > 0\}.
\]

It is asserted that
\[
(10) \quad \int_A f_n(x) \mu(dx) \geq 0.
\]

We define the following sets:
\[
P_j = \{x | f_j(x) \geq 0\} \quad j=0, 1, \ldots
\]
\[
A_N = \{x | \sup_{0 \leq j \leq N} f_n(x) \geq 0\} \quad N=0, 1, \ldots
\]
\[
C_{N, j} = P_N \cap \cdots \cap P_{j+1} \cap P_j \quad j=0, \ldots, N.
\]

Now (9) together with (3.1) imply that \( T_n P_j = P_j \) for \( k \leq j \). Consequently
\[
T_j C_{N, j} = C_{N, j} \quad \text{and} \quad \Phi(C_{N, j}) = \Phi(T_j^N C_{N, j}).
\]

Therefore
\[
2^j \Phi(C_{N, j}) = \sum_{k=0}^{j-1} \Phi(T_j^k C_{N, j}) = \Phi_j(C_{N, j})
\]
and
\[
2^j \Phi(C_{N, j}) = \int_{C_{N, j}} f_j(x) \mu_j(dx) \geq 0, \quad j=0, \ldots, N.
\]

Since the \( C_{N, j} \) are disjoint for \( j=0, \ldots, N \), we have \( \Phi(A_N) \geq 0 \) and by a limiting process we obtain (12).
Likewise if
\begin{equation}
B = \{x|\inf_{0 \leq n} f_n(x) \geq 0\},
\end{equation}
then
\begin{equation}
\int f_n(x) \mu(dx) \geq 0.
\end{equation}

Inasmuch as the preceding argument made no use of the finiteness of \( \Phi \), we may apply the result to the set function \( \Psi = \Phi - c\mu \) for any real \( c \). Since
\begin{equation}
\Psi_n(X) = \int (f_n(x) - c) \mu_n(dx)
\end{equation}
we deduce that for
\begin{equation}
A^c = \{x|\sup_{0 \leq n} f_n(x) \geq c\}
\end{equation}
we have
\begin{equation}
\Phi(A^c) \geq c \mu(A^c)
\end{equation}
and for
\begin{equation}
A_d = \{x|\inf_{0 \leq n} f_n(x) \leq d\}
\end{equation}
we have
\begin{equation}
\Phi(A_d) \leq d \mu(A_d).
\end{equation}

Let now for \( r > s \)
\begin{equation}
L^r_s = \{x|\lim_{n \to \infty} f_n(x) > r \text{ and } \lim_{n \to \infty} f_n(x) < s\}.
\end{equation}

From (10) we obtain
\begin{equation}
T^j_n L^r_s = L^r_s, \quad j = 0, 1, \cdots, 2^m - 1; \quad m = 0, 1, \cdots.
\end{equation}

Since \( L^r_s \) is invariant under each \( T_m \) we may consider it as a new space. The sets \( A^r \) and \( A_s \) relative to the new space are now the full space \( L^r \). Hence if we apply (16) and (18) we obtain
\begin{equation}
\Phi(L^r_s) \geq r \mu(L^r_s); \quad \Phi(L^r_s) \leq s \mu(L^r_s).
\end{equation}

The finiteness of \( \Phi \) together with the assumption \( r > s \) implies \( \mu(L^r_s) = 0 \). Thus \( \lim_{n \to \infty} f_n(x) \) exists almost everywhere \([\mu]\).

Property (i) of the limit function \( F(x) \) follows immediately from (10). Utilizing (i) the proofs of (ii) and (iii) are now identical with
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In the ergodic case.

The theorem for abstract Riemann sums analogous to the Hopf ergodic theorem is now deducible as a corollary.

**Corollary 1.** Let $T$ be a transformation such that (3.1) and (3.2) are satisfied and in addition

$$(21) \quad \mu(T_n X) = \mu(X) \quad n = 0, 1, \ldots.$$ 

Then for any integrable $f(x)$ with $f(Tx) = f(x)$ and any $g(x) > 0$ with $g(Tx) = g(x)$

$$
\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} f(T_k x)}{\sum_{k=0}^{n-1} g(T_k x)}
$$

exists for almost every $x [\mu]$. The limit function $h(x)$ is integrable, satisfies $h(T_n x) = h(x)$ for almost all $x [\mu]$, and for sets $Y$ with $\mu(Y) < \infty$ and $T_m Y = Y$, $m = 0, 1, \ldots$

$$(23) \quad \int_Y h(x)g(x)\mu(dx) = \int_Y f(x)\mu(dx).$$

**Proof.** Introduce the measure

$$\nu(X) = \int_X g(x)\mu(dx),$$

and the set function

$$F(X) = \int_X f(x)\mu(dx).$$

The function $F$ is absolutely continuous with respect to $\nu$ and is finite valued. Condition (21) implies that

$$F_n(X) = \int_X \sum_{k=0}^{n-1} f(T_k x)\mu(dx)$$

and

$$\nu_n(X) = \int_X \sum_{k=0}^{n-1} g(T_k x)\mu(dx).$$

Thus from the representation

$$F_n(X) = \int_X f_n(x)\nu_n(dx)$$
we deduce that
\[ f_n(x) = \frac{\sum_{k=0}^{2^n-1} f(T_{n,x}^k)}{\sum_{k=0}^{2^n-1} g(T_{n,x}^k)} \text{ almost everywhere } [\mu]. \]

The corollary is then an immediate consequence of Theorem 1.

The theorem of Jessen now follows from the version of Corollary 1 with \( g(x) = l \) with the \( T_n \) as noted in §2.

4. Invariant measure and two operators. It is possible for the conclusion of Corollary 1 to hold when \( g(x) = l \) but \( T \) does not satisfy (21). If we introduce

\[ R_n(A, Y) = 2^{-n} \sum_{k=0}^{2^n-1} \mu(Y \cap T_{n}^{-k}A) \]

we obtain the following theorem.

**Theorem 2.** If \( T \) is a transformation such that (3.1) and (3.2) are satisfied, then the following statements are equivalent:

(25.1) For every integrable \( f(x) \) with \( f(Tx) = f(x) \),

\[ \lim_{n \to \infty} 2^{-n} \sum_{k=0}^{2^n-1} f(T_{n,x}^k) \]

exists for almost every \( x \) [\( \mu \)].

(25.2) For each \( Y \) with \( \mu(Y) < \infty \), \( \lim_{n \to \infty} R_n(A, Y) \leq K \mu(A) \).

(25.3) For each \( Y \) with \( \mu(Y) < \infty \), \( \lim_{n \to \infty} R_n(A, Y) \leq K \mu(A) \).

(25.4) For an increasing sequence of sets \( Y_j \) with \( \bigcup_{j=1}^{\infty} Y_j = S \),

\[ \lim_{n \to \infty} R_n(A, Y_j) \leq K \mu(A) . \]

(25.5) There exists a countably additive measure \( \nu \) with the properties:

(i) \( 0 \leq \nu(X) \leq K \mu(X) \)

(ii) If \( A = T_n A \), \( n = 1, 2, \ldots \), \( \nu(A) = \mu(A) \)

(iii) \( \nu(A) = \nu(T_n A) \), \( n = 1, 2, \ldots \).

The proof is almost identical with that of Ryll-Nardzewski [7] in
the ergodic case, and is omitted. The existence of an invariant measure implies, as in the ergodic case [2], the following theorem with two operators (or two sequences of roots of the same operator).

**Theorem 3.** Let $T$ and $U$ each satisfy (3.1), (3.2), (3.3) and (25.1), and let

$$\sum_{k=0}^{2^{n-1}} \mu(T_n^k X)$$

be absolutely continuous with respect to

$$\mu_n(X) = \sum_{k=0}^{2^{n-1}} \mu(U_n^k X), \quad n=0, 1, \ldots$$

For any finite valued set function $\Phi$ absolutely continuous with respect to $\mu$ and with $\Phi(TX) = \Phi(X)$ form

$$\Phi_n(X) = \sum_{k=0}^{2^{n-1}} \Phi(T_n^k X).$$

Then in the representation

$$\Phi_n(X) = \int_X f_n(x) \mu_n(dx),$$

the averaging sequence of point functions $f_n(x)$ tends to a limit as $n \to \infty$ for almost every $x [\mu]$.

As a consequence of Theorem 3 we obtain the following corollary in the same fashion as Corollary 1 was derived from Theorem 1.

**Corollary 2.** Let $T$ and $U$ each satisfy (3.1) and (3.2), and in addition

$$\mu(V_n X) = \mu(X) \quad n=0, \ldots$$

for $V=T$ and $V=U$. Then for any integrable $f(x)$ with $f(Tx) = f(x)$ and any $g(x) > 0$ with $g(Ux) = g(x)$

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{2^{n-1}} f(T_n^k X)}{\sum_{k=0}^{2^{n-1}} g(U_n^k X)}$$

exists for almost all $x [\mu]$. 


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