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WEAK LOCALLY MULTIPLICATIVELY-CONVEX ALGEBRA

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Let E be an algebra over the reals or complex numbers, E' a total subspace of the algebraic dual E^* of vector space E . We first discuss the following natural questions: When is the weak topology $\sigma(E, E')$ defined on E by E' locally m -convex? When is multiplication continuous for $\sigma(E, E')$, that is, when is $\sigma(E, E')$ compatible with the algebraic structure of E ? We then apply our results to certain weak topologies on the algebra of polynomials in one indeterminate without constant term.

1. Weak topologies.

Let K be either the reals or complex numbers, E a K -algebra. A topology \mathcal{T} on E is *locally multiplicatively-convex* (which we abbreviate henceforth to “locally m -convex”) if it is a locally convex topology and if there exists a fundamental system of idempotent neighborhoods of zero (a subset A of E is idempotent if $A^2 \subseteq A$). Multiplication is then clearly continuous at $(0, 0)$ and hence everywhere, so \mathcal{T} is compatible with the algebraic structure of E . If A is idempotent, so is its convex envelope, its equilibrated envelope (a subset V of E is called equilibrated if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$), and its closure for any topology on E compatible with the algebraic structure of E . Hence if \mathcal{T} is locally m -convex, zero has a fundamental system of convex, equilibrated, idempotent, closed neighborhoods. (For proofs of these and other elementary facts about locally m -convex algebras, see §§ 1-3 of [8] or [1].) Henceforth, E' is a total subspace of the algebraic dual of E .

LEMMA 1. *Let W be a weak, equilibrated neighborhood of zero (that is, for the topology $\sigma(E, E')$), J a subspace of E , and $g \in E'$ such that $J \subseteq W \subseteq W \cup W^2 \subseteq \{g\}^0$. Then J , JE , and EJ are contained in the kernel of g .*

Proof. Let $x \in J$, $y \in E$. As W is equilibrated and absorbing, let $\lambda > 0$ be such that $\lambda y \in W$. For all positive integers m , $\lambda^{-1}mx \in J$, and therefore $mxy = (\lambda^{-1}mx)(\lambda y) \in JW \subseteq W^2 \subseteq \{g\}^0$; hence $|g(mxy)| \leq 1$ for all positive integers m , and therefore $g(xy) = 0$. Hence JE is contained in

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the kernel of g . Similarly for EJ . Also $|g(mx)| \leq 1$ for all $x \in J$ and all positive integers m , and therefore $g(x) = 0$ for all $x \in J$.

LEMMA 2. *Let V be a weak neighborhood of zero. Then $L = \bigcap [u^{-1}(0) \mid u \in V^0]$ is a weakly closed subspace of finite codimension.*

Proof. L is clearly a weakly closed subspace. By definition of $\sigma(E, E')$ there exist h_1, h_2, \dots, h_n in E' such that $\{h_1, h_2, \dots, h_n\}^0 \subseteq V$. Thus if $|h_i(z)| \leq 1$ for $1 \leq i \leq n$, then $|u(z)| \leq 1$ for all $u \in V^0$. Then if $x \in \bigcap_{i=1}^n h_i^{-1}(0)$, for any positive integer m $|h_i(mx)| = 0 < 1$ for $1 \leq i \leq n$ and hence $|u(mx)| \leq 1$, so $u(x) = 0$ for all $u \in V^0$. Hence $\bigcap_{i=1}^n h_i^{-1}(0) \subseteq L$. Since the codimension of $\bigcap_{i=1}^n h_i^{-1}(0)$ is at most n , so also the codimension of L is at most n .

LEMMA 3. *Let E_1, E_2, \dots, E_n be finite-dimensional, Hausdorff topological K -vector spaces, F a topological K -vector space. Any multilinear transformation from $E_1 \times E_2 \times \dots \times E_n$ into F is continuous.*

Proof. This lemma is well known, and follows from Theorem 2 of [3, p. 27] just as Corollary 2 of that theorem does.

THEOREM 1. *$\sigma(E, E')$ is a locally m -convex topology on E if and only if for all $g \in E'$, the kernel of g contains a weakly closed ideal of finite codimension.*

Proof. Necessity: Let $g \in E'$. Let V be a weakly closed, convex, equilibrated, idempotent neighborhood of zero such that $V \subseteq \{g\}^0$. Let $L = \bigcap [u^{-1}(0) \mid u \in V^0]$. Then clearly $L \subseteq V^{00}$, but since V is weakly closed, convex, and equilibrated, $V^{00} = V$ (see [4]). By Lemma 2 L is a weakly closed subspace of finite codimension. We assert L is an ideal: Let $x \in L$, $y \in E$. Choose $\lambda > 0$ such that $\lambda y \in V$. For all positive integers m , $\lambda^{-1}mx \in L$; hence $mxy = (\lambda^{-1}mx)(\lambda y) \in LV \subseteq V^2 \subseteq V$. Hence for all positive integers m and any $u \in V^0$, $|u(mxy)| \leq 1$; hence $u(xy) = 0$ for all $u \in V^0$, so $xy \in L$. Similarly $yx \in L$, so L is an ideal. Now let $J = L \cap g^{-1}(0)$. Then J is a weakly closed subspace of finite codimension contained in the kernel of g . It remains to show J is an ideal. Now $J \subseteq L \subseteq V = V \cup V^2 \subseteq \{g\}^0$; hence by Lemma 1 $JE \subseteq g^{-1}(0)$ and $EJ \subseteq g^{-1}(0)$. Also $JE \subseteq LE \subseteq L$ and $EJ \subseteq EL \subseteq L$. Therefore $JE \subseteq L \cap g^{-1}(0) = J$ and $EJ \subseteq L \cap g^{-1}(0) = J$, so J is an ideal.

Sufficiency: It clearly suffices to show that for all $g \in E'$ there

exists an idempotent neighborhood V of zero such that $V \subseteq \{g\}^0$. Let J be a weakly closed ideal of finite codimension contained in $g^{-1}(0)$. Then $F = E/J$ is a finite-dimensional algebra with a Hausdorff topology compatible with the vector space structure of F . Multiplication is a bilinear transformation from $F \times F$ into F , and hence by Lemma 3 multiplication is continuous. But also, any finite-dimensional, Hausdorff, K -vector space has its topology defined by a norm (this follows from Theorem 2 of [3, p. 27]); and by a familiar property of normed spaces with a continuous multiplication, the norm may be so chosen that F is a normed algebra [6, p. 50]. Let φ be the continuous canonical homomorphism from E onto F , and let $g = \bar{g} \circ \varphi$. \bar{g} is continuous on F , so we may select an idempotent neighborhood U of zero in F such that $v \in U$ implies $|\bar{g}(v)| \leq 1$. Then $V = \varphi^{-1}(U)$ is a neighborhood of zero for $\sigma(E, E')$. As U is idempotent and φ a homomorphism, V is idempotent. Finally, if $x \in V$ then $\varphi(x) \in U$, and therefore $|g(x)| = |\bar{g}(\varphi(x))| \leq 1$, so $x \in \{g\}^0$; hence $V \subseteq \{g\}^0$, and the theorem is completely proved.

THEOREM 2. *Multiplication in E is continuous for $\sigma(E, E')$ if and only if for all $g \in E'$, the kernel of g contains a weakly closed subspace J of finite codimension such that JE and EJ are also contained in the kernel of g .*

Proof. Necessity: Let $g \in E'$. Then since $\{g\}^0$ is a neighborhood of zero, we may choose a weakly closed, convex, equilibrated neighborhood W of zero such that $W \cup W^2 \subseteq \{g\}^0$. Let $L = \bigcap \{u^{-1}(0) \mid u \in W^0\}$. Then clearly $L \subseteq W^{00} = W$, since W is weakly closed, convex, and equilibrated. By Lemma 2 L is a weakly closed subspace of finite codimension. Let $J = L \cap g^{-1}(0)$. Then J is also a weakly closed subspace of finite codimension contained in the kernel of g . Also $J \subseteq L \subseteq W \subseteq W \cup W^2 \subseteq \{g\}^0$, so by Lemma 1, JE and EJ are contained in the kernel of g .

Sufficiency: It suffices to show that for any $g \in E'$ and any $a \in E$, there exist neighborhoods W and V of zero in E such that $W^2 \subseteq \{g\}^0$ and $Va \cup aV \subseteq \{g\}^0$ ([5, p. 49]). Let $I = g^{-1}(0)$ and let J be a weakly closed subspace of finite codimension contained in I such that $EJ \subseteq I$ and $JE \subseteq I$. Let φ and ψ respectively be the canonical maps from E onto E/J and from E onto E/I . Let $g = \bar{g} \circ \psi$. We assert the map $(\varphi(x), \varphi(y)) \rightarrow \psi(xy)$ is a well-defined bilinear map from $(E/J) \times (E/J)$ into E/I : If $x - x' \in J$ and $y - y' \in J$, then $xy - x'y \in JE \subseteq I$ and $x'y - x'y' \in EJ \subseteq I$; hence $xy - x'y' = (xy - x'y) + (x'y - x'y') \in I + I = I$. The map is therefore well-defined; bilinearity is easily seen. Both (E/J) and (E/I) are finite-dimensional Hausdorff topological K -vector spaces, so by

Lemma 3 the above bilinear map is continuous. Hence there exists a neighborhood U of zero in E/J such that if $\varphi(x), \varphi(y) \in U$, then $\psi(xy) \in \{\bar{g}\}^0$. If $W = \varphi^{-1}(U)$, then W is a neighborhood of zero for $\sigma(E, E')$; if $x, y \in W$, then $\varphi(x), \varphi(y) \in U$ and hence $|g(xy)| = |\bar{g}(\psi(xy))| \leq 1$, so $xy \in \{g\}^0$. Thus $W^2 \subseteq \{g\}^0$. Now let $a \in E$. We assert the maps $\varphi(x) \rightarrow \psi(ax)$ and $\varphi(x) \rightarrow \psi(xa)$ are well-defined, linear maps from E/J into E/I : For if $x - x' \in J$, then $ax - ax' \in EJ \subseteq I$ and $xa - x'a \in JE \subseteq I$, so the maps are well-defined. Linearity is immediate. Since E/J and E/I are finite dimensional and Hausdorff, again by Lemma 3 these maps are continuous. Hence we may choose a neighborhood P of zero in E/J such that if $\varphi(x) \in P$ then $\psi(ax), \psi(xa) \in \{\bar{g}\}^0$. Then $V = \varphi^{-1}(P)$ is a neighborhood of zero for $\sigma(E, E')$. If $x \in V$, then $\varphi(x) \in P$ and hence $|g(ax)| = |\bar{g}(\psi(ax))| \leq 1$ and similarly $|g(xa)| \leq 1$. Hence $aV \cup Va \subseteq \{g\}^0$, and the theorem is completely demonstrated.

Here is an example of a Banach algebra E with topological dual E' such that multiplication is not continuous for the associated weak topology $\sigma(E, E')$. Let E be the algebra of all continuous functions from the compact interval $[0, 1]$ into K with the uniform topology. If $\mu(f) = \int_0^1 f(t) dt$ (dt is the usual Lebesgue complex-valued measure if K is the complex numbers), then $\mu \in E'$. But μ does not satisfy the restrictions of Theorem 2: Let J be any weakly closed subspace contained in the kernel of μ such that $JE \subseteq \mu^{-1}(0)$. If $f \in J$, then $f\bar{f} \in JE \subseteq \mu^{-1}(0)$ ($\bar{f} = f$ if K is the reals); hence $\int_0^1 |f(t)|^2 dt = 0$ and so, since f is continuous, $f = 0$. Therefore $J = \{0\}$. But since E is infinite-dimensional, J is not of finite codimension. Hence by Theorem 2, multiplication is not continuous for $\sigma(E, E')$.

2. Algebras of polynomials. If E is any locally m -convex algebra, E' its topological dual, $\mathcal{M}(E)$ is the set of all continuous multiplicative linear forms, $\mathcal{M}^-(E)$ the set of all nonzero continuous multiplicative linear forms. $\mathcal{M}(E)$ and $\mathcal{M}^-(E)$ are topologized as subsets of E' ; $\sigma(E', E)$.

In [9] Šilov proved the following theorems:

(1) If E is a normed C -algebra (C is the complex numbers) with identity e , generated by e and another element x (that is, if all elements of E are of form $\alpha_0 e + \alpha_1 x + \dots + \alpha_n x^n$), then $\mathcal{M}^-(E)$ is homeomorphic with a compact subset of C whose complement is connected; (2) every such subset of C arises in this manner.

Here we give elementary analogues of these theorems for locally m -convex algebras.

Proposition 1. If E is a locally m -convex Hausdorff algebra generated by a single element x , then $f \rightarrow f(x)$ is a homeomorphism from $\mathcal{L}(E)$ onto a subset of K .

Proof. The map is surely continuous and is one-to-one since x generates E . To show $f(x) \rightarrow f$ is continuous, it suffices to show $f(x) \rightarrow f(z)$ is continuous for all $z \in E$; but as x generates E it suffices for this to show $f(x) \rightarrow f(x^n)$ is continuous for all positive integers n . But $f(x^n) = f(x)^n$, so $f(x) \rightarrow f(x^n)$ is simply a restriction of the map $\lambda \rightarrow \lambda^n$ from K into K , which is surely a continuous map. Hence $f \rightarrow f(x)$ is a homeomorphism into K .

Proposition 2. Let E be an algebra over any field F . The set M of nonzero multiplicative linear forms is a linearly independent subset of E^* , the algebraic dual of E .

Proof. In Theorem 12 of [2, p. 34], Artin proves that if G is a group, F a field, then the set of all nonzero homomorphisms from G into the multiplicative semi-group of F is a linearly independent subset of the vector space $\mathcal{S}(G, F)$ of all functions from G into F . The proof remains valid if "semi-group" replaces "group" in the statement of the theorem, and thus modified the theorem may be applied to the multiplicative semi-group of an algebra to yield the desired result.

Henceforth, $K[X]$ is the K -algebra of all polynomials in one indeterminate, E the subalgebra of those without constant term. $K[X]$ has a base $\{e_i\}_{i=0}^{\infty}$ with multiplication table $e_i e_j = e_{i+j}$; $\{e_i\}_{i=1}^{\infty}$ is a base for E . For $\lambda \in K$ we let f_λ be the linear form defined on E by: $f_\lambda(e_j) = \lambda^j$. Also for every positive integer i , g_i is the linear form defined on E by: $g_i(e_i) = 1$, $g_i(e_j) = 0$ for $j \neq i$.

LEMMA 4. The set of all multiplicative linear forms on E is $[f_\lambda \mid \lambda \in K]$.

Proof. $f_\lambda(e_j e_k) = f_\lambda(e_{j+k}) = \lambda^{j+k} = \lambda^j \lambda^k = f_\lambda(e_j) f_\lambda(e_k)$. This suffices to show f_λ is multiplicative. Conversely, if f is any multiplicative linear form, let $\lambda = f(e_1)$. Then for any positive integer i , $f(e_i) = f(e_1^i) = f(e_1)^i = \lambda^i$. Hence $f = f_\lambda$.

LEMMA 5. $\{f_\lambda\}_{\lambda \in K, \lambda \neq 0} \cup \{g_i\}_{i=1}^{\infty}$ is a linearly independent subset of E^* .

Proof. Suppose $\sum_{i=1}^n \alpha_i g_i + \sum_{j=1}^p \beta_j f_{\lambda_j} = 0$, where the λ_j are distinct from each other and all different from zero. Then for $m > n$, $g_i(e_m) = 0$

for $1 \leq i \leq n$, so $\sum_{j=1}^p \beta_j f_{\lambda_j}(e_n) = 0$. The subspace of E generated by $\{e_j\}_{j=n+1}^\infty$ is clearly a subalgebra; the restrictions of the f_{λ_j} , $1 \leq j \leq p$, to this algebra are again clearly distinct from each other and different from zero. Hence by Proposition 2 applied to this subalgebra, all $\beta_j = 0$. Hence $\sum_{i=1}^n \alpha_i g_i = 0$; but $\alpha_i = \alpha_i g_i(e_i) = \sum_{j=1}^n \alpha_j g_j(e_i) = 0$, so the lemma is proved.

LEMMA 6. *Let $\{\lambda_i\}_{i=1}^\infty$ be a denumerable family of distinct nonzero elements of K . Then $\{f_{\lambda_i}\}_{i=1}^\infty$ separates the points of E .*

Proof. For $\lambda \neq 0$, each f_λ has a unique extension to a multiplicative linear form on $K[X]$ obtained by setting $f_\lambda(e_0) = 1$. Let $x = \sum_{i=1}^n \alpha_i e_i \in E$. Then $x = \sum_{i=0}^n \alpha_i e_i$ in $K[X]$ where $\alpha_0 = 0$. Suppose $f_{\lambda_j}(x) = 0$ for $1 \leq j \leq n+1$. Then $\sum_{i=0}^n \alpha_i \lambda_i^j = 0$ for $1 \leq j \leq n+1$. But the determinant of the system of linear equations $\sum_{i=0}^n \zeta_i \lambda_i^j = 0$, $1 \leq j \leq n+1$, is

$$\begin{vmatrix} \lambda_1^0 & \lambda_1^1 & \cdots & \lambda_1^n \\ \lambda_2^0 & \lambda_2^1 & \cdots & \lambda_2^n \\ \vdots & \vdots & & \vdots \\ \lambda_{n+1}^0 & \lambda_{n+1}^1 & \cdots & \lambda_{n+1}^n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n+1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j \leq n+1} (\lambda_i - \lambda_j) \neq 0$$

(this is the Vandermonde determinant). Hence the above system of linear equations has only the trivial solution, and therefore $\alpha_i = 0$, $0 \leq i \leq n$, and hence $x = 0$. Thus the proof is complete.

Proposition 3. *If L is any subset of K containing zero, there is a Hausdorff, weak locally m -convex topology \mathcal{T} on E such that the canonical map $f_\lambda \rightarrow \lambda$ maps $\mathcal{N}(E)$ homeomorphically onto L . Further if L is an infinite set, \mathcal{T} may be so chosen that the completion of E ; \mathcal{T} is semi-simple; and if L is denumerable, \mathcal{T} is metrizable.*

Proof. *Case 1:* L is finite. Let $M = [f_\lambda \mid \lambda \in L]$, and let E' be the subspace of E^* generated by $\{g_i\}_{i=1}^\infty \cup M$. Clearly E' is a total subspace of E^* , and so, as E' has a denumerable linear base, $\sigma(E, E')$ is a metrizable weak topology on E . To show $\sigma(E, E')$ is locally m -convex, it clearly suffices to show that the condition of Theorem 1 holds for all members of a base of E' . The condition holds trivially for all $u \in M$, since the kernel of $u \in M$ is already a weakly closed ideal. Consider any g_i : The linear subspace generated by $\{e_j\}_{j=i+1}^\infty$ is clearly of finite codimension, and the multiplication table shows that it is actually an ideal. Further, it is identical with $\bigcap_{k=1}^i g_k^{-1}(0)$ and thus is

weakly closed and contained in the kernel of g_i . Hence by Theorem 1, $\sigma(E, E')$ is locally m -convex. By Lemma 5 the set of all multiplicative linear forms in E' is M . As the topological dual of E ; $\sigma(E, E')$ is E' (see [7]), M is the set of all continuous multiplicative linear forms on E ; $\sigma(E, E')$, and by Proposition 1 applied to $x=e_1$, M is homeomorphic with L .

Case 2: L is infinite. Again let $M=[f_\lambda | \lambda \in L]$, and let E' be the subspace of E^* generated by M . By Lemma 6, E' is total. The condition of Theorem 1 is trivially satisfied by E' , so $\sigma(E, E')$ is a Hausdorff, weak locally m -convex topology on E . If L is denumerable, E' has a countable base and so $\sigma(E, E')$ is metrizable. M is again the set of all continuous multiplicative linear forms on E ; $\sigma(E, E')$ and is homeomorphic with L . The completion of E for this topology is E'^* ([7]), and as M generates E' , M separates the points of E'^* ; thus the completion of E for this topology is semi-simple by Corollary 5.5 of [8].

It is easy to see that E has no divisors of zero and that zero is the only element having an adverse; thus the Jacobson radical is $\{0\}$ and E is semi-simple. If, in Proposition 3, $L=\{0\}$ and the scalar field is the complex numbers, E is a commutative, metrizable locally m -convex algebra with no continuous nonzero multiplicative linear forms; the completion \hat{E} of E then has no continuous nonzero multiplicative linear forms and hence by Corollary 5.5 of [8] is a radical algebra. Thus we have an example of a semi-simple metrizable algebra whose completion is a radical algebra. This phenomenon is also known even for normed algebras. For example, an elementary calculation shows the following is a norm on E :

$$\left\| \sum_{n=1}^m \alpha_n e_n \right\| = \sum_{n=1}^m \frac{|\alpha_n|}{n!}.$$

$\|(m-1)!e_m\|=1/m \rightarrow 0$, so $(m-1)!e_m \rightarrow 0$ for this norm topology. But for any $\lambda \neq 0$, $|f_\lambda((m-1)!e_m)|=(m-1)!|\lambda|^m \rightarrow \infty$, so f_λ is not continuous. Hence E has no continuous nonzero multiplicative linear forms and so, assuming the scalar field is the complex numbers, the completion of E for this norm is a radical algebra.

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