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ON A THEOREM OF S. BERNSTEIN

N. C. ANKENY AND T. J. RIVLIN

1. Introduction and proof of the main theorem. A result of S. Bernstein [4] is the following.

THEOREM A. *If $p(z)$ is a polynomial of degree n such that* $[\max|p(z)|,\ |z|=1]=1$, *then*

$$[\max|p(z)|,\ |z|=R>1]\le R^n, \tag{1}$$

with equality only for $p(z)=\lambda z^n$, *where* $|\lambda|=1$.

We propose to show here that if we restrict ourselves to polynomials of degree n having no zero within the unit circle the right hand member of (1) can be made smaller. In particular we have the following result.

THEOREM 1. *If $p(z)$ is a polynomial of degree n such that* $[\max|p(z)|,\ |z|=1]=1$, *and $p(z)$ has no zero within the unit circle, then*

$$[\max|p(z)|,\ |z|=R>1]\le \frac{1+R^n}{2},$$

with equality only for $p(z)=(\lambda+\mu z^n)/2$, *where* $|\lambda|=|\mu|=1$.

In order to prove Theorem 1 we use a conjecture of Erdös first proved by Lax [2] (See also [1]).

THEOREM B. *If $p(z)$ is a polynomial of degree n such that* $[\max|p(z)|,\ |z|=1]=1$, *and $p(z)$ has no zero within the unit circle, then*

$$[\max|p'(z)|,\ |z|=1]\le \frac{n}{2}.$$

Turning now to Theorem 1, let us assume that $p(z)$ does not have the form $(\lambda+\mu z^n)/2$. In view of Theorem B

$$|p'(e^{i\varphi})|\le \frac{n}{2}, \qquad 0\le\varphi<2\pi, \tag{2}$$

from which we may deduce that

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(3) $$|p'(re^{i\varphi})|<\frac{n}{2}r^{n-1}, \qquad 0\leq\varphi<2\pi, \qquad r>1,$$

by applying Theorem A to the polynomial $p'(z)/(n/2)$ and observing that we have the strict inequality in (3) because $p(z)$ does not have the form $(\lambda+\mu z^n)/2$. But for each φ, $0\leq\varphi<2\pi$, we have

$$p(Re^{i\varphi})-p(e^{i\varphi})=\int_1^R e^{i\varphi}p'(re^{i\varphi})dr.$$

Hence

$$|p(Re^{i\varphi})-p(e^{i\varphi})|\leq\int_1^R |p'(re^{i\varphi})|dr<\frac{n}{2}\int_1^R r^{n-1}dr=\frac{R^n-1}{2},$$

and

$$|p(Re^{i\varphi})|<\frac{R^n-1}{2}+|p(e^{i\varphi})|\leq\frac{1+R^n}{2}.$$

Finally, if $p(z)=(\lambda+\mu z^n)/2$, $|\lambda|=1$, then

$$[\max|p(z)|,\ |z|=R>1]=\frac{1+R^n}{2}.$$

As a corollary of Theorem 1 we may deduce

THEOREM 2. *If $p(z)$ is a polynomial of degree n with real coefficients having all zeros of nonpositive real part and if for some $R>1$*

$$p(R)>p(1)\left(\frac{R^k+R^n}{2}\right),$$

k a nonnegative integer, then $p(z)$ has at least $(k+1)$ zeros in $|z|<1$.

Proof. Suppose $p(z)$ has m zeros in $|z|<1$ and $m\leq k$. Let

$$p(z)=(z-z_1)\cdots(z-z_m)(z-z_{m+1})\cdots(z-z_n),$$

and suppose $|z_j|<1$, $(j=1,\cdots,m)$. Put

$$g(z)=(z-z_1)\cdots(z-z_m)$$

and

$$h(z)=(z-z_{m+1})\cdots(z-z_n).$$

The polynomials $p(z)$, $g(z)$ and $h(z)$ have positive coefficients, hence for all $R>1$

$$g(R)\leq g(1)R^m$$

and

$$h(R) \leq h(1)\left(\frac{1+R^{n-m}}{2}\right)$$

according to Theorems A and 1 respectively.
Thus

$$p(R)=h(R)g(R) \leq p(1)\left(\frac{R^m+R^n}{2}\right) \leq p(1)\left(\frac{R^k+R^n}{2}\right),$$

a contradiction, establishing Theorem 2.

2. The converse problem. The converse of Theorem 1 is false as the simple example $p(z)=(z+\frac{1}{2})(z+3)$ shows. However, the following result in the converse direction is valid.

THEOREM 3. *If $p(z)$ is a polynomial of degree n such that*

$$p(1)=[\max|p(z)|,\ |z|=1]=1$$

and

$$[\max|p(z)|,\ |z|=R>1] \leq \frac{1+R^n}{2}$$

for $0<R-1<\delta$, where δ is any positive number, then $p(z)$ does not have all its roots within the unit circle.

For the proof we need the following

LEMMA. *If*

$$q(z)=(z-z_1)\cdots(z-z_m)$$

where $|z_j|<1$, $(j=1, \cdots, m)$, then if $|a|=1$ we have

$$\left|\frac{q'(a)}{q(a)}\right|>\frac{m}{2}.$$

Proof. According to Laguerre's Theorem [3, p. 38]

$$\frac{q'(a)}{q(a)}=\frac{m}{a-w},$$

where $|w|<1$, hence $|a-w|<2$ and

$$\left|\frac{q'(a)}{q(a)}\right|>\frac{m}{2}.$$

We turn now to the proof of Theorem 3. Suppose $p(z)$ has all its zeros in $|z|<1$. Let

$$p(z)=a_0+a_1z+\cdots+a_nz^n,$$

put

$$\bar{p}(z)=\bar{a}_0+\bar{a}_1z+\cdots+\bar{a}_nz^n$$

and consider the polynomial $g(z)=p(z)\bar{p}(z)$ of degree $2n$. $g(z)$ is real for real z,

$$[\max|g(z)|,\ |z|=1]=g(1)=1,$$

$$|g(Re^{i\varphi})|\leq\left(\frac{1+R^n}{2}\right)^2\leq\frac{1+R^{2n}}{2}$$

and $g(z)$ has all its zeros in $|z|<1$. Now $g'(1)$ is not only real but positive. This is so since, given any $\eta>0$, we have $g(1-\eta)<g(1)$. Hence

$$g'(1)=\lim_{\eta\to 0}\frac{g(1-\eta)-g(1)}{-\eta}\geq 0.$$

Now $g'(1)\neq 0$, as all of the roots of $g(z)=0$ are inside the unit circle, hence, by Lucas' Theorem all roots of $g'(z)=0$ are within the convex closure of the unit circle namely the unit circle itself.

Given any $\varepsilon>0$, sufficiently small,

$$|g(1+\varepsilon)-g(1)|=g(1+\varepsilon)-g(1)\leq\frac{(1+\varepsilon)^{2n}+1}{2}-1=\frac{(1+\varepsilon)^{2n}-1}{2},$$

or

$$|g(1+\varepsilon)-g(1)|\leq n\varepsilon+O(\varepsilon^2), \qquad \text{as } \varepsilon\to 0$$

and $g'(1)\leq n$. Therefore $g'(1)/g(1)\leq n$ contradicting the lemma. Theorem 3 is established.

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THE JOHNS HOPKINS UNIVERSITY

THE USE OF FORMS IN VARIATIONAL CALCULATIONS

LOUIS AUSLANDER

Introduction. The purpose of this paper is to present a method of calculating the first and second variation which is suitable for spaces which have a Euclidean connection. I then use this method to calculate the first and second variations along a geodesic in a Finsler space in terms of differential invariants of the Finsler metric. In the special case of Riemannian geometry, this calculation has been carried out by Schoenberg in [4].

Indications as to how this calculation should be made are originally due to E. Cartan [1]. I wish to thank Prof. S. S. Chern for the privilege of seeing his calculations on this matter for Riemann spaces.

1. Algebraic Preliminaries. Let $I=[0, 1]$ and $0\leq\xi_1, \xi_2\leq 1$. Let M^n be an n-dimensional C^∞ manifold. Assume we have a one parameter family of mappings of I into M^n which we will denote by $f(\xi_1, \xi_2)$, where ξ_2 is taken as the parameter along I and ξ_1 parametrizes the family of mappings. Then we may define a mapping η: $I\times I\rightarrow M^n$ by the equation

$$\eta(\xi_1, \xi_2)=f(\xi_1, \xi_2).$$

We require that η shall also be a C^∞ mapping.

Let η_* denote the mapping induced by η on the tangent space to $I\times I$ into the tangent space to M^n. Let η^* denote the dual mapping induced on the cotangent spaces. Then we define two vector fields X_1 and X_2 over $\eta(I\times I)$ by

$$X_2=\eta_*(\partial/\partial\xi_2) \quad \text{and} \quad X_1=\eta_*(\partial/\partial\xi_1).$$

Then if w is any form in M^n we may write

$$\eta^*(w)=w_\delta d\xi_1+w_d d\xi_2,$$

where w_δ and w_d are defined by the equation.

LEMMA 1.1. *If $\langle X, w\rangle$ denotes the value that X takes on the covector w at each point, then*

$$w_\delta=\langle X_1, w\rangle$$

and

$$w_d=\langle X_2, w\rangle.$$

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Proof. $w_\delta = \langle \partial/\partial\xi_1, \eta^*(w)\rangle = \langle \eta^*(\partial/\partial\xi_1), w\rangle = \langle X_1, w\rangle$. The proof is analogous for w_d.

Let Ω be any two form and let X_1 and X_2 be any two vector fields. It is well known that $\Lambda^2(V)$ and $\Lambda^2(V^*)$ are dually paired. Let this pairing be denoted by

$$\langle X_1 \wedge X_2, \Omega\rangle .$$

Then if Ω can be decomposed as $w_1 \wedge w_2$, where w_1 and w_2 are one forms, we have that the pairing may be defined by the following expression:

$$\langle X_1 \wedge X_2, w_1 \wedge w_2\rangle = \langle X_1, w_1\rangle\langle X_2, w_2\rangle - \langle X_1, w_2\rangle\langle X_2, w_1\rangle .$$

THEOREM 1.1. $\langle X_1 \wedge X_2, w_1 \wedge w_2\rangle = w_{1\delta} w_{2d} - w_{1d} w_{2\delta}$.

The proof of this theorem is straightforward.

We define the symbols δw_d and dw_δ by the following equations:

$$\delta w_d = \partial/\partial\xi_1 \langle X_2, w\rangle ,$$
$$dw_\delta = \partial/\partial\xi_2 \langle X_1, w\rangle .$$

If f is any function of ξ_1 and ξ_2, we define

$$d^r \delta^s f = \frac{\partial^t f}{\partial \xi_2^r \partial \xi_1^s} ,$$

where $t = r+s$. Define $\delta^r d^s f$ similarly.

THEOREM 1.2. $\langle X_1 \wedge X_2, dw\rangle = \delta w_d - dw_\delta$.

Proof. Now, in terms of a local coordinate system $(x_1, \cdots, x_n)$,

$$\langle X_1 \wedge X_2, dw\rangle = \sum \left[\frac{\partial}{\partial\xi_1}\left(a_i \frac{\partial x_i}{\partial\xi_2}\right) - \frac{\partial}{\partial\xi_2}\left(a_i \frac{\partial x_i}{\partial\xi_1}\right)\right]$$

since

$$\sum a_i \frac{\partial^2 x_i}{\partial\xi_1 \partial\xi_2} = \sum a_i \frac{\partial^2 x_i}{\partial\xi_2 \partial\xi_1} .$$

This and the definition of δw_d and dw_δ prove the theorem.

2. The First Variation. Consider the integral

$$I = \int_a^b F(q_1, \cdots, q_n ; q_1', \cdots, q_n' ; t)dt \tag{2.1}$$

in a space M of $2n+1$ dimensions. Then in the cotangent space to the manifold M define the form w by the equation

(2.2) $$w=\sum \frac{\partial F}{\partial q_i'} dq -\left(\sum q_i' \frac{\partial F}{\partial q_i'} - F\right)dt .$$

Now let C be a curve in M^{2n+1} expressed by the equations

$$q_i=q_i(\xi_2) , \qquad q_i'=q_i'(\xi_2) , \qquad t=(b-a)\xi_2+a .$$

Assume further that $dq_i/d\xi_2=q_i'$ for all values of ξ_2. Let X_2 be the image of $\partial/\partial\xi_2$ under the mapping described above. Then

(2.3) $$X_2=\sum q_i' \frac{\partial}{\partial q_i} +\sum \frac{\partial q_i}{\partial \xi_2} \frac{\partial}{\partial q_i'} + (b-a)\frac{\partial}{\partial t} ,$$

and

$$w_d d\xi_2=F(q, q', t)\frac{dt}{(b-a)} .$$

Hence

(2.4) $$I=\int_0^1 w_d d\xi_2= \int_a^b F\big(q_1(t), \cdots, q_n(t) ; q_1'(t), \cdots, q_n'(t) ; t\big)dt .$$

Now consider a one parameter family of curves $f(\xi_1, \xi_2)$ each with the property described above. For each curve in the family we get a vector field which we will denote by $X_2(\xi_1)$. We may consider the variational problem for this family of curves. The crucial fact is that the requirement that $f(\xi_1, \xi_2)$ is a mapping of a *fixed* interval for each fixed value of ξ_1 enables us to treat the problem of variable end point without the necessity of differentiating limits of integration. We consider

$$I(\xi_1)=\int_0^1 \langle X_2(\xi_1), w\rangle d\xi_2$$

and

(2.5) $$\delta I=\frac{\partial I(\xi_1)}{\partial \xi_1}=\int_0^1 \delta w_d d\xi_2 .$$

If we add and subtract dw_δ under the integral sign we get

(2.6) $$\delta I=[w_\delta]_0^1+\int_0^1 (\delta w_d-dw_\delta)d\xi_2$$

(2.7) $$=[w_\delta]_0^1 +\int_0^1 w'(\delta, d)d\xi_2 ,$$

where

(2.8) $$w'(\delta, d)=\langle X_1\wedge X_2, dw\rangle ,$$

and

$$w'(d, \delta) = \langle X_2 \wedge X_1, dw \rangle .$$

It may be noted that $w'(\delta, d) = -w'(d, \delta)$. The term $[w_\delta]_0^1$ is called the transversality term.

THEOREM 2.1. *Assume* $[w_\delta]_0^1 = 0$. *Then a necessary and sufficient condition for* $\delta I = 0$ *for all variations is that* $dw = 0$ *along* C.

Proof. The condition is clearly sufficient. An equivalent form of the hypothesis is that

$$\int_0^1 \langle X_1 \wedge X_2, dw \rangle d\xi_2 = 0$$

for all vector fields X_1 along C. Assume dw does not equal zero along C. Then there exists an X_1 such that $\langle X_1 \wedge X_2, dw \rangle > 0$ for some open interval $a < \xi_2 < b$. Then we may choose a new vector field X_1 such that:

$$\bar{X}_1 = X_1 \quad \text{for} \quad a < \xi_2 < b$$
$$\bar{X}_1 = 0 \quad \text{for} \quad 0 \leq \xi_2 \leq a - \varepsilon \quad \text{or} \quad b + \varepsilon \leq \xi_2 \leq 1 ,$$

where ε may be chosen arbitrarily small. Then

$$\int_0^1 \langle \bar{X}_1 \wedge X_2, dw \rangle d\xi_2 = \int_a^b \langle X_1 \wedge X_2, dw \rangle d\xi_2 + \varepsilon' ,$$

where ε' depends on ε and $\lim_{\varepsilon \to 0} \varepsilon' = 0$. Hence we may choose ε in such a way that

$$\int_0^1 \langle \bar{X}_1 \wedge X_2, dw \rangle d\xi_2 > 0 .$$

This contradiction proves the theorem.

Remark: This is essentially the usual argument for the derivation of Euler's equation.

3. Application to Finsler Geometry. If we assume that our integral is of the Finsler type then we may proceed to calculate the second variation. For treating this special case we assume that the reader has a familiarity with Euclidean connections and we will use the Euclidean connection for a Finsler space as calculated by E. Cartan in [2] and Chern [3].

Let M be an n-dimensional differentiable manifold and let G be the principal bundle over M with fiber and group the n-dimensional orthogonal groups, $O_{(n)}$. Then in G, we have forms w_i, w_{ij}, where $w_{ij} + w_{ji} = 0$ and $i, j = 1, \cdots, n$. The equations of structure are

$$dw_i = w_j \wedge w_{ji} + \gamma_{ji\alpha} w_j \wedge w_{\alpha n} \tag{3.1}$$

$$dw_{ij} = w_{ik} \wedge w_{kj} + \Omega_{ij} , \tag{3.2}$$

where $\alpha = 1, \cdots, n-1$. (Henceforth we will assume that Greek indices run from 1 to $n-1$ and Latin indices run from 1 to n.) The $\gamma_{ij\alpha}$ are symmetric in all indices and zero if any index is n. Also

$$\Omega_{ij} = \frac{1}{2} \sum_{\alpha,\beta} Q_{ij\alpha\beta} w_{\alpha n} \wedge w_{\beta n} + \sum_{l,\alpha} P_{ijl\alpha} w_l \wedge w_{\alpha n} + \frac{1}{2} \sum_{l,k} R_{ijlk} w_l \wedge w_k . \tag{3.3}$$

Let C be any path in M^n. Choose any path in G with the property that if $e_1, \cdots, e_n$ represents a righthanded frame, that is, an element of $O_{(n)}$, then e_n is in the tangent direction to C. Then arc length along a path C is

$$I = \int_0^1 (w_n)_d d\xi_2 .$$

This follows from equation (2.4) and the definition of w_n (see [3]).

Now $X_2 = e_n$ and $X_1 = \sum k_i e_i$. Therefore $(w_n)_\delta = \langle X_1, w_n \rangle = k_n$. Hence if X_1 is perpendicular to the curve C, then the transversality term is zero. From equation (3.1), we have

$$dw_n = \sum w_\alpha \wedge w_{\alpha n} .$$

Hence

$$\delta I = [\delta(w_n)]_0^1 + \int_0^1 \sum \{(w_\alpha)_\delta (w_{\alpha n})_d - (w_\alpha)_d (w_{\alpha n})_\delta\} d\xi_2 , \tag{3.4}$$

where
$$(w_\alpha)_d = \langle w_\alpha , e_n \rangle = 0 .$$

It is clear from the last equation that the symbols δ and d and our indices make the notation awkward. Hence a w_d will be written as w and a w_δ will be written as ϕ. In this notation equation (3.4) becomes

$$I = [\phi_n]_0^1 + \int_0^1 \sum \phi_\alpha w_{\alpha n} d\xi_2 , \tag{3.5}$$

since $w_\alpha = 0$ along the path C.

From Theorem 2.1 we have the following theorem.

THEOREM 3.1. *The differential equations of a geodesic in Finsler geometry are*

$$w_\alpha = 0 , \qquad w_{\alpha n} = 0 , \qquad \alpha = 1, \cdots, n-1.$$

We will now compute the second variation along a geodesic. We have

$$\delta I = \int_0^1 \delta w_n d\xi_2 ,$$

and $\delta^2 I$ is the second variation. Hence we have to compute $\delta^2(w_n)$ along a geodesic. Now

(3.6) $$\delta^2(w_n) = \delta d(\phi_n) + \phi_\alpha \delta(w_{\alpha n})$$

since $w_{\alpha n}=0$ along the geodesic. We have

(3.7) $$\delta(w_{\alpha n}) - d(\phi_{\alpha n}) = \langle X_1 \wedge X_2, dw_{\alpha n} \rangle .$$

From equation (3.2) we obtain

$$\langle X_1 \wedge X_2, dw_{\alpha n} \rangle = \langle X_1 \wedge X_2, w_{\alpha\beta} \wedge w_{\beta n} \rangle + \langle X_1 \wedge X_2, \Omega_{\alpha n} \rangle .$$

By Theorem 1.1 and since C is a geodesic, we have

(3.8) $$\delta w_{\alpha n} = d\phi_{\alpha n} - w_{\alpha\beta}\phi_{\beta n} + \langle \Omega_{\alpha n}, X_1 \wedge X_2 \rangle .$$

Now by equation (3.2) and the facts that

$$R_{ijkl} = -R_{jikl} , \qquad R_{ij,kl} = R_{kl,ij}$$

we have

(3.9) $$\langle X_1 \wedge X_2, \Omega_{\alpha n} \rangle = \sum P_{n\alpha n\beta} w_n \phi_{\beta n} + \sum R_{n\alpha n\beta} \phi_\beta w_n .$$

Therefore, from equations (3.6), (3.8) and (3.9), we obtain

(3.10) $$\delta^2(w_n) = \delta d\phi_n + \sum \phi_\alpha [d\phi_{\alpha n} - \phi_{\beta n} w_{\alpha\beta} + P_{n\alpha n\beta} w_n \phi_{\beta n} + R_{n\alpha n\beta} \phi_\beta w_n] .$$

Now,

$$\delta d\phi_n = d\delta\phi_n \quad \text{and} \quad d(\phi_\alpha \phi_{\alpha n}) = \phi_{\alpha n}(d\phi_\alpha) + \phi_\alpha (d\phi_{\alpha n}) .$$

Hence

(3.11) $$\begin{aligned} \delta^2(w_n) = {} & d[\delta\phi_n + \phi_\alpha \phi_{\alpha n}] - \phi_{\alpha n} d\phi_\alpha \\ & + [-\phi_\alpha \phi_{\beta n} w_{\alpha\beta} + P_{n\alpha n\beta} \phi_\alpha \phi_{\beta n} + R_{n\alpha n\beta} \phi_\alpha \phi_\beta] w_n . \end{aligned}$$

But from equation (3.1) we have

(3.12) $$d\phi_\alpha = \delta w_\alpha + w_j \phi_{j\alpha} - \phi_j w_{j\alpha}$$

since

$$\gamma_{j\alpha\beta}[\phi_j w_{\beta n} - w_j \phi_{\beta n}] = 0$$

along the geodesic. Also $\delta w_\alpha = 0$ along the geodesic, since $w_\alpha \geq 0$ and equals zero along the geodesic and hence w_α must attain a minimum along a geodesic.

Hence

$$\delta^2 w_n = d[\delta\phi_n + \sum \phi_\alpha \phi_{\alpha n}] + \sum(\phi_{\alpha n}\phi_{\alpha n} + P_{n\alpha n\beta}\phi_\alpha\phi_{\beta n} + R_{n\alpha n\beta}\phi_\alpha\phi_\beta)w_n \,. \tag{3.13}$$

Hence the integral form of the second variation becomes

$$\delta^2 I = [\delta\phi_n + \sum \phi_\alpha \alpha_{\alpha n}]_0^1 + \int_0^1 \sum (\phi_{\alpha n}\phi_{\alpha n} + P_{n\alpha n\beta}\phi_\alpha\phi_{\beta n} + R_{n\alpha n\beta}\phi_\alpha\phi_\beta)w_n d\xi_2 \,.$$

For Riemannian geometry we have $P_{ijkl}=0$ and $\sum \phi_\alpha \phi_{\alpha n}$ represents the second fundamental form of the geodesic surface perpendicular to the geodesic at the point.

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YALE UNIVERSITY

ABSTRACT RIEMANN SUMS

PAUL CIVIN

1. Introduction. A theorem of B. Jessen [5] asserts that for $f(x)$ of period one and Lebesgue integrable on [0, 1]

$$(1) \qquad \lim_{n\to\infty} 2^{-n}\sum_{k=0}^{2^n-1} f(x+k2^{-n})=\int_0^1 f(t)dt \text{ almost everywhere}.$$

We show that the theorem of Jessen is a special case of a theorem analogous to the Birkhoff ergodic theorem [1] but dealing with sums of the form

$$(2) \qquad 2^{-n}\sum_{k=0}^{2^n-1} f(T^{k/2^n}x).$$

In this form T is an operator on a σ-finite measure space such that $T^{1/2^n}$ exists as a one-to-one point transformation which is measure preserving for $n=0, 1, \cdots$, and $f(x)$ is integrable with $f(x)=f(Tx)$. We also obtain in § 3 the analogues for abstract Riemann sums of the ergodic theorems of Hurewicz [4] and of Hopf [3].

We might remark that there is no use, due to the examples of Marcinkiewicz and Zygmund [6] and Ursell [8], in considering sums of the form

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^{k/n}x)$$

without further hypothesis on $f(x)$. However we may replace 2^n throughout by $m_1m_2\cdots m_n$ with m_j integral and $m_j\geq 2$ without altering any argument.

In § 4 necessary and sufficient conditions are obtained on a transformation T in order that the sums (2) have a limit as $n\to\infty$ for almost all x. These conditions are analogous to those of Ryll-Nardzewski [7] in the ergodic case. We use the necessary conditions to establish an analogue of a form of the Hurewicz ergodic theorem for two operators [2].

2. Notation. Let (S, Ω, μ) be a fixed σ-finite measure space. We consider throughout point transformations T which have measurable square roots of all orders, that is,

(3.1) *There exist one-to-one point transformations* T_n *so that*

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$$T_0 = T\,; \quad T_n^2 = T_{n-1} \qquad n=1,\ 2,\ \cdots.$$

(3.2) *If* $X \in \Omega$, *then* $T_n X \in \Omega$ *and* $T_n^{-1} X \in \Omega$, $\quad n=0,\ 1,\ \cdots.$

No requirement is made of the uniqueness of the sequence T_n. For example in the theorem of Jessen, T is the identity transformation while $T_n x = x + 2^{-n}$ (mod 1). We also suppose throughout that T is measure preserving

$$\mu(TX) = \mu(X) \quad \textit{for} \quad X \in \Omega. \tag{3.3}$$

3. **Limit theorems.** Let Φ be a finite valued set function defined on Ω and absolutely continuous with respect to μ. Form the sums

$$\Phi_n(X) = \sum_{k=0}^{2^n - 1} \Phi(T_n^k X) \qquad n=0,\ 1,\ \cdots, \tag{4}$$

and

$$\mu_n(X) = \sum_{k=0}^{2^n - 1} \mu(T_n^k X) \qquad n=0,\ 1,\ \cdots. \tag{5}$$

Then Φ_n is absolutely continuous with respect to μ_n and there exists an *averaging sequence of point functions* $f_n(x)$ so that

$$\Phi_n(X) = \int_X f_n(x) \mu_n(dx)\,, \qquad n=0,\ 1,\ \cdots. \tag{2}$$

THEOREM 1. *Let T be a transformation such that* (3.1), (3.2) *and* (3.3) *are satisfied. Let Φ be a finite valued set function defined on Ω, absolutely continuous with respect to μ and such that $\Phi(TX) = \Phi(X)$. Then for almost all $x[\mu]$ the averaging sequence of point functions defined by* (4), (5) *and* (6) *has a limit as $n \to \infty$. The limit function $F(x)$ has the following properties*:

(i) *$F(T_n x) = F(x)$ almost everywhere $[\mu]$,* $\quad n=0,\ 1,\ \cdots.$
(ii) *$F(x)$ is integrable over S.*
(iii) *For any set X with $T_n X = X$, $n=0,\ 1,\ \cdots$ and $\mu(X) < \infty$*

$$\int_X F(x) \mu(dx) = \int_X f(x) \mu(dx).$$

Proof. Note first that since $\Phi(TX) = \Phi(X)$,

$$\Phi_n(T_n X) = \sum_{k=0}^{2^n - 1} \Phi(T_n^{k+1} X) = \Phi(X). \tag{7}$$

Likewise

$$\mu_n(T_n X)=\mu_n(X)\,. \tag{8}$$

Therefore for all X

$$\int_X f_n(T_n x)\mu_n(dx)=\int_{T_n X} f_n(x)\,\mu_n(dx)=\int_X f_n(x)\,\mu_n(dx)$$

and consequently

$$f_n(T_n x)=f_n(x) \qquad \text{almost everywhere } [\mu_n]. \tag{9}$$

Relation (3.1) then implies

$$\left\{\begin{array}{l} \overline{\lim_{n\to\infty}}\, f_n(T_m^j x)=\overline{\lim_{n\to\infty}}\, f_n(x) \\ \underline{\lim_{n\to\infty}}\, f_n(T_m^j x)=\underline{\lim_{n\to\infty}}\, f_n(x) \end{array}\right. \quad \text{almost everywhere } [\mu] \quad \begin{array}{l} j=1,\cdots,2^m-1 \\ m=1,2,\cdots \end{array} \tag{10}$$

Let

$$A=\{x|\sup_{0\le n} f_n(x)\ge 0\}\,. \tag{11}$$

It is asserted that

$$\int_A f_0(x)\mu(dx)\ge 0\,. \tag{12}$$

We define the following sets:

$$P_j=\{x|f_j(x)\ge 0\} \qquad j=0,\ 1,\ \cdots$$

$$A_N=\{x|\sup_{0\le n\le N} f_n(x)\ge 0\} \qquad N=0,\ 1,\ \cdots$$

$$C_{N,\,j}=P'_N\cap\cdots\cap P'_{j+1}\cap P_j \qquad j=0,\ \cdots,\ N\,.$$

Now (9) together with (3.1) imply that $T_k P_j=P_j$ for $k\le j$. Consequently

$$T_j C_{N,\,j}=C_{N,\,j} \text{ and } \Phi(C_{N,\,j})=\Phi(T_j^k C_{N,\,j}).$$

Therefore

$$2^j\Phi(C_{N,\,j})=\sum_{k=0}^{2^j-1}\Phi(T_j^k C_{N,\,j})=\Phi_j(C_{N,\,j})$$

and

$$2^j\Phi(C_{N,\,j})=\int_{C_{N,\,j}} f_j(x)\mu_j(dx)\ge 0\,, \qquad j=0,\ \cdots,\ N\,.$$

Since the $C_{N,\,j}$ are disjoint for $j=0,\ \cdots,\ N$, we have $\Phi(A_N)\ge 0$ and by a limiting process we obtain (12).

Likewise if

(13) $$B=\{x\mid\inf_{0\leq n} f_n(x)\geq 0\},$$

then

(14) $$\int_B f_0(x)\mu(dx)\geq 0\,.$$

Inasmuch as the preceding argument made no use of the finiteness of Φ, we may apply the result to the set function $\Psi=\Phi-c\mu$ for any real c. Since

$$\Psi_n(X)=\int_X (f_n(x)-c)\mu_n(dx)$$

we deduce that for

(15) $$A^c=\{x\mid\sup_{0\leq n} f_n(x)\geq c\}$$

we have

(16) $$\Phi(A^c)\geq c\mu(A^c)$$

and for

(17) $$A_d=\{x\mid\inf_{0\leq n} f_n(x)\leq d\}$$

we have

(18) $$\Phi(A_d)\leq d\mu(A_d).$$

Let now for $r>s$

(19) $$L_s^r=\{x\mid\overline{\lim_{n\to\infty}} f_n(x)>r \text{ and } \underline{\lim_{n\to\infty}}\, f_n(x)<s\}\,.$$

From (10) we obtain

(20) $$T_m^j L_s^r=L_s^r \qquad j=0,\,1,\,\cdots,\,2^m-1\,;\; m=0,\,1,\,\cdots\,.$$

Since L_s^r is invariant under each T_m we may consider it as a new space. The sets A^r and A_s relative to the new space are now the full space L_s^r. Hence if we apply (16) and (18) we obtain

$$\Phi(L_s^r)\geq r\mu(L_s^r)\,;\quad \Phi(L_s^r)\leq s\mu(L_s^r)\,.$$

The finiteness of Φ together with the assumption $r>s$ implies $\mu(L_s^r)=0$. Thus $\lim_{n\to\infty} f_n(x)$ exists almost everywhere $[\mu]$.

Property (i) of the limit function $F(x)$ follows immediately from (10). Utilizing (i) the proofs of (ii) and (iii) are now identical with the corresponding proofs by Hurewicz [4, p. 201] in the ergodic case.

The theorem for abstract Riemann sums analogous to the Hopf ergodic theorem is now deducible as a corollary.

COROLLARY 1. *Let T be a transformation such that* (3.1) *and* (3.2) *are satisfied and in addition*

$$\mu(T_n X)=\mu(X) \qquad n=0,\ 1,\ \cdots. \tag{21}$$

Then for any integrable $f(x)$ with $f(Tx)=f(x)$ and any $g(x)>0$ with $g(Tx)=g(x)$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{2^n-1} f(T_n^k x)}{\sum_{k=0}^{2^n-1} g(T_n^k x)} \tag{22}$$

exists for almost every x $[\mu]$. The limit function $h(x)$ is integrable, satisfies $h(T_n x)=h(x)$ for almost all x $[\mu]$, and for sets Y with $\mu(Y)<\infty$ and $T_m Y=Y$, $m=0,\ 1,\ \cdots$

$$\int_Y h(x)g(x)\mu(dx)=\int_Y f(x)\mu(dx). \tag{23}$$

Proof. Introduce the measure

$$\nu(X)=\int_X g(x)\mu(dx),$$

and the set function

$$F(X)=\int_X f(x)\mu(dx).$$

The function F is absolutely continuous with respect to ν and is finite valued. Condition (21) implies that

$$F_n(X)=\int_X \sum_{k=0}^{2^n-1} f(T_n^k x)\mu(dx)$$

and

$$\nu_n(X)=\int_X \sum_{k=0}^{2^n-1} g(T_n^k x)\mu(dx).$$

Thus from the representation

$$F_n(X)=\int_X f_n(x)\,\nu_n(dx)$$

we deduce that

$$f_n(x)=\frac{\sum_{k=0}^{2^n-1} f(T_n^k x)}{\sum_{k=0}^{2^n-1} g(T_n^k x)} \quad \text{almost everywhere } [\mu].$$

The corollary is then an immediate consequence of Theorem 1.

The theorem of Jessen now follows from the version of Corollary 1 with $g(x)=1$ with the T_n as noted in § 2.

4. Invariant measure and two operators. It is possible for the conclusion of Corollary 1 to hold when $g(x)=1$ but T does not satisfy (21). If we introduce

$$R_n(A,\ Y)=2^{-n}\sum_{k=0}^{2^n-1}\mu(Y\cap T_n^{-k}A) \tag{24}$$

we obtain the following theorem.

THEOREM 2. *If T is a transformation such that* (3.1) *and* (3.2) *are satisfied, then the following statements are equivalent* :

(25.1) *For every integrable $f(x)$ with $f(Tx)=f(x)$,*

$$\lim_{n\to\infty} 2^{-n}\sum_{k=0}^{2^n-1} f(T_n^k x)$$

exists for almost every x $[\mu]$.

(25.2) *For each Y with $\mu(Y)<\infty$, $\lim_{n\to\infty} R_n(A,\ Y)\leq K\mu(A)$.*

(25.3) *For each Y with $\mu(Y)<\infty$, $\overline{\lim_{n\to\infty}}\, R_n(A,\ Y)\leq K\mu(A)$.*

(25.4) *For an increasing sequence of sets Y_j with $\bigcup_{j=1}^{\infty} Y_j=S$,*

$$\overline{\lim_{n\to\infty}}\, R_n(A,\ Y_j)\leq K\mu(A)\ .$$

(25.5) *There exists a countably additive measure ν with the properties*:

(i) $0\leq\nu(X)\leq K\mu(X)$

(ii) *If* $A=T_nA$, $\quad n=1,\ 2, \cdots$, $\nu(A)=\mu(A)$

(iii) $\nu(A)=\nu(T_nA)$, $\quad n=1,\ 2, \cdots$.

The proof is almost identical with that of Ryll-Nardzewski [7] in the ergodic case, and is omitted. The existence of an invariant measure implies, as in the ergodic case [2], the following theorem with two operators (or two sequences of roots of the same operator).

THEOREM 3. *Let T and U each satisfy* (3.1), (3.2), (3.3) *and* (25.1), *and let*

$$\sum_{k=0}^{2^n-1} \mu(T_n^k X)$$

be absolutely continuous with respect to

$$\mu_n(X)=\sum_{k=0}^{2^n-1} \mu(U_n^k X), \qquad n=0,\ 1,\ \cdots.$$

For any finite valued set function Φ absolutely continuous with respect to μ and with $\Phi(TX)=\Phi(X)$ form

$$\Phi_n(X)=\sum_{k=0}^{2^n-1} \Phi(T_n X).$$

Then in the representation

$$\Phi_n(X)=\int_X f_n(x)\mu_n(dx),$$

the averaging sequence of point functions $f_n(x)$ tends to a limit as $n\to\infty$ for almost every x $[\mu]$.

As a consequence of Theorem 3 we obtain the following corollary in the same fashion as Corollary 1 was derived from Theorem 1.

COROLLARY 2. *Let T and U each satisfy* (3.1) *and* (3,2), *and in addition*

$$\mu(V_n X)=\mu(X) \qquad n=0,\ \cdots \tag{26}$$

for $V=T$ and $V=U$. Then for any integrable $f(x)$ with $f(Tx)=f(x)$ and any $g(x)>0$ with $g(Ux)=g(x)$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{2^n-1} f(T_n^k X)}{\sum_{k=0}^{2^n-1} g(U_n^k X)}$$

exists for almost all x $[\mu]$.

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UNIVERSITY OF OREGON AND
THE INSTITUTE FOR ADVANCED STUDY

SOME ERGODIC THEOREMS INVOLVING TWO OPERATORS

PAUL CIVIN

1. Introduction. The object of the present note is to indicate how the ergodic theorem of W. Hurewicz [3] and E. Hopf [2] can be extended to theorems involving two operators. While for a finite measure space, the Hopf theorem for two operators is readily seen to be the consequence of the theorem for one operator and the Birkhoff ergodic theorem, in the general case the theorem for two operators is established via the extended form of the Hurewicz theorem. An application is made to the theory of Markov chains in § 4.

Let (S, Ω, μ) be a fixed measure space which is assumed to be σ-finite unless otherwise stated. Capital letters are reserved for elements of Ω. For a measure ξ and for point functions we write $f(x)=g(x)[\xi]$ for equality almost everywhere $[\xi]$.

We consider two one-to-one transformations of S onto itself, t and u, each of which is measurable in the sense that for $v=t$ and $v=u$, $M\in\Omega$ implies $vM\in\Omega$ and $v^{-1}M\in\Omega$, and if $\mu(M)=0$ then $\mu(v^{-1}M)=0$. We suppose throughout that neither t nor u has wandering sets of positive measure, that is,

(1) *For* $v=t$ *and* $v=u$, *if* $A\cap v^kA=0$, $k=1, 2, \cdots$, *then* $\mu(A)=0$.

2. The Hurewicz theorem. For any finite valued countably additive set function φ defined on Ω and absolutely continuous with respect to μ, form the set functions

$$\varphi_n(X)=\sum_{k=0}^{n}\varphi(t^kX), \qquad n=0, 1, \cdots, \tag{2}$$

and

$$\nu_n(X)=\sum_{k=0}^{n}\mu(t^kX), \qquad n=0, 1, \cdots. \tag{3}$$

Then φ_n and ν_n are countably additive set functions and φ_n is absolutely continuous with respect to ν_n so admits the representation

$$\varphi_n(X)=\int_X g_n(x)\mu_n(dx), \qquad n=0, 1, \cdots. \tag{4}$$

The Hurewicz theorem then asserts that $g_n(x)$ has a limit at all points except for a nullset with respect to t, that is for all points except a t-invariant set of μ measure zero.

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To formulate the theorem for two operators we introduce

$$(5)\qquad \mu_n(X)=\sum_{k=0}^{n}\mu(u^kX)\,, \qquad n=0,1,\cdots.$$

The set function μ_n is countably additive but φ_n is no longer automatically absolutely continuous with respect to μ_n. In order to have this absolute continuity for any countable additive set function φ absolutely continuous with respect to μ with the consequent representation

$$(6)\qquad \varphi_n(X)=\int_X f_n(x)\mu_n(dx)\,, \qquad n=0,1,\cdots$$

it is necessary for ν_n to be absolutely continuous with respect to μ_n. To see this simply take $\varphi=\mu$, whence $\varphi_n=\nu_n$. We therefore take as a basic hypothesis

(7) ν_n *is absolutely continuous with respect to* μ_n

with the consequent representation

$$(8)\qquad \nu_n(X)=\int_X c_n(x)\mu_n(dx)\,, \qquad n=0,1,\cdots.$$

We also assume that the operators t and u satisfy the Birkhoff ergodic theorem, that is,

(9) *For* $v=t$ *and* $v=u$, *if* $f(x)\in L^1(S)$, $\lim_{n\to\infty}\sum_{k=0}^{n} f(v^kx)/n$ *exists almost everywhere* $[\mu]$.

THEOREM 1. *Let t and u be one-to-one measurable transformations of S onto itself which have no wandering sets of positive measure. Let φ be a finite valued countably additive set function defined on Ω and absolutely continuous with respect to μ. If* (7) *and* (9) *are satisfied, then the "averaging sequence" $f_n(x)$ of point functions defined by* (2), (5) *and* (6) *converges everywhere except for the union of a t- and u-nullset as $n\to\infty$.*

Proof. We suppose first that $\mu(S)<\infty$. From the representations (4) and (8) we deduce that

$$\varphi_n(X)=\int_X g_n(x)c_n(x)\mu_n(dx)\,.$$

The comparison with (6) yields $f_n(x)=g_n(x)c_n(x)[\mu_n]$. The Hurewicz theorem implies that $g_n(x)$ has a finite limit except for a t-nullset. A result of C. Ryll-Nardzewski [4] shows that the hypothesis (9) that t satisfies the Birkhoff ergodic theorem implies the existence of a countably additive measure α with the additional properties:

(10.1) $0 \leq \alpha(X) \leq k\mu(X)$.

(10.2) *If* $X = t^{-1}X$, *then* $\alpha(X) = \mu(X)$.

(10.3) $\alpha(t^{-1}X) = \alpha(X)$.

Likewise, since u satisfies (9), there is a countably additive measure with the additional properties:

(11.1) $0 \leq \beta(X) \leq k\mu(X)$

(11.2) *If* $X = u^{-1}X$, *then* $\beta(X) = \mu(X)$

(11.3) $\beta(u^{-1}X) = \beta(X)$.

From (10.1) we note that α is absolutely continuous with respect to μ. Hence if

$$\alpha_n(X) = \sum_{k=0}^{n} \alpha(t^k X), \qquad n = 0, 1, \cdots \tag{12}$$

then α_n is absolutely continuous with respect to ν_n and we may write

$$\alpha_n(X) = \int_X a_n(x)\nu_n(dx), \qquad n = 0, 1, \cdots. \tag{13}$$

Likewise if

$$\beta_n(X) = \sum_{k=0}^{n} \beta(u^k X), \qquad n = 0, 1, \cdots, \tag{14}$$

β_n is absolutely continuous with respect to μ_n and

$$\beta_n(X) = \int_X \rho_n(x)\mu_n(dx) \qquad n = 0, 1, \cdots. \tag{15}$$

If $\beta(A) = 0$, then (11.3) implies $\beta\left(\bigcup_{-\infty}^{\infty} u^k A\right) = 0$, and since $\bigcup_{-\infty}^{\infty} u^k A$ is u-invariant (11.2) implies $\mu\left(\bigcup_{-\infty}^{\infty} u^k A\right) = 0$ and thus $\mu(A) = 0$. Hence we also have the representation

$$\mu_n(X) = \int_X b_n(x)\beta_n(dx), \qquad n = 0, 1, \cdots. \tag{16}$$

If we combine (13), (8) and (16) we obtain

$$\alpha_n(X) = \int_X a_n(x)c_n(x)b_n(x)\beta_n(dx). \tag{17}$$

By the use of (10.3) and (11.3), (17) simplifies to

$$\alpha(X) = \int_X a_n(x)c_n(x)b_n(x)\beta(dx), \qquad n = 0, 1, \cdots. \tag{18}$$

Since $c_0(x) = 1[\mu]$, we find that

(19) $$a_n(x)c_n(x)b_n(x)=a_0(x)b_0(x)[\mu]\,, \qquad n=0, 1, \cdots$$

Since we are supposing at present that $\mu(S)<\infty$, the Hurewicz theorem can be applied to (13) and (16), and thus $a_n(x)$ has a limit $a(x)$ as $n\to\infty$, except for a t-nullset and $b_n(x)$ has a limit $b(x)$ as $n\to\infty$, except for a u-nullset. By a further conclusion of the Hurewicz theorem, not already stated, we know that $a(x)$ is t-invariant and that

$$\int_X a(x)\mu(dx)=\int_X a_0(x)\mu(dx)$$

for every invariant set X. Hence for $Z=\{x|a(x)=0\}$, $\alpha(Z)=0$ and since Z is t-invariant, $\mu(Z)=0$ by (10.2). The identical argument shows that $b(x)$ is not zero except for a u-nullset. If we also observe that the sets where $a_0(x)=\infty$ and $b_0(x)=\infty$ are t- and u-nullsets respectively, as are the sets where $a_0(x)=0$ and $b_0(x)=0$, we conclude that for all x except the union of a t- and u-nullset $c_n(x)$ has a finite limit as $n\to\infty$. Thus $f_n(x)$ has a finite limit excepting the union of a t- and u-nullset.

If the measure space (S, Ω, μ) is σ-finite, let $k(x)$ be a bounded positive function integrable over S. Let

$$\lambda(X)=\int_X k(x)\mu(dx)$$

and form

$$\lambda_n(X)=\sum_{j=0}^{n}\lambda(u^jX)\,.$$

The measure space (S, Ω, λ) is a finite measure space, and φ is absolutely continuous with respect to λ. Hence by the first part of the proof, if

$$\varphi_n(X)=\int_X h_n(x)\lambda_n(dx)\,,$$

then $h_n(x)$ has a finite limit at all points other then the union of a t- and u-nullset in the λ measure and hence also in the μ measure. Thus if we let

(20) $$\lambda_n(X)=\int_X k_n(x)\mu_n(dx)\,, \qquad n=0, 1, \cdots$$

we have

$$\varphi_n(X)=\int_X h_n(x)k_n(x)\mu_n(dx)\,,$$

and consequently $f_n(x)=h_n(x)k_n(x)[\mu_n]$. The Hurewicz theorem applied to (20) asserts that $k_n(x)$ has a finite limit except for a u-nullset, which implies the conclusion of the theorem.

THEOREM 2. *If in addition to the hypotheses of Theorem* 1 $\mu(S)<\infty$ *and* t *and* u *commute, then* $f(x)=\lim_{n\to\infty} f_n(x)$ *has the properties*

(i) $f(tx)=f(x)$

(ii) $\int_X f(x)\mu(dx)=\int_X f_0(x)\mu(dx)$ *for any t-invariant set* X.

Proof. We use the same notation as in the proof of Theorem 1. From (10.1) we see that any function integrable with respect to α is also integrable with respect to μ. Hence the counterpart of (9) is satisfied with $v=u$ and μ replaced by α. By a further use of the results of C. Ryll-Nardzewski we find the existence of a countably additive measure γ, defined as a Banach-Mazur limit

$$\gamma(X)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}(u^{-j}X)$$

and having the additional properties:

(21.1) $0\le\gamma(X)\le k_1\alpha(X)$

(21.2) *If* $X=u^{-1}X$, $\gamma(X)=\alpha(X)$

(21.3) $\gamma(u^{-1}X)=\gamma(X)$.

Since α is t-invariant and t and u commute we have $\alpha(u^{-j}tX)=\alpha(u^{-j}X)$, and thus the definition of $\gamma(X)$ shows

(2.14) $\gamma(X)=\gamma(t^{-1}X)$.

We similarly obtain a countably additive measure with the properties:

(22.1) $0\le\delta(X)\le k_1\beta(X)$

(22.2) *If* $X=t^{-1}X$, *then* $\delta(X)=\beta(X)$

(22.3) $\delta(t^{-1}X)=\delta(X)$

(22.4) $\delta(u^{-1}X)=\delta(X)$.

From (21.1) we obtain

$$\gamma(X)=\int_X m(x)\alpha(dx). \tag{23}$$

An earlier argument showed that $\delta(X)=0$ implies $\beta(X)=0$, hence

$$\beta(X)=\int_X n(x)\delta(dx). \tag{24}$$

The combination of (23), (18), (19) and (24) then yields

$$\gamma(X)=\int_X m(x)a_0(x)b_0(x)n(x)\delta(dx).$$

Since γ and δ are both t- and u-invariant, the integrand must be both t- and u-invariant. With the aid of (10.2), (11.2), (21.2) and (22.2) it is then seen that $m(x)a_0(x)b_0(x)n(x)=1[\delta]$.

Likewise the t-invariance of γ and α shows that $m(x)=1[\alpha]$, and the u-invariance of β and δ shows that $n(x)=1[\beta]$. Since a set of measure zero in any of the measures α, β, δ, and μ is also of measure zero in any of the other measure, we conclude that $a_0(x)b_0(x)=1[\mu]$.

The Hurewicz theorem, applied to (13), implies that for any t-invariant set X, if we let $a(x)=\lim_{n\to\infty} a_n(x)$

$$\int_X a(x)\mu(dx)=\int_X a_0(x)\mu(dx)=\alpha(X). \tag{25}$$

If we combine (10.2) with (25) we find

$$\mu(X)=\int_X a(x)\mu(dx).$$

The t-invariance of $a(x)$ then yields $a(x)=1[\mu]$. A repetition of the argument shows that $\lim_{n\to\infty} b_n(x)=1[\mu]$, consequently $\lim_{n\to\infty} c_n(x)=1[\mu]$. The conclusions of the theorem now follow from the corresponding conclusions of the Hurewicz theorem applied to (4).

3. The Hopf theorem.

THEOREM 3. *Let t and u be one-to-one measure preserving transformations of S onto itself. Let $f(x)\in L^1(S)$ and $g(x)>0$, then for almost all x the quotient*

$$\sum_{j=0}^{n} f(t^jx) \Big/ \sum_{j=0}^{n} g(u^jx)$$

has a limit as $n\to\infty$.

Proof. Let

$$\lambda(X)=\int_X g(x)\mu(dx), \qquad \lambda_n(X)=\sum_{j=0}^{n}\lambda(u^jX), \qquad \rho_n(X)=\sum_{j=0}^{n}\lambda(t^jX).$$

Then ρ_n is absolutely continuous with respect to λ_n and

$$\varphi(X)=\int_X f(x)\lambda(dx)$$

is a finite valued countably additive set function absolutely continuous with respect to λ. We form $f_n(x)$ according to (4) and (5) with μ replaced by λ. Now

$$\lambda_n(X)=\int_X \sum_{j=0}^{n} g(u^jx)\mu(dx),$$

so

$$\varphi_n(X)=\int_X f_n(x)\lambda_n(dx)=\int_X f_n(x)\sum_{j=0}^{n} g(u^j x)\mu(dx)\ .$$

But by definition

$$\varphi_n(x)=\int_X \sum_{j=0}^{n} f(t^j x)\mu(dx)\ .$$

Thus

$$f_n(x)=\sum_{j=0}^{n} f(t^j x)\Big/\sum_{j=0}^{n} g(u^j x)$$

and the conclusion follows from Theorem 1.

4. An application. In a recent note [1] T. E. Harris and Herbert Robbins used the Hopf ergodic theorem to obtain results concerning Markov chains admitting an infinite invariant measure. We indicate below the corresponding results that are obtainable by the use of Theorem 3.

Consider the real valued Markov chain $\cdots, x_{-1}, x_0, x_1, \cdots$ with a stationary transition probability function

$$h(u, B)=\text{prob}\ (x_{n+1}\in B|x_n=u)\ .$$

It is assumed that there is a measure II on the real Borel sets, which does not vanish identically, is finite for bounded Borel sets and satisfies

$$\text{II}(B)=\int_{-\infty}^{\infty} h(u, B)\text{II}(du)\ .$$

Let Φ be the class of real Borel sets, S the space of sequences of real numbers $x=(\cdots, x_{-1}, x_0, x_1, \cdots)$ and Ω the Borel extension of the cylinder sets, in S. If $A\in\Omega$ is determined by the coordinates $x_k, x_{k+1}, \cdots, x_r$ then $q(A|x_k=u)$ will denote the probability of A relative to the Markov chain starting with $x_k=u$, as specified by h.

A measure is established [1] in Ω by the relation

$$m(A)=\int q(A|x_j=u)\,\text{II}\,(du) \qquad j\leq k$$

for cylinder sets determined by $x_k, \cdots, x_r$.

We shall apply Theorem 3, with t the ath shift transformation, $(tx)_i=x_{i+a}$, and u the bth shift transformation. If $\Gamma\in\Phi$, let R_Γ be the event that $x_n\in\Gamma$ infinitely often. The assumption

(26) *If* $\Gamma\in\Phi$, *then* $q(R_\Gamma|x_0=u)=1[\text{II}]$,

then yields [1] that t and u are m measure preserving and that neither t nor u has wandering sets of positive measure.

THEOREM 4. *If* (26) *is satisfied, $h(u)$ is* II *summable and* $k(u)>0$, *then for almost all* x_0[II]

$$\lim_{n\to\infty} \frac{h(x_c)+h(x_{a+c})+\cdots+h(x_{na+c})}{k(x_d)+h(x_{b+d})+\cdots+k(x_{nb+d})}$$

exists with probability one.

THEOREM 5. *Let $y_1, y_2, \cdots$ be independent random variables with a common distribution function. Suppose that for any interval I*

$$\text{prob}\,(y_c+y_{a+c}+\cdots+y_{na+c}\in I \textit{ infinitely often})=1$$

and

$$\text{prob}\,(y_d+y_{b+d}+\cdots+y_{nb+d}\in I \textit{ infinitely often})=1.$$

Then for $h(u)$ Lebesgue integrable $k(u)>0$ *and almost all m*

$$\lim_{n\to\infty} \frac{\sum_{p=0}^{n} h\left(m+\sum_{j=0}^{p} y_{ja+c}\right)}{\sum_{p=0}^{n} k\left(m+\sum_{j=0}^{p} y_{jb+d}\right)}$$

exists with probability one.

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UNIVERSITY OF OREGON AND
THE INSTITUTE FOR ADVANCED STUDY

THE NUMBER OF SOLUTIONS OF CERTAIN CUBIC CONGRUENCES

ECKFORD COHEN

1. Introduction. In this paper we shall be concerned with cubic congruences of the form

$$n \equiv a_1x_1^3 + \cdots + a_sx_s^3 \pmod{m}, \tag{1.1}$$

where n is arbitrary, $m > 1$, and the a_i are integers prime to m. The number of sets of solutions $(x_1, \cdots, x_s)$ of (1.1), distinct modulo m, will be denoted by $N_s(n, m)$. Our discussion of $N_s(n, m)$ is limited to the cases $s=2$ and $s=3$; however, we emphasize that the method involved can be extended to arbitrary s.

Suppose that m has the factorization $m = p_1^{\lambda_1}, \cdots, p_l^{\lambda_l}$ as a product of powers of distinct primes $p_1, \cdots, p_l$. Then it follows easily that

$$N_s(n, m) = N_s(n, p_1^{\lambda_1}) \cdots N_s(n, p_l^{\lambda_l}). \tag{1.2}$$

Thus the determination of $N_s(n, m)$ reduces to the problem of determining $N_s(n, p^\lambda)$ where p is a prime. We accordingly limit ourselves to the case of a prime-power modulus p^λ.

If we denote by t the largest integer $\leq \lambda$ such that $n \equiv 0 \pmod{p^t}$, then one may write

$$n = p^t\xi, \qquad (\xi, p) = 1, \qquad 0 \leq t \leq \lambda. \tag{1.3}$$

We observe, in case $\lambda > t$, that ξ is uniquely determined $\pmod{p}$. Our main goal will be to obtain exact formulas for the number of solutions $N_2(n, p^\lambda, t) = N_2$ of

$$n \equiv ax^3 + by^3 \pmod{p^\lambda}, \tag{1.4}$$

and the number of solutions $N_3(n, p^\lambda, t) = N_3$ of

$$n \equiv ax^3 + by^3 + cz^3 \pmod{p^\lambda}, \tag{1.5}$$

where n is arbitrary of the form (1.3), and the following conditions are satisfied:

$$p \equiv 1 \pmod{3}, \qquad abc \not\equiv 0 \pmod{p}. \tag{1.6}$$

The restriction $p \equiv 1 \pmod{3}$ is natural, since other primes are special in the case of cubic congruences.

The method of the paper is based on elementary properties of

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finite exponential sums. These are listed for the cubic case as preliminary lemmas in § 2. The principal formula for N_2 is contained in Theorem 1 (§ 3) and the corresponding result for N_3 in Theorem 2 (§ 4). Both results involve the pair of integers (A, B), determined uniquely by the relations [7],

$$4p=A^2+27B^2, \qquad A\equiv 1 \pmod 3, \quad B>0. \tag{1.7}$$

However, in the special case $t\not\equiv 0 \pmod 3$, the value of N_2 is given explicity (§ 3, Corollary 2).

On the basis of these formulas, solvability criteria for (1.4) and (1.5) are developed in § 5. In fact, it is shown in Theorem 5 that (1.5) is *always* solvable ($N_3>0$). As for N_2, the following criterion is established: *If* $p\neq 7$, *then* (1.4) *is insolvable if and only if* $t\not\equiv 0$ (mod 3), $t<\lambda$, *and* a *and* b *belong to different cubic character classes* (mod p). (For the exceptional case $p=7$, see the complete statement of the criterion in Theorem 6). Approximations to N_2 and N_3 are also given in § 5 (Theorems 3 and 4, respectively).

Regarding previous research on cubic congruences, we note the work of Gauss who evaluated N_2 in the case of a prime modulus p [4]. More recently, Dickson determined N_3 for a prime modulus, with $a=b=c=1$ [3, p. 167]. In addition, Skolem [9] and Selmer [8] have considered such congruences in their treatment of cubic Diophantine equations. Some of these results were deduced by the author in an earlier note anticipating the present paper [2].

2. Notation and preliminary lemmas.

The cubic Gauss sum $G(n, m)$ is defined by

$$G(n, m)=\sum_{\mu \pmod m} \varepsilon(n^3, m), \tag{2.1}$$

where the summation is over a complete residue system (mod m), and ε is defined for integral α, by

$$\varepsilon(\alpha, m)=e^{2\pi i\alpha/m}. \tag{2.2}$$

Expansion of $N_s(n, m)$ into a Fourier sum [1, § 5] reveals immediately the relation between $N_s(n, m)$ and the Gauss sum (2.1):

LEMMA 1. *The number of solutions of* (1.1) *is given by*

$$N_s(n,m)=\frac{1}{m}\sum_{\mu \pmod m} \varepsilon(n, m)\prod_{i=1}^{s} G(-a_i\mu, m). \tag{2.3}$$

We next note two reduction formulas for G [6].

LEMMA 2.

$$G(nm', mn')=m' G(n, m)\,. \tag{2.4}$$

LEMMA 3. *If* $(\nu, p)=1$, *then*

$$G(\nu, p^k)=\begin{cases} p^{2j} & (k=3j)\,, \\ p^{2j}G(\nu, p) & (k=3j+1)\,, \\ p^{2j+1} & (k=3j+2)\,. \end{cases} \tag{2.5}$$

Closely related to $G(n, p^k)$ are the two Gauss-Kummer sums defined by

$$\tau_i^{(k)}(n)=\sum_{\substack{\nu (\mathrm{mod}\ p^k) \\ (\nu,\ p)=1}} \chi^i(\nu)\varepsilon(n, p^k)\,, \qquad (i=1,\ 2)\,, \tag{2.6}$$

where $\chi(\nu)$ and $\chi^2(\nu)$ denote the two non-principal cubic characters (mod p), the summation being over a reduced residue system (mod p^k). In order to differentiate between the two non-principal characters, we write

$$\theta_1=\frac{1}{2}(A+3B\sqrt{-3})\,, \qquad \theta_2=\bar{\theta}_1\,, \qquad (\theta_1\theta_2=p)\,, \tag{2.7}$$

where A and B are defined by (1.7). Then one may define $\chi(\alpha)$, for integers α prime to p, to be that cube root of unity satisfying

$$\chi(\alpha)\equiv\alpha^{(p-1)/3} \qquad (\mathrm{mod}\ \theta_1)\,. \tag{2.8}$$

The relation (2.8) is the cubic extension of the Euler criterion [5, p. 455]. In our discussion, the primitive cube roots of unity will be denoted by ω and ω^2, with $\omega=\frac{1}{2}(-1+\sqrt{-3})$.

We place further,

$$\tau_i(n)=\tau_i^{(1)}(n)\,, \quad \tau_i=\tau_i(1)\,, \qquad (i=1,\ 2).$$

With this notation, we state the following reduction formula for $\tau_i^{(k)}(n)$.

LEMMA 4. *If* $k\geq 1$ *and* $i=1$ *or* 2, *then*

$$\tau_i^{(k)}(n)=\begin{cases} p^{k-1}\tau_i(\xi) & (n=p^{k-1}\xi\,, \quad (\xi, p)=1)\,, \\ 0 & (\text{otherwise}) \end{cases} \tag{2.9}$$

The important relation connecting $G(\nu, p)$, $\tau_1(\nu)$, and $\tau_2(\nu)$ is contained in the following lemma.

LEMMA 5. *If* $(\nu, p)=1$, *then*

$$G(\nu, p)=\tau_1(\nu, p)+\tau_2(\nu, p)\,. \tag{2.10}$$

The sums $\tau_1(\nu)$, $\tau_2(\nu)$ have the following fundamental properties [5],

$$\tau_1(\nu)=\chi^2(\nu)\tau_1\,,\quad \tau_2(\nu)=\chi(\nu)\tau_2\,, \qquad (\nu,\,p)=1\,, \tag{2.11}$$

$$\tau_1\tau_2=p\,, \tag{2.12}$$

$$\tau_1^3=p\theta_1\,,\qquad \tau_2^3=p\theta_2\,, \tag{2.13}$$

θ_1 and θ_2 being defined by (2.7).

Corresponding to the principal character (mod p), we have the familiar (Ramanujan) sum,

$$C(n,p^k)=\sum_{\substack{\nu (\mathrm{mod}\ p^k)\\ (\nu,\,p)=1}} \varepsilon(n\nu,\,p^k)\,, \tag{2.14}$$

which has the evaluation ($k>0$),

$$C(n,\,p^k)=\begin{cases} p^{k-1}(p-1) & (p^k \mid n\,, \\ -p^{k-1} & (p^{k-1} \mid n,\ p^k \nmid n)\,, \\ 0 & (p^{k-1}\nmid n)\,. \end{cases} \tag{2.15}$$

Also of importance in this paper are the functions,

$$T(\alpha)=\frac{1}{p}\,(\chi^2(\alpha)\tau_1^3+\chi(\alpha)\tau_2^3)\,, \tag{2.16}$$

$$J(\alpha)=\begin{cases} A & (\chi(\alpha)=1)\,, \\ \frac{1}{2}(9h(\alpha)B-A) & (\chi(\alpha)\neq 1)\,, \end{cases} \tag{2.17}$$

where $h(\alpha)$ is defined for cubic non-residues α (mod p) by

$$h(\alpha)=1 \quad \text{or} \quad -1\,, \tag{2.18}$$

according as $\chi(\alpha)=\omega$ or ω^2.

Application of (2.13) gives

LEMMA 6.

$$T(\alpha)=J(\alpha)\,. \tag{2.19}$$

The following notation will be needed.

$$q=\left[\frac{t-1}{3}\right],\qquad r=\left[\frac{t}{3}\right],\qquad s=\left[\frac{t-2}{3}\right], \tag{2.20}$$

$$Q=\left[\frac{\lambda-1}{3}\right],\qquad R=\left[\frac{\lambda}{3}\right],\qquad S=\left[\frac{\lambda-2}{3}\right], \tag{2.21}$$

where $[\beta]$ indicates the largest integer $\leq\beta$; and for $i=0,\,1,\,2$,

$$L_i(t)=\begin{cases} 1 & (t\equiv i \ (\mathrm{mod}\ 3),\quad t<\lambda)\,, \\ 0 & (\text{otherwise})\,. \end{cases} \tag{2.22}$$

3. The number of solutions of (1.4). In this section we use the notation,

$$\zeta = ab\xi \,, \tag{3.1}$$

where ξ is defined by (1.3), and

$$\eta = \chi(a)\chi^2(b) + \chi(b)\chi^2(a) = 2 \quad \text{or} \quad -1 \,, \tag{3.2}$$

according as $\chi(a) = \chi(b)$ or $\chi(a) \neq \chi(b)$.

The main result on (1.4) is contained in

THEOREM 1. *The number of solutions of* (1.4) *is given by*

$$\begin{aligned} N_2(n, p^\lambda, t) = p^{\lambda-1}\{ & p^r J(\zeta) L_0(t) + p^{q+1}\eta(1 - L_0(t)) \\ & + p^{r+1}(1 - L_2(t)) + p^{s+1}(1 - L_1(t)) - (\eta + 1)\} \,, \end{aligned} \tag{3.3}$$

where t is defined by (1.3), *J by* (2.17), *q, r, s by* (2.20), *the $L_i(t)$ by* (2.22), *and ζ, η by* (3.1) *and* (3.2) *respectively.*

Proof. By Lemma 1 it follows immediately that

$$N_2 = \frac{1}{p^\lambda} \sum_{(\mathrm{mod}\ p^\lambda)} \varepsilon(n\mu, p^\lambda) G(-a\mu, p^\lambda) G(-b\mu, p^\lambda) \,. \tag{3.4}$$

The residue system $\mu \pmod{p^\lambda}$ may be assumed to be the set $\mu = \nu p^{\lambda - k}$ where k ranges over the values $0 \leq k \leq \lambda$, and for each k, ν ranges over a reduced residue system (mod p^k). Thus (3.4) becomes, using (2.4),

$$N_2 = p^\lambda \sum_{k=0}^{\lambda} \frac{1}{p^{2k}} \sum_{\substack{\nu (\mathrm{mod}\ p^k) \\ (\nu, p) = 1}} \varepsilon(\nu n, p^k) G(-a\nu, p^k) G(-b\nu, p^k) \,. \tag{3.5}$$

We now break up the k summation according as $k \equiv 1$, 0, or 2 (mod 3), and apply Lemma 3 to obtain

$$N_2 = U_1 + U_2 + U_3 \,, \tag{3.6}$$

where

$$U_1 = p^{\lambda-2} \sum_{j=0}^{Q} \frac{1}{p^{2j}} \sum_{\substack{\nu (\mathrm{mod}\ p^{3j+1}) \\ (\nu, p) = 1}} \varepsilon(\nu n, p^{3j+1}) G(-a\nu, p) G(-b\nu, p) \,, \tag{3.7}$$

$$U_2 = p^\lambda \sum_{j=0}^{R} \frac{1}{p^{2j}} C(n, p^{3j}) \,, \qquad U_3 = p^{\lambda-2} \sum_{j=0}^{S} \frac{1}{p^{2j}} C(n, p^{3j+2}) \,. \tag{3.8}$$

Applying Lemma 5 and (2.11) to (3.7), and expanding, U_1 may be written

$$U_1 = U_{11} + U_{12} + U_{13}\,, \tag{3.9}$$

where

$$U_{11} = p^{\lambda-2}\chi^2(ab)\tau_1^2 \sum_{j=0}^{Q} \frac{1}{p^{2j}} \tau_1^{(3j+1)}(n)\,,$$

$$U_{12} = p^{\lambda-2}\chi(ab)\tau_2^2 \sum_{j=0}^{Q} \frac{1}{p^{2j}} \tau_2^{(3j+1)}(n)\,,$$

$$U_{13} = p^{\lambda-2}\tau_1\tau_2\eta \sum_{j=0}^{Q} \frac{1}{p^{2j}} C(n, p^{3j+1})\,.$$

Application of (2.11) and Lemmas 4 and 6 to U_{11} and U_{12} gives

$$U_{11} + U_{12} = p^{\lambda-1+r} J(\zeta) L_0(t)\,, \tag{3.10}$$

while U_{13} becomes, on the basis of (2.12) and (2.15),

$$U_{13} = p^{\lambda-1}\eta\{p^{q+1}(1 - L_0(t)) - 1\}\,. \tag{3.11}$$

Also, using (2.15) and summing, we get

$$U_2 = p^{\lambda+r}(1 - L_2(t))\,, \qquad U_3 = p^{\lambda-1}\{p^{s+1}(1 - L_1(t)) - 1\}\,. \tag{3.12}$$

The theorem follows on combining (3.6), (3.9), (3.10), (3.11), and (3.12).

Three main cases of Theorem 1 are distinguished according as, (i) $\lambda > t$, $t \equiv 0 \pmod 3$, (ii) $\lambda > t$, $t \not\equiv 0 \pmod 3$, or (iii) $\lambda = t$ $(n=0)$. Corresponding to these cases, one may deduce the following corollaries from (3.3).

COROLLARY 1. *If* $\lambda > t = 3e$, *then*

$$N_2(n, p^\lambda, 3e) = p^{\lambda-1}\{p^e(J(\zeta) + p + 1) - \eta - 1\}\,. \tag{3.13}$$

COROLLARY 2. *If* $\lambda > t \not\equiv 0 \pmod 3$, *then*

$$N_2(n, p^\lambda, t) = p^{\lambda-1}(p^{e+1} - 1)(\eta + 1)\,, \tag{3.14}$$

where $t = 3e+1$ *or* $3e+2$, *according as* $t \equiv 1$ *or* $2 \pmod 3$.

COROLLARY 3. $(n=0)$. *If* $\lambda = t = 3e + j$, $(j = 0, 1, 2)$, *then*

$$N_2(n, p^\lambda, \lambda) = p^{\lambda-1}\{(\eta+1)(p^{e+\gamma} - 1) + p^{e+j+1-2\gamma}\}\,, \tag{3.15}$$

where $\gamma = 0$ *or* 1 *according as* $t \equiv 0$ *or* $t \not\equiv 0 \pmod 3$.

4. The number of solutions of (1.5). The elements of the set $(a, b, c, \xi) = H$ may be distributed among the three cubic character classes (mod p) in essentially four different ways. These four distributions, denoted by H_1, H_2, H_3, and H_4, are defined as follows: (H_1)

Every class contains at least one element of H; (H_2) One class contains two elements of H and a second class contains the other two; (H_3) One class contains three elements of H but not all four; (H_4) All four elements lie in the same class.

Using this notation we define the function,

$$\delta(H)=0,\ 3,\ -3,\ \text{or}\ 6, \tag{4.1}$$

according as the elements of H have a distribution of type H_1, H_2, H_3, or H_4.

We will also make use of the following notation:

$$\theta=abc\,;\quad \eta_1=\eta_1(a,b,c)=\chi(a)\chi^2(bc)+\chi(b)\chi^2(ac)+\chi(c)\chi^2(ab)\,,\quad \eta_2=\bar{\eta}_1\,, \tag{4.2}$$

$\bar{\eta}_1$ denoting the complex conjugate of η_1:

$$\varDelta(H)=\eta_1\chi(\xi)+\eta_2\chi^2(\xi)\,. \tag{4.3}$$

On the basis of the above notation, one may deduce

LEMMA 7.

$$\varDelta(H)=\delta(H)\,. \tag{4.4}$$

We now state the main theorem for $N_3(n, p^\lambda, t)$.

THEOREM 2. *The number solutions of* (1.5) *is given by*

$$\begin{aligned} N_3(n, p^\lambda, t)=p^{2\lambda-2}\{[(p-1)(q+1)-L_0(t)]J(\theta)+p\delta(H)L_0(t) \\ -L_1(t)-pL_2(t)+(p-1)(pr+s+1)+p^2\}\,, \end{aligned} \tag{4.5}$$

where $\delta(H)$ *is defined by* (4.1), θ *by* (4.2), *and the rest of the notation has the same meaning as in Theorem* 1.

Proof. As in the proof of Theorem 1, we may express N_3 as a Fourier sum and apply Lemmas 2 and 3 to obtain

$$N_3=V_1+V_2+V_3\,, \tag{4.6}$$

where

$$V_1=p^{2\lambda-3}\sum_{j=0}^{Q}\frac{1}{p^{3j}}\sum_{\substack{\nu(\mathrm{mod}\ p^{3j+1})\\(\nu,p)=1}}\varepsilon(\nu n, p^{3j+1})G(-a\nu, p)G(-b\nu, p)G(-c\nu, p), \tag{4.7}$$

$$V_2=p^{2\lambda}\sum_{j=0}^{R}\frac{1}{p^{3j}}C(n, p^{3j})\,,\qquad V_3=p^{2\lambda-3}\sum_{j=0}^{S}\frac{1}{p^{3j}}C(n, p^{3j+2})\,. \tag{4.8}$$

Application of Lemma 5 and (2.11) to (4.7) yields

$$V_1=V_{11}+V_{12}+V_{13}\,, \tag{4.9}$$

where

$$V_{11}=p^{2\lambda-2}T(\theta)\sum_{j=0}^{Q}\frac{1}{p^{3j}}C(n,\ p^{3j+1}),$$

$$V_{12}=p^{2\lambda-3}\tau_1^2\tau_2\eta_1\sum_{j=0}^{Q}\frac{1}{p^{3j}}\tau_2^{(3j+1)}(n),$$

$$V_{13}=p^{2\lambda-3}\tau_1\tau_2^2\eta_2\sum_{j=0}^{Q}\frac{1}{p^{3j}}\tau_1^{(3j+1)}(n).$$

Using (2.15) and Lemma 6 in case of V_{11}, one obtains

$$V_{11}=p^{2\lambda-2}J(\theta)\{(p-1)(q+1)-L_0(t)\}. \tag{4.10}$$

V_{12} and V_{13} may be transformed by (2.11), (2.12), and Lemmas 4 and 7, to give

$$V_{12}+V_{13}=p^{2\lambda-1}\delta(H)L_0(t). \tag{4.11}$$

As for V_2 and V_3, application of (2.15) gives

$$V_2=p^{2\lambda-2}\{p^2+pr(p-1)-pL_2(t)\}, \tag{4.12}$$

$$V_3=p^{2\lambda-2}\{(p-1)(s+1)-L_1(t)\}. \tag{4.13}$$

Combination of the results in (4.6) and formulas (4.10) through (4.13) leads to the theorem.

Corresponding to the corollaries of Theorem 1, we may deduce the following results as special cases of Theorem 2.

COROLLARY 1. *If* $\lambda>t=3e$, *then*

$$N_3(n,\ p^\lambda,\ 3e)=p^{2\lambda-2}\{(pe-e-1)J(\theta)+e(p^2-1)+p^2+p\delta(H)\}. \tag{4.14}$$

COROLLARY 2. *If* $\lambda>t\not\equiv 0 \pmod 3$, *then*

$$N_3(n,\ p^\lambda,\ t)=p^{2\lambda-2}(p-1)(e+1)(J(\theta)+p+1), \tag{4.15}$$

where $t=3e+1$ *or* $3e+2$.

COROLLARY 3 ($n=0$). *If* $\lambda=t$, *then*

$$N_3(n,\ p^\lambda,\ \lambda)=p^{2\lambda-2}\{(p-1)[J(\theta)(e+\mu_1)+e(p+1)+\mu_2]+p^2\}, \tag{4.16}$$

where $\mu_1=\mu_2=0$ *if* $t=3e>0$; $\mu_1=1$, $\mu_2=0$ *if* $t=3e+1$, *and* $\mu_1=\mu_2=1$ *if* $t=3e+2$.

5. Solvability criteria. First we establish some bounds for N_2 and N_3. To do this, note by Definition (1.7) that $|A|<2\sqrt{p}$, and by a simple process of maximalization, that $|9h(\alpha)B-A|<4\sqrt{p}$, $(h(\alpha)=\pm 1)$. Thus we have

LEMMA 8.

$$|J(\alpha)| < 2\sqrt{p} \; . \tag{5.1}$$

By means of this Lemma and Corollary 1 of § 3, we get the following estimate for $N_2(n, p^\lambda, 3e)$.

THEOREM 3. *If $\lambda > t = 3e$, then*

$$p^e(p+1-2\sqrt{p})-\eta-1 < \frac{N_2}{p^{\lambda-1}} < p^e(p+1+2\sqrt{p})-\eta-1 \, . \tag{5.2}$$

Similarly, we may deduce bounds for N_3 on the basis of Corollaries 1 and 2 of § 4.

THEOREM 4. *If $\lambda > t$, then in case $t = 3e$,*

$$\begin{aligned} p^2+e(p^2-1)-2(pe-e-1)\sqrt{p}+p\delta(H) &< p^{2(1-\lambda)}N_3 \\ &< p^2+e(p^2-1)+2(pe-e-1)\sqrt{p}+p\delta(H) \, , \end{aligned} \tag{5.3}$$

and in case $t = 3e+1$ or $3e+2$,

$$p+1-2\sqrt{p} < \frac{p^{2(1-\lambda)}N_3}{(p-1)(e+1)} < p+1+2\sqrt{p} \; . \tag{5.4}$$

We are now in a position to establish precise criteria for the solvability of (1.4) and (1.5).

THEOREM 5. *The congruence* (1.5) *has a solution for every integer n.*

Proof. To prove this theorem it suffices to show that the lower bounds in (5.3) and (5.4) are positive. This follows immediately in the case of (5.4). Rewriting the lower bound in (5.3) in the form,

$$ep^{3/2}(\sqrt{p}-2)+e(2\sqrt{p}-1)+p(p+2p^{-1/2}+\delta) \, ,$$

and remembering that the minimal values of p, $\delta(H)$, and e are $p=7$, $\delta=-3$, and $e=0$, we see that $N_3 > 0$ also in the case $\lambda > t \equiv 0 \pmod 3$.

THEOREM 6. *The congruence* (1.4) *has no solution if and only if either $t \not\equiv 0 \pmod 3$, $t < \lambda$, and $\chi(a) \neq \chi(b)$, or if $p=7$, $t=0$, $\chi(a)=\chi(b)$ and $\zeta = ab\xi \equiv \pm 3 \pmod 7$.*

Proof. If $\lambda > t \not\equiv 0 \pmod 3$, it follows directly from Corollary 2 of § 3, that $N_2 = 0$ if and only if $\eta = -1 (\chi(a) \neq \chi(b))$. In the remainder of the proof we suppose, therefore, that $\lambda > t \equiv 0 \pmod 3$. Now the lower bound in (5.2) is positive in case $\eta=-1$ and also in case $\eta=2$, $e>0$. In the remaining case ($\eta=2$, $e=0$), the lower bound is $p-2-2\sqrt{p}$, which is positive if $p>7$. But if $p=7$, $e=0$, $\eta=2$, then substitution in (3.13) shows that $N_2=0$ if and only if $\chi(\zeta)=\omega^2$, which implies that $\zeta\equiv\pm 3 \pmod 7$.

As a corollary of Theorem 6, we have the following result [8], [9]:

COROLLARY (Skolem-Selmer). *If $p \nmid abc$, then the congruence*

$$ax^3+by^3+cz^3\equiv 0 \pmod{p^\lambda} \tag{5.5}$$

always has a non-trivial solution (x, y, z not all $\equiv 0 \pmod p$).

Proof. With $z=1$, $c=-n$, Theorem 6 shows that (5.5) has a non-trivial solution (X, Y, 1) unless $p=7$, $\chi(a)=\chi(b)$. In the latter case, however, there exists a solution (X, 1, 0), because an integer α is a cubic residue (mod p^λ) if and only if it is a residue (mod p).

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THE UNIVERSITY OF SOUTH CAROLINA

SPECIALIZATIONS OVER DIFFERENCE FIELDS

RICHARD M. COHN

Introduction. We consider a system S of algeraic difference equations with coefficients in a difference field $\mathscr{F}$ and involving also parameters λ_i. Well-known results concerning systems of algebraic equations and systems of algebraic differential equations would lead one to expect that, if S has solutions in some extension of the difference field formed by adjoining the parameters λ_i to $\mathscr{F}$, then the system resulting from S by assigning special values to the λ_i has solutions, provided only that the special values are chosen so as not to annul a certain difference polynomial. But the examples in [5, p. 510] show that this is not so.

The difficulty in these examples arises from the fact that a difference field $\mathscr{F}$ may have incompatible extensions, that is to say, extensions which cannot both be embedded isomorphically in any one of its extensions. In particular, it may happen that one can express in terms of a solution of the system S an element α, independent of the λ_i. Then α will be contained in the difference field formed by adjoining to $\mathscr{F}$ a solution of any system (possessing solutions) which arises by specializing the parameters of S. It will then not be possible to find solutions if one specializes the λ_i in such a way that the extension of $\mathscr{F}$ formed by adjoining the specialized values is incompatible with that formed by adjoining α.

The principal result of this paper is that one can restore the expected result concerning the specialization of parameters of S by imposing a suitable condition of compatibility. If the system S has solutions, then, in order to assure that the system obtained from S by specializing the parameters has solutions, it suffices to choose the specializations from an extension of $\mathscr{F}$ compatible with a certain extension $\mathscr{G}$ of $\mathscr{F}$ and not annulling a certain difference polynomial. In particular, if $\mathscr{F}$ is algebraically closed it has no incompatible extensions so that it suffices to choose specializations of the parameters not annulling a certain difference polynomial. Hence, in this case, one has the same freedom of specialization as with systems of algebraic equations. Even in the general case, there is considerable freedom as the compatibility condition will evidently be satisfied if the specialized values are chosen from $\mathscr{G}$ itself or any extension of $\mathscr{G}$. We turn now to a formal discussion of this theorem.

We consider a difference field $\mathscr{F}$ and extensions $\mathscr{G}$ and $\mathscr{H}$ of

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$\mathscr{F}$. Let a set S of elements α_i be selected from $\mathscr{G}$ and a set $\bar{S}$ of elements $\bar{\alpha}_i$ from $\mathscr{H}$, where the index i has the same range, finite or infinite, in each case. We shall say that the $\bar{\alpha}_i$ constitute a *specialization over* $\mathscr{F}$ of the α_i if there is a homomorphism of the difference ring[1] $\mathscr{F}\{S\}$ onto the difference ring $\mathscr{F}\{\bar{S}\}$, this homomorphism leaving the elements of $\mathscr{F}$ fixed and carrying each α_i into $\bar{\alpha}_i$.

We wish to discuss the following question. Let $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$ be a set of elements lying in an extension of the difference field $\mathscr{F}$ and such that no nonzero difference polynomial in $\mathscr{F}\{u_1, \cdots, u_q\}$ vanishes when we substitute β_i for u_i, $i=1, \cdots, q$. Let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ constitute a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q$. Under what circumstances do there exist elements $\bar{\gamma}_1, \cdots, \bar{\gamma}_p$ such that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_p$ constitute a specialization over $\mathscr{F}$ of the set $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$? If such elements $\bar{\gamma}_j$ exist we shall say that the specialization of the β_i can be *extended* to a specialization of the β_i and γ_j. We have already indicated that, in order to insure the possibility of the extension, we must impose a condition of compatibility. Our principal result is contained in the following theorem.

THEOREM 1. *Given a difference field $\mathscr{F}$ and an extension*

$$\mathscr{H} = \mathscr{F}\langle\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p\rangle$$

of $\mathscr{F}$ which is such that the degree of transformal transcendence of $\mathscr{G} = \mathscr{F}\langle\beta_1, \cdots, \beta_q\rangle$ over $\mathscr{F}$ is q, there exists a nonzero element δ in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization $\bar{\beta}_1, \cdots, \bar{\beta}_q$ over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q$ with the properties that

(a) *$\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{H}$ are compatible extensions of $\mathscr{F}$,*
(b) *the specialization of δ is not zero,*

can be extended to a specialization $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_p$ over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$.

It is evident that Theorem 1 may be applied to show that *zeros of a reflexive prime difference ideal may be found for all assignments of values to its parametric indeterminates (if any) which lie in an extension of a certain field and do not annul a certain nonzero difference polynomial in the parametric indeterminates.*

The condition that $\beta_1, \cdots, \beta_q$ annul no nonzero difference polynomial

[1] For this and similar notations see [**5**, pp. 508 and 513]. Basic definitions will be found in [**9**], [**8**], [**1**] and [**5**].

with coefficients in $\mathscr{F}$ is essential in Theorem 1. Let β be an element transcendental over the field $\Re$ of rational numbers, and consider the difference field $\Re\langle\beta\rangle$, whose elements are their own transforms. We may extend $\Re\langle\beta\rangle$ to $\Re\langle\beta, \gamma\rangle$, where $\gamma^2=\beta$, $\gamma_1=-\gamma$ (subscripts now denote transforms). Then β may be specialized to the square of an element of $\Re$. No such specialization can be extended to γ. It is evident that this implies that no element δ exists with the properties prescribed in Theorem 1.

We give the proof of Theorem 1 in § 2 using preliminary lemmas proved in § 1. It is possible for a set S of elements α_i, $i=1, \cdots, n$, to specialize to a set $\bar{S}$ such that $\mathscr{F}\langle S\rangle$ and $\mathscr{F}\langle\bar{S}\rangle$ are incompatible extensions of $\mathscr{F}$. In § 3 we give an example of such a specialization and prove a theorem to the general effect that such specializations are scarce.

1. Proof of two lemmas.

1. 1. *Absolutely irreducible polynomials.* Let there be given a set S of elements λ_i, where the index i ranges over a suitable set of ordinals, and the λ_i lie in an extension of a field (not a difference field) $\mathscr{F}$ of characteristic 0. Let P be an absolutely irreducible polynomial in $\mathscr{F}(S)[x_1, \cdots, x_n]$. We shall show that almost every specialization of the λ_i specializes P into an absolutely irreducible polynomial. Specifically, we shall prove the following result.

LEMMA 1. *There is a nonzero element γ in $\mathscr{F}[S]$ snch that for any specialization of the λ_i over $\mathscr{F}$ for which γ does not specialize to zero, specializations of the coefficients of P are defined, and the polynomial $\bar{P}$ which is obtained by replacing the coefficients in P by their specializations is of the same degree as P in x_n and is absolutely irreducible.*

Proof. Using a device due to Kronecker [11, VI, p. 129] we introduce an auxiliary variable t and replace each x_i in P by $t^{m^{i-1}}$, where m is an integer exceeding the degree of P in any x_i. Then P goes over into a polynomial P^* in t. In the algebraic closure of $\mathscr{F}(S)$, P^* factors into (not necessarily distinct) linear factors

$$P^*=P_1 \cdots P_r .$$

Let S_i^*, $i=1, \cdots, 2^r-2=\nu$, denote the products of all subsets of from 1 to $r-1$ of the P_i. Let $T_i^*=P^*/S_i^*$.

In each S_i^* and T_i^* the powers of t may be replaced in a unique way by power products of the x_i which correspond to them by the substitution of the preceding paragraph and are of degree less than m in each x_i. Let polynomials S_i and T_i result from these replacements.

The absolute irreducibility of P is equivalent to its irreducibility in the algebraic closure of $\mathscr{F}(S)$ and this, in turn, is equivalent to the statement that none of the polynomials $Q_i=P-S_iT_i$, $i=1, \cdots, \nu$, is zero. Let ϕ_i, $i=1, \cdots, \nu$, be the coefficient of a term which appears effectively in Q_i. Let $\phi=\phi_1 \cdots \phi_r$. Let θ be the coefficient of a term of P which is of highest degree in x_n.

There exists an element γ in $\mathscr{F}[S]$ such that for any specialization of the λ_i for which γ does not specialize to zero:

(a) Specializations exist for all the coefficients of P,
(b) The specialization may be extended so as to define specializations for each coefficient occurring in the S_i and T_i,
(c) $\phi\theta$ does not specialize to zero under the extended specialization.

γ has the properties claimed in the statement of Lemma 1. For the existence of specializations of the coefficients of P is guaranteed in (a). The equality of the degrees of P and $\overline{P}$ in x_n follows from (c). It follows from (b) that polynomials $\overline{Q_i}$, $\overline{S_i}$, $\overline{T_i}$, $\overline{S_i}^*$ and $\overline{T_i}^*$ may be defined as the polynomials resulting by replacements of the coefficients of the Q_i, S_i, T_i, S_i^* and T_i^* respectively by their specializations. By Condition (c) no $\overline{Q_i}$ is zero. This implies the absolute irreducibility of $\overline{P}$. For $\overline{P^*}=\overline{P_1} \cdots \overline{P_r}$, where the $\overline{P_i}$ (which coincide with certain $\overline{S_i}^*$) result from the specialization of the coefficients of the P_i. Since the $\overline{P_i}$ are of degree zero[2] or one in t, factors of $\overline{P}$ in any extension of its coefficient field can be found by the method of Kronecker from the $\overline{P_i}$ of first degree. The $\overline{Q_i}$ relate to the $\overline{P_i}$ in the same way as the Q_i to the P_i. Hence if $\overline{P}$ had a proper factorization in any field, then some $\overline{Q_i}$ would be zero. This completes the proof of Lemma 1.

1. 2. *Absolutely irreducible manifolds.* Let Σ be a prime p.i.[3] (polynomial ideal) in $\mathscr{F}(S)[u_1, \cdots, u_q; x_1, \cdots, x_p]$, the u_i constituting a set of parametric indeterminates for Σ. Let $A_1, \cdots, A_p$ be a characteristic set of Σ with A_i introducing x_i, $i=1, \cdots, p$. We suppose that the manifold $\mathfrak{M}$ of Σ is absolutely irreducible. Then the following generalization of Lemma 1 may be proved.

LEMMA 2. *There is a nonzero element γ in $\mathscr{F}[S]$ such that for any*

[2] Actually no $\overline{P_i}$ is of zero degree, for this would imply that some $\overline{Q_i}=0$.

[3] We use this term as in [**9**, Chapter IV], to designate ideals in polynomial rings as distinguished from difference ideals.

specialization $\bar{\lambda}_i$ *of the* λ_i *for which* γ *does not specialize to zero, specializations of the coefficients of* $A_1, \cdots, A_p$ *are defined, and the polynomials* $\bar{A}_1, \cdots, \bar{A}_p$ *which are obtained by replacing the coefficients of* $A_1, \cdots, A_p$*, respectively, by their specializations form a characteristic set of a prime* p.i. $\bar{\Sigma}$ *in* $\mathscr{F}(\bar{S})[u_1, \cdots, u_q; x_1, \cdots, x_p]$*, where* $\bar{S}$ *denotes the set of* $\bar{\lambda}_i$. *The manifold of* $\bar{\Sigma}$ *is absolutely irreducible. Each* $\bar{A}_i$ *is of the same degree as* A_i *in* x_i. *The* $\bar{\lambda}_i$ *and a generic zero of* $\bar{\Sigma}$ *constitute a specialization over* $\mathscr{F}$ *of the* λ_i *and a generic zero of* Σ.

Proof. Let

$$w=\sum_{i=1}^{p} a_i x_i ,$$

the a_i integers, be a resolvent unknown for Σ; let G be the corresponding resolvent, Π the prime p. i.

$$\left(\Sigma , w-\sum_{i=1}^{p} a_i x_i\right) .$$

Then Π contains polynomials $M_i x_i - N_i$, $i=1, \cdots, p$, where the M_i and the N_i are polynomials in w and the u_i of lower degree in w than G, and the M_i are nonzero.

$$(1) \qquad G; \quad M_1x_1-N_1, \cdots, M_px_p-N_p$$

is a characteristic set of Π corresponding to the ordering $u_1, \cdots, u_q$; w; $x_1, \cdots, x_p$ of the indeterminates, which we use throughout the following discussion.

G is absolutely irreducible. For, by [5, p. 514], the reducibility of G in any field would imply the reducibility of Σ in some extension of $\mathscr{F}$. Hence, by the preceding lemma, there is a nonzero element γ_1 of $\mathscr{F}[S]$ such that, for all specializations of the λ_i for which γ_1 does not vanish the coefficients of G specialize, and the polynomial $\bar{G}$ which is obtained by replacing the coefficients of G by their specializations is absolutely irreducible and is of the same degree as G in w.

Each coefficient of the M_i, N_i and A_i may be written as a quotient of elements of $\mathscr{F}[S]$. Let δ be the product of the denominators of these quotients. Let η_i/δ, $i=1, \cdots, p$, the $\eta_i \neq 0$, be coefficients of terms of the M_i. Let η be the product of the η_i.

Let I be the product of the initials of the A_i, and J the remainder of I with respect to (1). J is a nonzero polynomial in w and the u_i. Some coefficient of J has the form κ/δ^t, where $\kappa \neq 0$ is in $\mathscr{F}[S]$ and t is a positive integer.

We let $\gamma = \gamma_1 \delta \eta \kappa$. For any specialization of the λ_i to a set $\bar{S}$ of elements $\bar{\lambda}_i$ for which γ does not specialize to zero we may define polynomials $\overline{M}_i$, $\overline{N}_i$, $\overline{A}_i$ and $\overline{G}$ which result from the M_i, N_i, A_i and G respectively by the specialization of their coefficients. $\overline{G}$ is absolutely irreducible. The $\overline{M}_i$ are not zero and are reduced with respect to $\overline{G}$ since $\overline{G}$ is of the same degree as G in w. Hence

$$(2) \qquad \overline{G}\,; \quad \overline{M}_1x_1-\overline{N}_1, \cdots, \overline{M}_px_p-\overline{N}_p$$

is a characteristic set of a prime p.i. $\overline{\Pi}$ in $\mathscr{F}(\overline{S})[u_1, \cdots, u_q; w; x_1, \cdots, x_p]$. Each $\overline{A}_i$ is of the same degree as A_i in x_i, and its initial results from the specialization of the coefficients in the initial of A_i. The $\overline{A}_i$ are in $\overline{\Pi}$. For the A_i are in Π and hence have zero remainders with respect to (1). The equations which express this go over upon specialization into equations which show that the $\overline{A}_i$ have zero remainders with respect to (2). In saying this we make use of the fact that each coefficient in these equations may be written as an element of $\mathscr{F}[S]$ divided by a power of δ.

Let $\overline{\Sigma}$ denote the prime p.i. consisting of those polynomials of $\overline{\Pi}$ which are free of w. The $\overline{A}_i$ are in $\overline{\Sigma}$. Let $B_1, \cdots, B_p$ be a characteristic set of $\overline{\Sigma}$ with B_i introducing x_i. The product of the degrees of the B_i in the indeterminates they introduce equals the degree of $\overline{G}$ in w. This is the degree of G in w and hence equals the product of the degrees of the A_i in the respective x_i. This product, in turn, equals the product of the degrees of the $\overline{A}_i$ in the respective x_i. Hence the product of the degrees of the B_i in their respective x_i equals the corresponding product formed for the $\overline{A}_i$. It follows that the chain $B_1, \cdots, B_p$ cannot be lower than the chain $\overline{A}_1, \cdots, \overline{A}_p$. The latter is therefore a characteristic set of $\overline{\Sigma}$.

The absolute irreducibility of the manifold of $\overline{\Sigma}$ is a consequence of the absolute irreducibility of $\overline{G}$ since $\overline{G}$ is a resolvent for $\overline{\Sigma}$. It remains only to prove the last statement of the lemma. Let P be any polynomial of Σ whose coefficients are in $\mathscr{F}[S]$. On specialization of its coefficients P becomes a polynomial $\overline{P}$. The equation which shows that the remainder of P with respect to $A_1, \cdots, A_p$ is 0 goes over into an equation showing that the remainder of $\overline{P}$ with respect to $\overline{A}_1, \cdots, \overline{A}_p$ is 0. Hence $\overline{P}$ is in $\overline{\Sigma}$. This is equivalent to the statement that any algebraic relation between the λ_i and a generic zero of Σ goes over on specialization into a relation between the $\bar{\lambda}_i$ and a generic zere of $\overline{\Sigma}$. This completes the proof of Lemma 2.

1. 3. *Adjunction of a generic zero.* Our application of the preceding lemma will arise in the following situation. Let Π be a prime p.i. in $\mathscr{F}[u_1, \cdots, u_q; y_1, \cdots, y_p]$, the u_i constituting a parametric set. Let $\mathscr{H}$ be the field obtained by adjoining a generic zero $u_i=\alpha_i$, $i=1, \cdots, q$; $y_j=\beta_j$, $j=1, \cdots, p$, of Π to $\mathscr{F}$. The manifold $\mathfrak{M}$ of Π is the union of manifolds $\mathfrak{M}_1, \cdots, \mathfrak{M}_r$ irreducible over $\mathscr{H}$. Let $\mathfrak{M}_1$ be an $\mathfrak{M}_i$ containing the generic zero named above. Then $\mathfrak{M}_1$ is absolutely irreducible.

To prove this statement we consider the field $\mathscr{G}$ consisting of those elements of $\mathscr{H}$ which are algebraic over $\mathscr{F}$. Let $\mathfrak{M}'$ be the least manifold over $\mathscr{G}$ which contains $\mathfrak{M}_1$, and let Π' be the ideal of $\mathfrak{M}'$. Evidently Π' is prime. $u_i=\alpha_i$, $i=1, \cdots, q$; $y_j=\beta_j$, $j=1, \cdots, p$, is a zero of Π'. Now Π' contains Π and hence is of at most the dimension q of Π. Since the degree of transcendence of

$$\mathscr{G}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_p)=\mathscr{H}$$

with respect to $\mathscr{G}$ is the same as the degree of transcendence of $\mathscr{H}$ with respect to $\mathscr{F}$, and the latter is q, it follows that $u_i=\alpha_i$, $i=1, \cdots, q$; $y_j=\beta_j$, $j=1, \cdots, p$ is actually a generic zero of Π', and that Π' is of dimension q.

It suffices to prove the absolute irreducibility of $\mathfrak{M}'$. For, since $\mathfrak{M}'$ is of the same dimension as $\mathfrak{M}_1$ and contains $\mathfrak{M}_1$, its absolute irreducibility would imply that it coincides with $\mathfrak{M}_1$, and hence that the latter is absolutely irreducible.

Suppose $\mathfrak{M}'$ is not absolutely irreducible. Then there is an element γ algebraic over $\mathscr{G}$ such that $\mathfrak{M}'$ is reducible over $\mathscr{G}(\gamma)$. Let γ be of degree d over $\mathscr{G}$. Then γ is also of degree d over $\mathscr{H}$ because every element of $\mathscr{H}$ algebraic over $\mathscr{G}$ is in $\mathscr{G}$.[4] Evidently γ is also of degree d over $\mathscr{G}_1=\mathscr{G}(\alpha_1, \cdots, \alpha_q)$. Let e be the degree of $\mathscr{H}$ over $\mathscr{G}_1$, and let f be the degree of $\mathscr{G}_1(\gamma; \beta_1, \cdots, \beta_p)$ with respect to $\mathscr{G}_1(\gamma)$. The reducibility of $\mathfrak{M}'$ over $\mathscr{G}(\gamma)$ implies that $f<e$. On the other hand the degree of $\mathscr{G}_1(\gamma; \beta_1, \cdots, \beta_p)$ with respect to $\mathscr{G}_1$ is given both by de and df, so that $e=f$. This is a contradiction which establishes our claim that $\mathfrak{M}_1$ is absolutely irreducible.

2. Proof of Theorem 1.

2. 1. *A special case.* We return to the notation in which Theorem 1 was stated. We treat first the case that $p=1$, and that γ_1, which we shall now denote by η, using subscripts to denote its transforms, is algebraic over $\mathscr{G}$. Without loss of generality we may suppose that $\mathscr{F}$ is inversive.[5]

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[4] One applies Lemma 2 of [5] to the prime p.i. in $\mathscr{G}[y]$ whose generic zero is γ.

[5] This is an easy consequence of the fact, proved in [3], that every difference field has an inversive extension.

Suppose first that η and, hence, its transforms are algebraic over $\mathcal{F}$. Then the compatibility of the extension $\mathcal{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathcal{H}$ implies the existence of a field[6] $\mathcal{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \eta\rangle$. We say that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \eta$ constitutes a specialization of $\beta_1, \cdots, \beta_q; \eta$. For, if P is a polynomial in $\mathcal{F}\{u_1, \cdots, u_q; y\}$ which vanishes when we replace u_i by β_i, $i=1, \cdots, q$, and y by η, then each coefficient of P as a polynomial in the u_{ij} is a difference polynomial of $\mathcal{F}\{y\}$ which has the zero η. If this were not the case a set of β_{ij} would be algebraically dependent over $\mathcal{F}\langle\eta\rangle$ and hence over $\mathcal{F}$, which is not so. It follows that P vanishes when we replace the u_i by the corresponding $\bar{\beta}_i$ and y by η. Hence in this case the assertion of Theorem 1 holds with $\delta=1$.

2. 2. *Conclusion of the algebraic case.* We proceed to complete the proof of the algebraic case by induction. We shall suppose that the conclusion of Theorem 1 has be been verified for algebraic functions of the β_{ij}, $i=1, \cdots, q$; $j=0, 1, \cdots, n-1$. Let η be algebraic over the field formed by adjoining to $\mathcal{F}$ the β_{ij}, $i=1, \cdots, q$; $j=0, 1, \cdots, n$.

We denote by S_k, $k=0, 1, \cdots$, the set of β_{ij}, $i=1, \cdots, q$; $j=k, \cdots, k+n-1$[7]; and by T_k the set of β_{ij}, $i=1, \cdots, q$; $j=k, \cdots, k+n$. Then η is algebraic over $T(=T_0)$. Let those elements of $\mathcal{H}=\mathcal{G}\langle\eta\rangle$ which are algebraic over any $\mathcal{F}(S_k)$ be adjoined to $\mathcal{G}$. There results a difference field whose inversive extension we denote by $\mathcal{G}'$. Let η be of degree d over $\mathcal{G}'$. Evidently there is an element σ of $\mathcal{H}$, algebraic over $\mathcal{F}(S)$, such that some transform η_t of η is of degree d over $\mathcal{G}\langle\sigma\rangle$. Let $\mathcal{G}^*$ be the difference field formed by adjoining to $\mathcal{G}$ elements whose tth transforms are respectively σ and the β_i, $i=1, \cdots, q$. Then η is of degree d over $\mathcal{G}^*$. Let Π be the reflexive prime difference ideal in $\mathcal{G}^*\{y\}$ whose generic zero is η. We claim that the characteristic set of Π consists of a single polynomial.

Evidently, the first polynomial of this characteristic set is of order zero and degree d in y. To prove that it is the only polynomial of the characteristic set we must show that, for any $r>0$, the degree of η_r over $\mathcal{G}^*(\eta_0, \cdots, \eta_{r-1})$ is d. For any $r>0$, η_r satisfies an irreducible algebraic equation of degree $d'\leq d$ with coefficients in $\mathcal{G}^*(\eta_0, \cdots, \eta_{r-1})$. Since η_r is algebraic over $\mathcal{F}(T_r)$ these coefficients may be chosen to be algebraic over $\mathcal{F}(T_r)$. The coefficients are rational combinations with coefficients in $\mathcal{F}$ of certain transforms and inverse transforms of the β_i and σ, and of $\eta_0, \cdots, \eta_{r-1}$. The β_{ij} involved, either directly, or

[6] The field $\mathcal{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \eta\rangle$, and other fields arising in similar situations, is not necessarily determined to within isomorphisms. It is a field which contains and is generated by subfields isomorphic to $\mathcal{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathcal{F}\langle\eta\rangle$. Our notation is intended to indicate that one such field is selected and held fixed throughout the discussion.

[7] S_k is to denote the empty set if $n=0$.

because the σ_k involved or $\eta_0, \cdots, \eta_{r-1}$ are algebraic functions of them, are finite in number. We may specify a positive integer p such that for all β_{ij} involved, we have $-t \leq j \leq p$.

We now specialize the β_{ij}, $-t \leq j < r$, to integers. This is to be a specialization in the sense of algebra only; the operation of transforming need not be preserved by the specialization. If the integer values are appropriately chosen the specialization may be extended to the β_{ij}, $r \leq j \leq p$, and to the σ_k involved in the coefficients and $\eta_0, \cdots, \eta_r$ in such a way that these β_{ij} remain algebraically independent over $\mathscr{F}$. It follows that the coefficients of the irreducible equation for η_r, the σ_k involved, $k \geq r$, and η_r itself are unaltered by the specialization. That is to say, the specializations of these elements, and of the β_{ij}, $j \geq r$, satisfy precisely the same set of algebraic relations over $\mathscr{F}$ as did the corresponding unspecialized elements.

The σ_k, $-t \leq k < r$, and $\eta_0, \cdots, \eta_{r-1}$ specialize to elements algebraic over $\mathscr{F}(S_r)$. There is then an element λ, algebraic over $\mathscr{F}(S_r)$ such that these specializations lie in $\mathscr{F}(S_r, \lambda)$. Evidently, then, η_r is of degree at most d' over $\mathscr{G}^*(\lambda)$. Hence if d'' donotes the degree of η_r over $\mathscr{G}'(\lambda)$ we must certainly have $d'' \leq d'$.

Now λ is algebraic over the field $\mathscr{K}$ consisting of elements of $\mathscr{G}'$ which are algebraic over $\mathscr{F}(S_r)$. Let its degree over $\mathscr{K}$ be h. Every element of $\mathscr{G}'(\eta_r)$ algebraic over $\mathscr{K}$ is in $\mathscr{K}$, as follows from the descriptions of these fields. Hence λ is also of degree h over $\mathscr{G}'(\eta_r)$ in consequence of Lemma 2 of [5]. Then λ is also of degree h over $\mathscr{G}'$. Hence the degree of $\mathscr{G}'(\eta_r, \lambda)$ with respect to $\mathscr{G}'$ must equal dh and also hd''. Hence $d'' = d' = d$. Thus we have shown that the characteristic set of Π consists of a single polynomial. We denote this polynomial by F. We may choose F so that its coefficients are in $\mathscr{F}\{\beta_{1,-t}, \cdots, \beta_{q,-t}; \sigma_{-t}\}$.

Let μ denote the initial of F. Some transform of μ is algebraic over $\mathscr{F}\langle\beta_1, \cdots, \beta_q\rangle$. Hence there is an element $\delta_1 \neq 0$ of $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization of the β_{ij} in the sense of algebra which does not annul δ_1 cannot be extended to a specialization to zero of this transform of μ. By the induction hypothesis, if $n \geq 1$, or by the special case proved in 2.1 if $n=0$, there is a $\delta_2 \neq 0$ in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization $\bar{\beta}_1, \cdots, \bar{\beta}_q$ of $\beta_1, \cdots, \beta_q$ such that δ_2 does not specialize to zero and that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{G}\langle\sigma\rangle$ are compatible extensions of $\mathscr{F}$, can be extended to a specialization $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\sigma}$ of $\beta_1, \cdots, \beta_q; \sigma$ over $\mathscr{F}$.

Let $\delta = \delta_1\delta_2$. We shall show that δ has the properties specified in Theorem 1. Let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ be any specialization of $\beta_1, \cdots, \beta_q$ such that δ does not specialize to zero and that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{H}$

are compatible extensions of $\mathscr{F}$. Since σ is in $\mathscr{H}$, $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{G}\langle\sigma\rangle$ are compatible extensions of $\mathscr{F}$. Hence the specialization of the β_i to the $\bar{\beta}_i$ can be extended to a specialization of σ to $\bar{\sigma}$. Let F become $\bar{F}$ when we replace σ by $\bar{\sigma}$ in its coefficients.[8] Because δ_1 does not specialize to zero $\bar{F}$ is of the same degree as F and its initial $\bar{\mu}$ is the specialization of μ. We let $\bar{\eta}$ be any solution of the difference equation $\bar{F}=0$. We shall show that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\eta}$ constitutes a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q; \eta$.

Let P be any polynomial in $\mathscr{F}\{u_1, \cdots, u_q; y\}$ which vanishes when we put $u_i=\beta_i$, $i=1, \cdots, q$; $y=\eta$. When the u_i are replaced by the β_i, P goes over into a polynomial P' of $\mathscr{G}\{y\}$, and η is a zero of P'. Hence P' is in Π. Then $\phi P'$, where ϕ is a product of powers of transforms of μ, is a linear combination of F and its transforms with coefficients which are polynomials of $\mathscr{G}^*\{y\}$. By a consideration of the process of forming the remainder we see that these coefficients are actually in $\mathscr{F}\langle\sigma_{-t}\rangle\{\beta_{1,-t}, \cdots, \beta_{q,-t}; y\}$ and hence, since the transforms of σ are algebraic over $\mathscr{G}$, they are in $\mathscr{F}\{\sigma_{-t}; \beta_{1,-t}, \cdots, \beta_{q,-t}; y\}$. Hence specializations may be defined for them.

From the relation of the preceding paragraph we obtain on specializing the β_i to the $\bar{\beta}_i$ and σ to $\bar{\sigma}$ an expression for $\bar{\phi}\bar{P}'$, where $\bar{\phi}\neq 0$ is the specialization of ϕ, and $\bar{P}'$ is the polynomial obtained from P by replacing the u_i by the $\bar{\beta}_i$, as a linear combination of $\bar{F}$ and its transforms. Hence $\bar{\eta}$ is a zero of $\bar{P}'$. This implies that P vanishes when the u_i, $i=1, \cdots, q$, are replaced by the corresponding $\bar{\beta}_i$ and y is replaced by $\bar{\eta}$. Thus Theorem 1 is proved in the algebraic case.

2. 3. *Completion of the proof of Theorem 1.* We now revert to the situation in which there are no restrictions on $\gamma_1, \cdots, \gamma_p$. We shall show that, without loss of generality, we may assume that each γ_j is transformally algebraic over $\mathscr{G}$. For, if this is not so, let, say, $\gamma_1, \cdots, \gamma_k$ constitute a basis of transformal transcendency[9] for the γ_j. If the theorem can be proved under the restriction just mentioned there is a $\delta'\neq 0$ in $\mathscr{F}\{\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k\}$ such that any specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k$ for which δ' does not specialize to zero, and which is such that $\mathscr{H}$ and the field formed by adjoining the specialized elements to $\mathscr{F}$ are compatible extensions of $\mathscr{F}$, can be extended to a specialization of $\gamma_{k+1}, \cdots, \gamma_p$. We write δ' as a polynomial in $\gamma_1, \cdots, \gamma_k$ with coefficients in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$. Let $\delta\neq 0$ be a coefficient

8 There is no difficulty in defining any needed inverse transforms of $\bar{\sigma}$.

9 A basis of transformal transcendency of a set of elements (over a given difference field) is a maximal subset of the elements not annulling any nonzero difference polynomial with coefficients in the field.

of this polynomial. Then δ has the properties specified in Theorem 1. For, let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ be a specialization over $\mathcal{F}$ of $\beta_1, \cdots, \beta_q$ which is such that δ does not specialize to 0 and that $\mathcal{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathcal{H}$ are compatible extensions of $\mathcal{F}$. We extend $\mathcal{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ by means of successive transformally transcendental adjunctions of elements $\bar{\gamma}_1, \cdots, \bar{\gamma}_k$. Then it is evident that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_k$ constitutes a specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k$ such that δ' does not specialize to zero, and that $\mathcal{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_k\rangle$ and $\mathcal{H}$ are compatible extensions of $\mathcal{F}$. Hence it is possible to extend this specialization to a specialization of $\gamma_{k+1}, \cdots, \gamma_p$. We shall deal henceforth only with the restricted case.

Since the case that no β_i exist is trivial and may be dismissed, $\mathcal{G}$ contains an element, β_1, which is distinct from all its transforms. Because of this and the restriction that the γ_j are transformally algebraic over $\mathcal{G}$ the Theorem of [4] implies that $\mathcal{H}$ contains an element

$$\theta=\sum_{j=1}^{p} \mu_j \gamma_{js},$$

$s \geq 0$ an integer, the μ_j in $\mathcal{F}\{\beta_1, \cdots, \beta_q\}$, such that θ is of equal order and effective order over $\mathcal{G}$, and for some integer $k \geq s$, and each j, $j=1, \cdots, p$, γ_{jk} is in $\mathcal{G}\langle\theta\rangle$. There exist difference polynomials P_j, $j=1, \cdots, p$, and Q in $\mathcal{G}\{w\}$ such that θ is not a zero of Q and that each quotient P_j/Q becomes γ_{jk} when w is replaced by θ. We may and shall choose the P_j and Q to be in $\mathcal{F}\{\beta_1, \cdots, \beta_q; w\}$.

Let Π denote the reflexive prime difference ideal in $\mathcal{G}\{w\}$ with generic zero θ, and let $A_0, A_1, \cdots$ be a characteristic sequence for Π. A_0 is of equal order and effective order. We choose an integer m such that the order m' of A_m is not less than the order of the last polynomial of a characteristic set of Π and also not less than the order of Q. Let $h=m'-m$. Then A_0 is of order h. We may assume without loss of generality that the coefficients of

$$(1) \qquad A_0, A_1, \cdots, A_m$$

are in $\mathcal{F}\{\beta_1, \cdots, \beta_q\}$. For, if this is not the case, it can be brought about by multiplying these polynomials by a suitable element of $\mathcal{F}\{\beta_1, \cdots, \beta_q\}$.

Let $\mathcal{G}'$ denote the subfield of $\mathcal{H}$ consisting of those of its elements which are algebraic over $\mathcal{G}$. By the Theorem of [6] there is an element[10] τ in $\mathcal{G}'$ such that $\mathcal{G}'=\mathcal{G}\langle\tau\rangle$. Since τ and its trans-

[10] We see from [6] that there is a finite set of elements τ_i which generate $\mathcal{G}'$ when adjoined, together with their transforms, to $\mathcal{G}$. Since the τ_i are algebraic over $\mathcal{G}$ it follows that there is a linear combination of them which will serve as τ.

forms are algebraic over $\mathscr{G}$, $\mathscr{G}\langle\tau\rangle=\mathscr{G}\{\tau\}$.

The manifold of A_0, regarded as a manifold over $\mathscr{G}'$, is the union of components of which at least one contains θ. Let Π' denote a reflexive prime difference ideal in $\mathscr{G}'\{w\}$ whose manifold is this component. Π' contains A_0 and is of order and effective order h.

We shall construct a beginning

$$C_0, \cdots, C_m \tag{2}$$

of a characteristic sequence of Π' in such a way that the coefficients of each polynomial of (2) are in the ring

$$\mathscr{R}=\mathscr{F}\{\beta_1, \cdots, \beta_q;\ \tau\}$$

and that each is obtained from the preceding by the procedure described in [1, pp. 142–145], all polynomials entering the computations having coefficients in $\mathscr{R}$. There is no trouble about C_0. We need merely start with the first polynomial of a characteristic sequence for Π and multiply it by a suitable element of $\mathscr{R}$. We specify that C_0 is to be irreducible. Suppose $C_0, \cdots, C_i$ have already been determined. Let B_{i+1} denote the remainder of the transform C_{i1} of C_i with respect to $C_0, \cdots, C_i$ considered as a chain of algebraic polynomials. Then

$$B_{i+1}=D_iC_{i1}-\sum_{j=0}^{i} L_{ij}C_j,$$

where D_i is a product of powers of initials of $C_0, \cdots, C_i$, and the L_{ij} are polynomials of $\mathscr{G}'\{w\}$. An examination of the remainder process shows that the L_{ij} and B_{i+1} are actually in $\mathscr{R}\{w\}$.

Now C_{i+1} is either equal to B_{i+1}, so that

$$C_{i+1}=D_iC_{i1}-\sum_{j=0}^{i} L_{ij}C_j, \tag{3}$$

or there is a relation

$$E_i[T_iB_{i+1}-C_{i+1}H_i]=\sum_{j=0}^{i} M_{ij}C_j, \tag{4}$$

where E_i is a product of powers of initials of $C_0, \cdots, C_i$, the M_{ij} are in $\mathscr{G}'\{w\}$, H_i is in $\mathscr{G}'\{w\}$ and is of order $h+i+1$, T_i is in $\mathscr{G}'\{w\}$ and is of order $h+1$, and C_{i+1} and T_i are reduced with respect to $C_0, \cdots, C_i$, while H_i is a product of polynomials reduced with respect to this chain. We see that by multiplying the polynomials defined by (4) by suitable elements of $\mathscr{R}\{w\}$ it is possible to obtain a relation of the form of (4) in which all polynomials present are in $\mathscr{R}\{w\}$. We assume this to be done. Then $C_0, \cdots, C_m$ as defined by relations (3) or (4) have the stated properties.

We now treat the w_i as a set of indeterminates in the sense of algebra, and the difference fields as fields. The polynomials of Π which are of order not exceeding m' form a prime p.i. Π_m in $\mathscr{G}\{w_0, \cdots, w_{m'}\}$, while the polynomials of Π' of order not exceeding m' form a prime p.i, Π'_m in $\mathscr{G}'\{w_0, \cdots, w_{m'}\}$. Both Π_m and Π'_m have dimension h. $A_0, \cdots, A_m$ is a characteristic set for Π_m, and $C_0, \cdots, C_m$ is a characteristic set for Π'_m.

We say that the manifold of Π'_m is absolutely irreducible. For, by the definition of τ, every element of $\mathscr{G}'(\theta_0, \cdots, \theta_{m'})$ not in $\mathscr{G}'$ is transcendental over $\mathscr{G}'$. Hence $C_0, \cdots, C_m$ is the characteristic set of a prime p.i. Π''_m in $\mathscr{G}'(\theta_0, \cdots, \theta_{m'})[w_0, \cdots, w_{m'}]$ whose manifold is that of Π'_m.[11] But $\theta_0, \cdots, \theta_{m'}$ is a generic zero of Π'_m and a zero of Π''_m, since it annuls $C_0, \cdots, C_m$ but not the initials of these polynomials. By the remark after the proof of Lemma 2 above it follows that the manifold of Π''_m is absolutely irreducible.

Lemma 2 now shows that $\mathscr{R}$ contains an element δ_0 such that for any specialization in the sense of algebra of the β_{ij} and the transforms of τ for which δ_0 does not vanish, (2) specializes to a characteristic set $\overline{C}_0, \cdots, \overline{C}_m$ of a prime p.i. over the field formed by adjoining to $\mathscr{F}$ the specializations of the β_{ij} and the transforms of τ. Now C_0 is absolutely irreducible. For it follows from the absolute irreducibility of the manifold of Π'_m that C_0 has no factors other than itself which involve w_h, whatever extension of $\mathscr{G}'$ is used as the coefficient field; while the irreducibility of C_0 in $\mathscr{G}'$ shows that in no field does it have factors other than field elements which are free of w_h. Hence, by Lemma 1, there is a δ_1 in $\mathscr{R}$ such that for any specialization of the β_{ij} and the transforms of τ for which δ_1 does not vanish, C_0 specializes to an absolutely irreducible polynomial.

Since $\delta_0\delta_1$ is algebraic over $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$, this ring contains a δ_2 such that any specialization in the sense of algebra of the β_{ij} for which δ_2 does not specialize to 0 cannot be extended to a specialization of $\delta_0\delta_1$ in which this product specializes to 0.

By the special case of Theorem 1 proved in 2.2 there is a δ_3 in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization of the β_i over $\mathscr{F}$ to elements $\overline{\beta}_1, \cdots, \overline{\beta}_q$ such that $\mathscr{F}\langle\overline{\beta}_1, \cdots, \overline{\beta}_q\rangle$ and $\mathscr{G}'=\mathscr{F}\langle\beta_1, \cdots \beta_q; \tau\rangle$ are compatible extensions of $\mathscr{F}$, and that δ_3 does not specialize to 0, can be extended to a specialization of the β_i and τ.

The polynomials of Π' which are in $\mathscr{G}\{w\}$ form a reflexive prime

[11] To prove the identity of the manifolds we consider a generic zero of a component of the manifold of Π''_m which is irreducible over $\mathscr{G}'$ $(\theta_0, \cdots, \theta_{m'})$. This generic zero must annul the C_i. Because the dimension of the component equals the dimension of Π'_m the generic zero cannot annul the initial of any C_i. Hence it annuls the polynomials of Π''_m. Hence the component is contained in the manifold of Π''_m. Our statement follows readily from this.

difference ideal of dimension h with zero θ. Evidently this must be Π. Hence Q and the product J of the initials of the polynomials of (1), which are not in Π, are not in Π'. Let S be the product of the T_i and the initials of the H_i of (4). Then S is not in Π'. The remainder R of JQS with respect to the chain (2) is therefore not 0. Let $\delta_4 \neq 0$ be a coefficient of R. Then δ_4 is in the ring $\mathscr{R}$, and there is a δ_5 in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization of $\beta_1, \cdots, \beta_q$ for which δ_5 does not vanish cannot be extended to a specialization of δ_4 to zero.

We let $\delta = \delta_2\delta_3\delta_5$. We shall show that δ has the properties specified in the statement of Theorem 1.

Let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ be a specialization of $\beta_1, \cdots, \beta_q$ over $\mathscr{F}$ which is such that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{H}$ are compatible extensions of $\mathscr{F}$ and that δ does not vanish under the specialization. Then $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{G}'$ are compatible extensions of $\mathscr{F}$, and δ_3 does not vanish under the specialization. Hence there is a $\bar{\tau}$ such that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\tau}$ constitutes a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q; \tau$. Let the polynomials of (2) become $\bar{C}_0, \cdots, \bar{C}_m$ when their coefficients are subjected to this specialization. The non-vanishing of δ_2 shows that $\bar{C}_0, \cdots, \bar{C}_m$ is a characteristic set of a prime p.i. over the field $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\tau}\rangle$, with $w_0, \cdots, w_{h-1}$ constituting a set of parametric indeterminates, and that $\bar{C}_0$ is irreducible. The initials of the C_i specialize to the initials of the $\bar{C}_i$.

Because δ_5 does not vanish the specialization carries R into a nonzero polynomial $\bar{R}$ reduced with respect to $\bar{C}_0, \cdots, \bar{C}_m$. Hence J, Q and S are carried by the specialization of their coefficients into polynomials $\bar{J}$, $\bar{Q}$ and $\bar{S}$ respectively which are annulled by no regular zero of the chain $\bar{C}_0, \cdots, \bar{C}_m$. Hence the T_i and the initials of the H_i do not vanish when their coefficients are specialized, so that the relations (3) and (4) are carried by the specialization into relations of the same type. It follows that $\bar{C}_0, \cdots, \bar{C}_m$ is the beginning of a characteristic sequence of one or more reflexive prime difference ideals whose manifolds are components of the general solution of $\bar{C}_0$. Let $\bar{\Pi}$ be one of these ideals, and $\bar{\theta}$ a generic zero of $\bar{\Pi}$. Evidently $\bar{J}$ and $\bar{Q}$ do not have $\bar{\theta}$ as a zero.

Let the P_i, $i=1, \cdots, p$, be carried into polynomials $\bar{P}_i$ by the specialization of their coefficients. We define $\bar{\gamma}_{is}$, $i=1, \cdots, p$, to be the result obtained by replacing w by $\bar{\theta}$ in $\bar{P}_i/\bar{Q}$.

We say that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_{1s}, \cdots, \bar{\gamma}_{ps}$ constitutes a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q; \gamma_{1s}, \cdots, \gamma_{ps}$. For let F be a polynomial of

$\mathscr{F}\{u_1, \cdots, u_q; y_1, \cdots, y_p\}$ which is free of y_{ij}, $j<s$, and which vanishes when we replace each u_i, $i=1, \cdots, q$, by β_i, and each y_{jk}, $j=1, \cdots, p$; $k=s, s+1, \cdots$, by γ_{jk}. Let the u_i in F be replaced by the β_i, the y_{js} by P_j/Q, and their transforms by transforms of these expressions. After multiplication by a suitable product of powers of transforms of Q there results a polynomial G in $\mathscr{F}\{\beta_1, \cdots, \beta_q; w\}$. Evidently G is in Π.

Let $\overline{G}$ denote the polynomial obtained from G by specializing its coefficients, and let $\overline{A}_0, \cdots, \overline{A}_m$ denote the polynomials so obtained from the polynomials of (1). The A_i have 0 remainder with respect to (2) considered as a chain of polynomials in the indeterminates $w_0, \cdots, w_{h+m}$. By specialization we see that the $\overline{A}_j$ have 0 remainder with respect to the chain $\overline{C}_0, \cdots, \overline{C}_m$, and hence have the zero $\overline{\theta}$. Similarly G has zero remainder with respect to the chain $A_0, \cdots, A_m$. By specialization we see that $\overline{J}\,\overline{G}$ has the zero $\overline{\theta}$. Hence $\overline{G}$ has the zero $\overline{\theta}$. If we replace the u_i in F by the $\overline{\beta}_i$ and the y_{js} and their transforms by the $\overline{P}_j/\overline{Q}$ and their transforms we shall also obtain $\overline{G}$. Hence F has zero $u_i=\overline{\beta}_i$, $y_{jk}=\overline{\gamma}_{jk}$, $k\geq s$. This proves our statement concerning the $\overline{\gamma}_{js}$. If we define $\overline{\gamma}_j$, $j=1, \cdots, p$, as an element whose sth transform is $\overline{\gamma}_{js}$, we see that $\overline{\beta}_1, \cdots, \overline{\beta}_q; \overline{\gamma}_1, \cdots, \overline{\gamma}_p$ is a specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$. The proof of Theorem 1 is now complete.

2. 4. *Corollaries to Theorem 1.*

Corollary 1. *The specialization of $\gamma_1, \cdots, \gamma_p$ whose existence is proved in Theorem 1 may be made in such a way that if a basis of transformal transcendency for $\gamma_1, \cdots, \gamma_p$ is selected in advance, then its elements specialize into a basis of transformal transcendency for $\overline{\gamma}_1, \cdots, \overline{\gamma}_p$. Furthermore the effective order of $\gamma_1, \cdots, \gamma_p$ with respect to the pre-assigned basis equals the effective order of $\overline{\gamma}_1, \cdots, \overline{\gamma}_p$ with respect to the basis obtained by specialization.*

Proof. Let $\gamma_1, \cdots, \gamma_k$ be the pre-assigned basis of transformal transcendency. The first statement follows immediately from the construction used in the proof of Theorem 1, since $\overline{\gamma}_1, \cdots, \overline{\gamma}_k$ are so chosen as to annul no nonzero difference polynomial with coefficients in $\mathscr{F}\langle\overline{\beta}_1, \cdots, \overline{\beta}_q\rangle$. The second statement follows from the fact that Π and Π' are of equal effective order.[12]

[12] No such statement holds for orders. For let $\mathscr{F}$ be an inversive difference field containing an aperiodic element. Let $u=\beta$, $y=\gamma$ be a generic zero of the ideal $\{y_1-u\}$ of $\mathscr{F}\{u, y\}$. Then $\mathscr{F}\langle\beta, \gamma\rangle$ is of first order over $\mathscr{F}\langle\beta\rangle$, but if β is specialized to an element of $\mathscr{F}$, γ specializes to an element of $\mathscr{F}$. The specialization of β can be chosen so as not to annul any pre-assigned nonzero element of $\mathscr{F}\{\beta\}$.

Corollary 2. *Let $\mu \neq 0$ be an element of $\mathscr{F}\{\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p\}$. For an appropriate choice of δ of Theorem 1 the specialization of $\gamma_1, \cdots, \gamma_p$ whose existence is proved in Theorem 1 may be made in such a way that μ does not specialize to 0 and that the requirements of Corollary 1 are satisfied.*

Proof. Let A be a polynomial of $\mathscr{F}\{u_1, \cdots, u_q; y_1, \cdots, y_p\}$ which goes into μ when the u_i, $i=1, \cdots, q$, are replaced by the β_i and the y_j, $j=1, \cdots, p$, by the γ_j. Suppose first that each[13] γ_j is transformally algebraic over $\mathscr{G}$. If we replace the u_i in A by the β_i and the y_j by the P_j/Q, and multiply the result by a suitable product of powers of transforms of Q, we obtain a polynomial T of $\mathscr{G}\{w\}$ which is not in Π. We redefine R as the remainder of $JQST$ with respect to the chain (2), and redefine δ_4 and δ_5 correspondingly. Evidently $\delta=\delta_2\delta_3\delta_5$ has the desired properties.

To complete the proof of the corollary we proceed, as in the proof of Theorem 1, to obtain a δ'. In consequence of what has just been proved δ' may be so chosen that any specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k$ for which δ' does not vanish, and which satisfies the usual compatibility requirement, can be extended to a specialization of the remaining γ_j in such a way that μ does not specialize to 0. We form δ from δ' as in the proof of Theorem 1.

3. Proof of a partial converse.

3. 1. *A counterexample.* It is not necessarily the case that the extensions of a ground field $\mathscr{F}$ generated by a set of elements and by one of its specializations over $\mathscr{F}$ are compatible. To show this we take for the ground field the field $\mathfrak{R}$ of rational numbers and consider polynomials in $\mathfrak{R}\{y\}$. Let A be the polynomial $1+y^2$ and let F be $A^2+A_1^2$. Then y_2-y is a factor of F_1-F. Hence F, y_2-y is a characteristic set of a reflexive prime difference ideal[14] Π in $\mathfrak{R}\{y\}$. Let η be a generic zero of Π.

To $\mathfrak{R}$ we adjoin an element i such that $i^2=-1$, and define the transform of i to be itself. Then $\mathfrak{R}\langle i\rangle$ and $\mathfrak{R}\langle\eta\rangle$ are incompatible. For $1+\eta^2\neq 0$, since η, as a generic zero of Π, satisfies no zero order difference equation. Hence $\mathfrak{R}\langle\eta\rangle$ contains an element

$$\lambda=(1+\eta_1^2)/(1+\eta^2).$$

Since

$$(1+\eta^2)^2+(1+\eta_1^2)^2=0$$

[13] We are here using the symbolism of the proof of Theorem 1.

[14] It is easy to establish the irreducibility of F. Then one applies Theorem 3 of [1].

we see that $\lambda^2=-1$. From $\eta_2=\eta$ we readily derive the relation $\lambda\lambda_1=1$. These imply $\lambda_1=-\lambda$. Hence[15], $\mathfrak{R}\langle\lambda\rangle$ is incompatible with $\mathfrak{R}\langle i\rangle$. Evidently this implies that $\mathfrak{R}\langle\eta\rangle$ is incompatible with $\mathfrak{R}\langle i\rangle$. But i is a specialization of η over $\mathfrak{R}$. For the substitution $y=i$ annuls F and y_2-y, but does not annul their initials. Hence it annuls every polynomial of Π.

3. 2. *Compatibility of "most" specializations.* Let $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ be an extension of the difference field $\mathscr{F}$. The following theorem provides a restriction on the specializations of $\eta_1, \cdots, \eta_n$ over $\mathscr{F}$ which generate extensions of $\mathscr{F}$ incompatible with $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$.

THEOREM 2. *There is an element $\gamma\neq 0$ in $\mathscr{F}\{\eta_1, \cdots, \eta_n\}$ such that if $\bar{\eta}_1, \cdots, \bar{\eta}_n$ is a specialization over $\mathscr{F}$ of $\eta_1, \cdots, \eta_n$, and the corresponding specialization of γ is not 0, then $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ and $\mathscr{F}\langle\bar{\eta}_1, \cdots, \bar{\eta}_n\rangle$ are compatible extensions of $\mathscr{F}$.*

Proof. Let Π be the reflexive prime difference ideal in $\mathscr{F}\{y_1, \cdots, y_n\}$ with generic zero $\eta_1, \cdots, \eta_n$. Theorem 2 is equivalent to the statement that there is a polynomial Q in $\mathscr{F}\{y_1, \cdots, y_n\}$, but not in Π, such that if $\lambda_1, \cdots, \lambda_n$ is a zero of Π but not of Q, then $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are compatible extensions of $\mathscr{F}$. We shall prove this statement

We denote by $\mathscr{G}$ the subfield of $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ consisting of those of its elements which are algebraic over $\mathscr{F}$. By [6] there is a finite set of elements of $\mathscr{G}$ which generate $\mathscr{G}$ when adjoined, with their transforms, to $\mathscr{F}$. Since these elements are algebraic over $\mathscr{F}$ it follows that there is an element δ such that $\mathscr{G}=\mathscr{F}\langle\delta\rangle$. There exist polynomials P, Q in $\mathscr{F}\{y_1, \cdots, y_n\}$, Q not in Π, such that δ is obtained by replacing the y_i in P/Q by the corresponding η_i. We shall show that Q has the properties claimed in the preceding paragraph.

Let Σ be the reflexive prime difference ideal in $\mathscr{F}\{w\}$ with generic zero δ. Let $B_0, B_1, \cdots, B_r$ be a characteristic set for Σ. When the w_i, $i=0, 1, \cdots$, are replaced by P_i/Q_i in the polynomials $B_0, \cdots, B_r$ and the resulting expressions are multiplied by an appropriate product of powers of transforms of Q, there results a set, $C_0, \cdots, C_r$, of polynomials of Π.

For a zero $\lambda_1, \cdots, \lambda_n$ of Π which is not a zero of Q we define the element δ' to be the result of replacing the y_i in P/Q by the corresponding λ_i. Since the λ_i annul $C_0, \cdots, C_r$ it is easy to see that δ' is a zero of $B_0, \cdots, B_r$. Because B_0 is of zero order, any zero of B_0 is

[15] The incompatibility of these extensions of $\mathfrak{R}$ is discussed in Ritt [**10**].

the generic zero of a prime ideal whose manifold is an ordinary manifold of B_0. Hence δ' is the generic zero of an ideal Σ' of this description. But Σ' must be Σ. For no other such ideal contains every B_i, $i=0, \cdots, r$.

It follows that $\mathscr{G}$ and $\mathscr{F}\langle\delta'\rangle$ are isomorphic under a mapping which leaves fixed the elements of $\mathscr{F}$. Let $\mathscr{G}'$ denote the field consisting of those elements of $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ which are algebraic over $\mathscr{F}$. Evidently $\mathscr{G}'$ is an extension of $\mathscr{F}\langle\delta'\rangle$. It follows from the definition of compatibility and from the preceding statement that $\mathscr{G}$ and $\mathscr{G}'$ are compatible extensions of $\mathscr{F}$. By results obtained in proving Theorem 1 of [5] this implies that $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are compatible extensions of $\mathscr{F}$. This proves Theorem 2.

3. 3. *Alternate proof of Theorem 2.* We give another proof of Theorem 2 in the case that Π has dimension $n-1$. This proof has the advantage of furnishing a polynomial Q explicitly.

Let $y_1, \cdots, y_{n-1}$ be a parametric set of indeterminates for Π. With the ordering $y_1, \cdots, y_n$ of the indeterminates, let F be the first polynomial of a characteristic set of Π. *Then the y_n-separant of F may be used as Q.*

Proof. Let S denote this separant and h the effective order of F in y_n. Let $\lambda_1, \cdots, \lambda_n$ be a zero of Π which is not a zero of S.

In the field $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ the manifold of F has a component $\mathfrak{M}$ containing $\lambda_1, \cdots, \lambda_n$. Let Σ be the reflexive prime difference ideal in $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle\{y_1, \cdots, y_n\}$ whose manifold is $\mathfrak{M}$. Let $\alpha_1, \cdots \alpha_n$ be a generic zero of Σ. Since $\lambda_1, \cdots, \lambda_n$ is not a zero of S, $y_1, \cdots, y_{n-1}$ constitute a parametric set for Σ, and Σ is of effective order h in y_n. We denote by Σ' the reflexive prime difference ideal $\Sigma \cap \mathscr{F}\{y_1, \cdots, y_n\}$.

Since Σ' contains F its manifold is either a component of F or is properly contained in a component of F. The latter case is impossible because Σ' contains no nonzero polynomial of effective order less than h in y_n or free of y_n. Since $\lambda_1, \cdots, \lambda_n$ is a zero of both Π and Σ', but not a zero of S, it follows from [7] that Π and Σ' are identical. Hence $\alpha_1, \cdots, \alpha_n$ is a zero of Π, and evidently a generic zero. Then $\mathscr{F}\langle\alpha_1, \cdots, \alpha_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are isomorphic under a mapping which leaves fixed the elements of $\mathscr{F}$ and carries α_i into η_i, $i=1, \cdots, n$. Since $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n;\ \alpha_1, \cdots, \alpha_n\rangle$ is defined this implies that $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are compatible extensions of $\mathscr{F}$. This completes the proof.

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Rutgers University

PSEUDO-DISCRIMINANT AND DICKSON INVARIANT

JEAN DIEUDONNÉ

1. Let E be a vector space of finite dimension over a field K. To a bilinear symmetric form $f(x, y)$ defined over $E \times E$ is attached classically the notion of *discriminant*: it is an element of K which is not entirely defined by f; however, it is entirely determined when in addition a basis of E is chosen, and when the basis is changed, the discriminant is multiplied by a *square* in K. More precisely, let u be a linear mapping of E into E, and let $f_1(x, y)=f(u(x), u(y))$ the form "transformed" by u; if $\Delta(f)$, $\Delta(f_1)$ are the discriminants of f and f_1 with respect to the *same* basis of E, and $D(u)$ the determinant of u with respect to that basis, then one has the classical relation

$$\Delta(f_1)=(D(u))^2\Delta(f) . \tag{1}$$

When K has characteristic $\neq 2$, the preceding results may be expressed in terms of the "quadratic form" $f(x, x)$ associated to $f(x, y)$. However, when K has characteristic 2, the one-to-one association between bilinear symmetric forms and quadratic forms no longer subsists. More precisely, to a given *alternate* symmetric form $f(x, y)$ (that is, $f(x, x)=0$ for all $x \in E$) is associated a whole family of quadratic forms $Q(x)$, satisfying the fundamental identity

$$Q(x+y)=Q(x)+Q(y)+f(x, y) \tag{2}$$

and to all these Q is associated the same discriminant of f (with respect to a given basis).

Now C. Arf [1] has introduced an element $\Delta(Q)$ attached to Q and to a given *symplectic basis* of E (with respect to the form f) which we shall call the *pseudo-discriminant* of Q. He proved moreover that under a change of symplectic basis, $\Delta(Q)$ is transformed in the following way: let $\wp$ be the homomorphism $\xi \to \xi+\xi^2$ of the additive group K into itself; then the pseudo-discriminants of Q with respect to two different symplectic bases have a *difference* which has the form $\wp(\lambda)$. Arf's proof is rather lengthy and proceeds by induction on n. We propose to show how the pseudo-discriminant is related to the *Clifford algebra* of Q in a way which parallels the well-known relation between the discriminant of f and the Clifford algebra of f over a field of characteristic $\neq 2$. At the same time, this will clear up the origin of a curiously isolated result obtained by L. E. Dickson for the orthogonal

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group $O_n(K, Q)$ over a finite field of characteristic 2: the transformations u of that group are defined by the condition $Q(u(x))=Q(x)$, and Dickson showed [4, p. 206] that a certain bilinear polynomial $D(u)$ in the elements of the matrix of u (with respect to a symplectic basis), turns out to be always equal to 0 or 1 for elements of $O_n(K, Q)$ (the first case occurring if and only if u is a product of an *even* number of transvections of $O_n(K, Q)$; see [6, p. 301]). Now the connection with the Clifford algebra which we mentioned above leads one in a natural way to form the polynomial $D(u)$ for an arbitrary *symplectic* transformation u; if $Q_1(x)=Q(u(x))$ is then the "transformed" of Q by u, and $\Delta(Q)$, $\Delta(Q_1)$ and $D(u)$ are computed with respect to the *same* symplectic basis, we will prove the following identity, which can be considered as the counter-part of (1)

$$\Delta(Q_1)=\Delta(Q)+\wp(D(u)). \tag{3}$$

Dickson's result follows obviously from this relation.

2. **We shall** always suppose that the alternate form f is nondegenerate, which implies that $n=2m$ is *even*, and that the forms Q associated with f are *nondefective* [5, p. 39–40]. For the definition of the *Clifford algebra* $C(Q)$ of a quadratic form Q associated to f, we refer the reader to [3] or [6]. If $(e_i)_{1\leq i\leq n}$ is a symplectic basis of E, such that

$$f(e_i, e_{m+j})=\delta_{ij}, \quad f(e_i, e_j)=0, \quad f(e_{m+i}, e_{m+j})=0 \qquad 1\leq i, j\leq m,$$

then the unit element and the e_i $(1\leq i\leq n)$ constitute a system of generators for $C(Q)$, with the relations

$$\left\{\begin{array}{lll} e_i^2=Q(e_i), & e_{m+i}^2=Q(e_{m+i}), & e_ie_j=e_je_i \\ e_{m+i}e_{m+j}=e_{m+j}e_{m+i}, & e_ie_{m+j}+e_{m+j}e_i=\delta_{ij} & \qquad 1\leq i, j\leq m. \end{array}\right. \tag{4}$$

From this it follows that $C(Q)$ is an algebra of rank 2^{2m} over K. Moreover, the elements of *even degree* of $C(Q)$ (generated by the products of an even number of the e_i's) constitute a subalgebra $C^+(Q)$ of rank 2^{2m-1} over K, and it can be shown that the center Z of that algebra has rank 2 over K [3, p. 44]. Now, it is readily verified from (4) that the element

$$z=e_1e_{m+1}+e_2e_{m+2}+\cdots+e_me_{2m} \tag{5}$$

commutes with all products e_he_k, and therefore constitutes with the unit element a basis for Z over K. From (4) it follows that $z^2+z=\Delta(Q)$, where

$$\Delta(Q)=Q(e_1)Q(e_{m+1})+Q(e_2)Q(e_{m+2})+\cdots+Q(e_m)Q(e_{2m}) \tag{6}$$

is precisely the *pseudo-discriminant* of Q relative to the basis (e_i) considered by Arf. Now the fact that $\Delta(Q)$ has the form $\wp(\lambda)$ expresses the fact that the equation $z^2+z=\Delta(Q)$ has a solution in K, in other words, that Z is not a field. When Z is a field, it is a separable quadratic field over K, and if it is generated by the roots of any equation $t^2+t=\mu$, then μ and $\Delta(Q)$ differ by an element of the form $\wp(\lambda)$ [2, p. 177, exerc. 8]. This proves immediately that when the pseudo-discriminant is computed with respect to two different symplectic bases, the values obtained have a difference of the form $\wp(\lambda)$.

3. **We are now** going to make the above result more precise by proving (3). If u is a symplectic transformation, the elements $u(e_i)$ $(1 \leq i \leq 2m)$ constitute again a symplectic basis for E, hence also a system of generators for the Clifford algebra $C(Q)$, satisfying relations similar to (4) (with $Q(u(e_i))$ replacing $Q(e_i)$). The element

$$z' = u(e_1)u(e_{m+1}) + \cdots + u(e_m)u(e_{2m}) \tag{7}$$

constitutes therefore, with the unit element, a basis for Z over K, in other words, z' has the form $p+qz$, where p, q are in K. Now it is easy to compute z' as a function of the coefficients of the matrix of u with respect to (e_i) : let

$$u(e_i) = \sum_{j=1}^m a_{ij}e_j + \sum_{j=1}^m b_{ij}e_{m+j}$$
$$u(e_{m+i}) = \sum_{j=1}^m c_{ij}e_j + \sum_{j=1}^m d_{ij}e_{m+j} .$$

Let on the other hand $Q(e_i)=\alpha_i$, $Q(e_{m+i})=\beta_i$. Then z' is a linear combination of elements e_he_k, and it follows from (4) and (5) that we need only consider among those elements the squares e_i^2 and the products e_ie_{m+i}, $e_{m+i}e_i$ since we know in advance that z' can contain no other elements from the basis of $C^+(Q)$. We thus obtain

$$p = \sum_{i=1}^m \sum_{j=1}^m (\alpha_j a_{ij}c_{ij} + \beta_j b_{ij}d_{ij} + b_{ij}c_{ij}) \tag{8}$$

$$q = \sum_{i=1}^m (a_{ij}d_{ij} + b_{ij}c_{ij}) . \tag{9}$$

But it follows, from the fact that the transposed matrix of u is again the matrix of a symplectic transformation, that $q=1$. The expression on the right of (8) is the *Dickson invariant* $D(u)$; as the relation $z'=p+z$ yields $z'^2+z'=z^2+z+p^2+p$, the identity (3) follows immediately from (6).

4. **We cannot expect,** of course, that the mapping $u \to D(u)$ should be a homomorphism of the symplectic group $Sp_{2m}(K)$ into the additive group of K, if only because we know that $Sp_{2m}(K)$ is a simple group. However, there are some relations between the Dickson invariants of

two symplectic transformations u, v and the Dickson invariant of their product. In fact, it follows immediately from the expression of z' obtained in § 3, that we have

$$D(vu)=D(u)+D_u(v) \tag{10}$$

where $D(u)$ and $D(vu)$ are the Dickson invariants of u and vu with respect to the basis (e_i), and $D_u(v)$ the Dickson invariant of v *with respect to the basis* $(u(e_i))$. This general formula takes a simpler shape when u is an *orthogonal transformation*, because then $Q(u(e_i))=Q(e_i)$ for $1\leq i\leq 2m$; on the other hand, the matrix of v with respect to the basis $(u(e_i))$ is the same as the matrix of $u^{-1}vu$ with respect to (e_i), and we thus obtain

$$D(vu)+D(u^{-1}vu)=D(u)\,. \tag{11}$$

But in this identity we can replace v by uvu^{-1}; therefore we also have

$$D(uv)=D(u)+D(v) \tag{12}$$

when u is an *orthogonal* transformation, v an arbitrary *symplectic* transformation ($D(u)$ being equal to 0 or 1, as recalled above).

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Northwestern University

Added in proof (November 1955): Since this paper was submitted for publication, the following papers, containing substantially the result of § 2, have appeared:

M. Kneser, *Bestimmung des Zentrums der Cliffordschen Algebren einer quadratischen Form über einem Körper der Charakteristik* 2, J. Reine Angew. Math., **193** (1954), 123-125.

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A COMPARISON THEOREM FOR EIGENVALUES OF NORMAL MATRICES

KY FAN

The following interesting theorem was recently obtained by H. Wielandt (Oral communication, see also J. Todd [3]):

Let M, N be two normal matrices of order n, and let r denote the rank of $M-N$. Let D be an arbitrary closed circular disk in the complex plane, If D contains exactly p eigenvalues of M, and exactly q eigenvalues of N, then $|p-q|\leq r$.

It is then natural to raise the following question: Without considering the rank of $M-N$, is it possible to compare the eigenvalues of M and N in a manner similar to that of Wielandt's theorem? The purpose of this Note is to present such a rank-free comparison theorem which includes Wielandt's theorem stated above.

THEOREM. *Let M, N be two normal matrices*[1] *of order n and let r be an integer such that $0\leq r<n$. Let $\varepsilon\geq 0$ be such that ε^2 is not less than the $(r+1)$th eigenvalue of $(M-N)^*(M-N)$, when the eigenvalues of $(M-N)^*(M-N)$ are arranged in descending order.*[2] *If a closed circular disk*

$$|z-z_0|\leq\rho$$

contains p eigenvalues of M, then the concentric disk

$$|z-z_0|\leq\rho+\varepsilon$$

contains at least $p-r$ eigenvalues of N.

While Wielandt's proof of his theorem uses geometric arguments involving convexity, the proof of our theorem will be based on an inequality (Lemma below). This difference in methods explains why our result is of more quantitative character than Wielandt's theorem.

LEMMA. *Let A, B be any two matrices*[3] *of order n. If $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$ are the eigenvalues of A^*A, B^*B and $(A+B)^*(A+B)$ respectively, each arranged in descending order*

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[1] The elements of all matrices considered here are real or complex numbers.

[2] As usual, the adjoint of a matrix A is denoted by A^*.

[3] Here A, B need not be normal.

$$\alpha_i \geq \alpha_{i+1}, \quad \beta_i \geq \beta_{i+1}, \quad \gamma_i \geq \gamma_{i+1}, \qquad (1 \leq i \leq n-1)$$

then the inequality

$$\sqrt{\gamma_{i+j+1}} \leq \sqrt{\alpha_{i+1}} + \sqrt{\beta_{j+1}}$$

holds for any two nonnegative integers i, j such that $i+j+1 \leq n$.

A more general form of this lemma (valid for completely continuous linear operators in a Hilbert space) has been given in [2], and is a generalization of a classical inequality of H. Weyl [4, p. 445] concerning eigenvalues of sum of two symmetric kernels of linear integral equations.

Proof of the theorem. Let $\{\mu_i\}$, $\{\nu_i\}$ denote the eigenvalues of M, N respectively and so arranged that

$$|\mu_i - z_0| \geq |\mu_{i+1} - z_0|, \quad |\nu_i - z_0| \geq |\nu_{i+1} - z_0|, \qquad (1 \leq i \leq n-1).$$

Let

$$A = M - z_0 I, \qquad B = N - M.$$

Then $A+B = N - z_0 I$. Let $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$ denote the eigenvalues of A^*A, B^*B and $(A+B)^*(A+B)$, each arranged in descending order. As M, N are both normal, we have

$$\alpha_i = |\mu_i - z_0|^2, \qquad \gamma_i = |\nu_i - z_0|^2, \qquad (1 \leq i \leq n).$$

By the above Lemma, we have

$$|\nu_{i+r} - z_0| \leq |\mu_i - z_0| + \sqrt{\beta_{r+1}}, \qquad (1 \leq i \leq n-r).$$

Using our hypothesis $\beta_{r+1} \leq \varepsilon^2$, we obtain

$$(1) \qquad |\nu_{i+r} - z_0| \leq |\mu_i - z_0| + \varepsilon, \qquad (1 \leq i \leq n-r).$$

Let p denote the number of eigenvalues μ_i of M contained in the disk $|z - z_0| \leq \rho$, and q the number of eigenvalues ν_i of N contained in the concentric disk $|z - z_0| \leq \rho + \varepsilon$. We shall prove that

$$(2) \qquad q \geq p - r.$$

If $n-q-r<1$, then $q \geq n-r \geq p-r$. Thus we may assume $1 \leq n-q-r$. By (1),

$$|\nu_{n-q} - z_0| \leq |\mu_{n-q-r} - z_0| + \varepsilon.$$

But, according to the definition of q, we have

$$|\nu_{n-q} - z_0| > \rho + \varepsilon.$$

Therefore

$$|\mu_{n-q-r}-z_0|>\rho ,$$

which implies $n-q-r\leq n-p$ or (2). Our theorem is thus proved.

COROLLARY. *Let M, N be two normal matrices of order n and let r be an integer such that $0\leq r<n$. Let $x_1, x_2, \cdots, x_{n-r}$ be $n-r$ orthonormal vectors in the unitary n-space. If a closed circular disk $|z-z_0|\leq\rho$ contains p eigenvalues of M, then the concentric disk*

$$(3) \qquad |z-z_0|\leq\rho+\Big(\sum_{i=1}^{n-r}\|(M-N)x_i\|^2\Big)^{\frac{1}{2}}$$

contains at least $p-r$ eigenvalues of N.

Proof. By a minimum property of eigenvalues of Hermitian matrices [1, Theorem 1], the expression

$$\sum_{i=1}^{n-r}\|(M-N)x_i\|^2=\sum_{i=1}^{n-r}\Big((M-N)^*(M-N)x_i,\ x_i\Big)$$

is not less than the sum of the last $n-r$ eigenvalues of $(M-N)^*(M-N)$, and consequently not less than the $(r+1)$th eigenvalue of $(M-N)^*(M-N)$. Thus the corollary follows directly from the theorem.

In case r is the rank of $M-N$, we can choose $n-r$ orthonormal vectors $x_1, x_2, \cdots, x_{n-r}$ such that

$$(M-N)x_i=0 \qquad (1\leq i\leq n-r).$$

Then the disk (3) becomes $|z-z_0|\leq\rho$ and the corollary reduces to Wielandt's theorem.

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THE AMERICAN UNIVERSITY AND
UNIVERSITY OF NOTRE DAME

ON THE CONVERGENCE BEHAVIOUR OF TRIGONOMETRIC INTERPOLATING POLYNOMIALS

RICHARD P. GOSSELIN

1. Grünwald [1] and Marcinkiewicz [2] have shown by examples the existence of continuous functions for which the sequence of Lagrange interpolating polynomials taken at the Tchebysheff abcissas diverges at each point of $[-1, 1]$. Marcinkiewicz constructed a function which actually proved an equivalent proposition, the existence of continuous functions for which the sequence of trigonometric interpolating polynomials taken at an even number of equidistant points centered at the origin diverges everywhere.

A similar result is known if for the nth polynomial the interpolating points are of the form $2\pi i/(2n+1)$, $i=0, \pm 1, \cdots$; the sequence of interpolating polynomials corresponding to a certain continuous function, $f(x)$, diverges for every $x \not\equiv 0 \pmod{2\pi}$. The point $x=0$ must be excluded because it is of the form $2\pi i/(2n+1)$ for each n, and hence the nth polynomial must equal $f(x)$ there. (cf. Zygmund [3, p. 75]). We shall consider more generally the following sets of points

$$\alpha + \frac{2i}{2n+1}\pi, \qquad i=0, \pm 1, \pm 2, \cdots \tag{1}$$

where α is any real number which is held fixed as n varies. The points (1) are called the fundamental points of interpolation. We shall denote the nth trigonometric interpolating polynomial, that is, the uniquely defined polynomial of order not greater than n which agrees with a given periodic function $f(x)$ at the points of (1), by $I_n^{(\alpha)}(x; f)$, except that we write $I_n(x; f)$ for $I_n^{(0)}(x; f)$.

In this paper, by refinements of the Marcinkiewicz example, along with adjustments for the new set of fundamental points, we show the strong dependence of the convergence behaviour of $I_n^{(\alpha)}(x; f)$ for certain functions $f(x)$ on the number α. For proper choice of α, the convergence behaviour may be the worst possible, divergence for all $x \not\equiv 0$ (mod 2π), whereas for the same function, another choice of α will lead to uniform convergence of the above sequence. We make these notions precise in the statements of our theorems.

THEOREM 1. *For any real number α, irrational with respect to π, there is a continuous function $f(x)$ for which the sequence $I_n(x; f)$ diverges for all $x \not\equiv 0$* (mod 2π), *but for which the sequence $I_n^{(\alpha)}(x; f)$ converges uniformly.*

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2. The function in question, $f(x)$, is of the form

$$\sum_{i=1}^{\infty} n_i^{-1/2} f_{n_i}(x) .$$

We shall define each $f_n(x)$ on certain points and impose some further general conditions which ensure that $I_s(x;\, f)$ diverges for every $x \not\equiv 0 \pmod{2\pi}$. This part of the construction is quite similar to that of the Marcinkiewicz example and is discussed in [3], so that our remarks here will be brief.

Each function $f_n(x)$ satisfies the following conditions: it is continuous and bounded by 1; smooth enough so that $I_s(x;\, f_n)$ converges uniformly to it; but such that there is a bounded integral-valued function $\mu(x)$ defined on the closed intervals $[1/n,\ \pi-1/n]$, $[\pi+1/n,\ 2\pi-1/n]$, and also for $x=\pi$ for which $|I_{\mu(x)}(x;\, f_n)|>n$. Thus for each n, we choose m so large that

$$(2)\qquad \int_{2\pi/m}^{1/n} \frac{d\omega_{2s+1}(\theta)}{2\sin(\theta/2)} \geq M(n)\,,\quad s\geq m\,;\qquad \begin{array}{l} \omega_{2s+1}(\theta)=\theta_i^{(s)},\quad \theta_i^{(s)}\leq\theta<\theta_{i+1}^{(s)}\,, \\ \theta_i^{(s)}=\dfrac{2\pi i}{2s+1},\quad i=0,\ \pm1,\ \pm2,\cdots . \end{array}$$

$M(n)$ is some function of n which we may take as large as we wish. Let $p_1, p_2, \cdots, p_m$ be integers all depending upon n such that $m\leq p_i$ and

$$m(2p_i+3)^3<2p_{i+1}+1\,,\qquad i=1,\ 2,\cdots,\ m-1.$$

For each p, let S_p be the system of points $\theta_i^{(p)}$, $i=0,\ 1,\ 2,\cdots,\ 2p$; and let $S_p(u)$ be the intersection of S_p with the interval $[u,\ 2\pi]$. We define $f_n(x)$ on S_{p_1} as follows:

$$f_n(\theta_i^{(p_1)})=\begin{cases}(-1)^i, & \theta_i^{(p_1)}\in S_{p_1}(2\pi/m)\\ 0\,, & \theta_i^{(p_1)}\in S_{p_1}-S_{p_1}(2\pi/m).\end{cases}$$

Since S_{p_1} and S_{p_1+1} are disjoint, except for the point $x=0$, we may define $f_n(\theta)$ in the same way in S_{p_1+1}; that is equal to $(-1)^i$ if $\theta_i^{(p_1+1)} \in S_{p_1+1}(2\pi/m)$ and 0 elsewhere in S_{p_1+1}. Suppose now that $f_n(\theta)$ has been defined for $\theta\in S_{p_i}\cup S_{p_i+1}$, $i=1,\ 2,\cdots,\ k-1$. For the points of $S_{p_k}\cup S_{p_k+1}$ which coincide with points of $S_{p_i}\cup S_{p_i+1}$, $i=1,\ 2,\cdots,\ k-1$, the original definition holds. For the remaining points of $S_{p_k}\cup S_{p_k+1}$, we define $f_n(\theta)$ as follows:

$$f_n(\theta_i^{(p_k)})=\begin{cases}(-1)^i, & \theta_i^{(p_k)}\in S_{p_k}(2\pi k/m)\\ 0, & \theta_i^{(p_k)}\in S_{p_k}-S_{p_k}(2\pi k/m)\end{cases};$$

$$f_n(\theta_i^{(p_k+1)})=\begin{cases}(-1)^i, & \theta_i^{(p_k+1)}\in S_{p_k+1}(2\pi k/m)\\ 0, & \theta_i^{(p_k+1)}\in S_{p_k+1}-S_{p_k+1}(2\pi k/m)\end{cases}$$

Thus by recurrence, $f_n(\theta)$ is defined for all points of $S_{p_i}\cup S_{p_i+1}$, $i=1, 2, \cdots, m-1$. $f_n(\theta)$ may be defined arbitrarily elsewhere except that it must be continuous, bounded by 1, and smooth enough to ensure the uniform convergence of $I_s(x;\ f_n)$.

Every point x of the interval $[1/n,\ 2\pi-1/n]$ belongs to some interval $[2(k-1)\pi/m,\ 2k\pi/m]$, $k=1,\ 2, \cdots, m-1$. For x in the kth interval, we may write according to a well known formula

$$I_s(x;\ f_n)=\frac{1}{\pi}\int_0^{2\pi} f_n(\theta)D_s(x-\theta)d\omega_{2s+1}(\theta)$$

$$=\frac{\sin (s+1/2)x}{\pi}\int_0^{2\pi} f_n(\theta)\frac{\cos(s+1/2)\theta}{2\sin\left(\dfrac{x-\theta}{2}\right)}d\omega_{2s+1}(\theta)\,, \qquad s=p_k,\ p_k+1$$

where

$$D_s(x)=\frac{\sin (s+1/2)x}{2\sin (x/2)}\,.$$

By arguments similar to those in [2], we may show, using (2), that

$$|I_s(x;\ f_n)|\geq\left|\frac{\sin (s+1/2)x}{\pi}\right|M(n)+O(n)\,, \qquad s=p_k,\ p_k+1\,.$$

If $x=\pi$, then $\sin (s+1/2)x=\pm 1$. If x belongs to one of the intervals $[1/n,\ \pi-1/n,\ \pi+1/n,\ 2\pi-1/n]$, then either $|\sin (p_k+1/2)x|\geq 1/\pi n$ or $|\sin(p_k+3/2)x|\geq 1/\pi n$. This shows with suitable choice of $M(n)$ that

$$|I_s(x,\ f_n)|>n, \qquad x\in[1/n,\ \pi-1/n],\ [\pi+1/n,\ 2\pi-1/n],\ \text{or } x=\pi$$

where s is chosen to be p_k or p_k+1 for some k. The n_i's are spread out so sparsely that the following conditions are satisfied:

$$\sum_{i=1}^{\infty} n_i^{-1/2}<\infty\ ; \quad p_{m-1}(n_i)\sum_{k=i+1}^{\infty} n_k^{-1/2}<1\ ; \quad \text{and} \quad |I_s(x;\ f_{n_k})|<2,\ m(n_i)\leq s,\ k<i.$$

Because of the first condition, $f(x)$ is continuous, and the last can be satisfied by the uniform convergence of $I_s(x;\ f_n)$. By well known arguments (cf., for example, Zygmund [3, pp. 79, 80]), these conditions are sufficient to make $I_s(x;\ f)$ diverge for every $x\not\equiv 0 \pmod{2\pi}$.

3. **Our own proof** depends upon defining more explicitly each $f_n(x)$ throughout the interval $[0, 2\pi]$. Let $T(n)$ denote the set of points where $f_n(x)$ has already been defined to be ± 1. Let $r=r_n$ be the number of points of $T(n)$. For each s, there exists an integer $\nu(s)$ such that all the points

$$x_j(s)=\alpha+\frac{2i\pi}{2s+1}+\frac{2\nu(s)\pi}{2s+1}, \qquad j=0,\ 1,\ 2,\ \cdots,\ 2s$$

belong to the interval $[0, 2\pi]$. All of the numbers

$$x_j(s)-\frac{2i\pi}{2p+1}, \qquad \begin{array}{l} j=0,1,\cdots,2s; \quad s=1,2,\cdots,s_1 \\ i=0,1,\cdots,2p;\ p=p_1,p_2,\cdots,p_{m-1},p_1+1,\cdots,p_{m-1}+1 \end{array}$$

are different from 0, else our hypothesis about α would be violated. Let $\gamma(s_1)$, depending upon n, be the minimum of the absolute value of these numbers. Choose disjoint intervals I_ξ of length $2\eta=2\eta_n$ centered about each point ξ of $T(n)$. We choose η so small that I_ξ contains no other points of $S_{p_i}\cup S_{p_i+1}$, $i=1, 2, \cdots, m-1$. Let s_1 be so large that (a) $2r/\eta(2s+1)<1$ for $s\geq s_1$. Now in each interval I_ξ, we let $f_n(x)$ equal a "roof" function, $\pm\lambda_\delta(x-\xi)$ where $\lambda_\delta(0)=1$; $\lambda_\delta(x)=0$ if $|x|\geq\delta$, and $\lambda_\delta(x-\xi)$ is linear from $\xi-\delta$ to ξ and from ξ to $\xi+\delta$. The plus or minus sign is of course chosen in accordance with the original definition of $f_n(x)$ at ξ. Let $\delta=\delta_n$ be so small that (b) $r\delta<\eta$ and (c) $\delta<\gamma(s_1)$. Elsewhere, we define $f_n(x)$ to be 0. Condition (c) guarantees that $f_n(x)$ will be 0 at all points $x_j(s)$, $j=0, 1, \cdots, 2s$; $s\leq s_1$. Since $f_n(x)$ satisfies a Lipschitz condition, both $I_s(x; f_n)$ and $I_s^{(\alpha)}(x; f_n)$ converge uniformly to $f_n(x)$.

4. **We now proceed** to show that $|I_s^{(\alpha)}(x; f_n)|<A$ for all x in $[0, 2\pi]$ where A is a constant independent of x, s, and n. We have

$$I_s^{(\alpha)}(x;\ f_n)=\frac{1}{\pi}\int_{x-\pi}^{x+\pi} f_n(\theta)D_s(x-\theta)d\omega_{2s+1}(\theta)$$

$$=\frac{2}{2s+1}\sum_j f_n(x_j(s))D_s(x-x_j(s)).$$

If $s\leq s_1$, then $I_s^{(\alpha)}(x; f_n)$ is 0 by condition (c) on δ. If $s>s_1$, we write

$$I_s^{(\alpha)}(x;\ f_n)=I_{s,1}^{(\alpha)}(x;\ f_n)+I_{s,2}^{(\alpha)}(x;\ f_n)=\frac{2}{2s+1}\{\textstyle\sum_1+\sum_2\}f_n(x_j(s))D_s(x-x_j(s))$$

where $I_{s,2}^{(\alpha)}(x; f_n)$ consists of those terms of the sum corresponding to the points $x_j(s)$ which belong to the interval I_ξ containing x for some

ξ (if x belongs to one such interval), and $I^{(\alpha)}_{s,1}(x;\, f_n)$ consists of the remaining terms of the sum. We have

$$|I^{(\alpha)}_{s,1}(x;\, f_n)| \leq \frac{\pi}{2s+1} \sum\nolimits_1 \frac{1}{|x-x_j(s)|} .$$

Let $\eta-\delta=\beta=\beta_n$. Then $|x-x_j(s)|\geq\beta$ for the terms of $\sum_1$, whether x belongs to any interval I_ξ or not. Now let $k_n(s)$ be the number of terms of $\sum_1$. It follows that

$$(3) \qquad |I^{(\alpha)}_{s,1}(x;\, f_n)| \leq \frac{\pi k_n(s)}{(2s+1)\beta} .$$

To estimate $k_n(s)$, consider all of the intervals I_ξ, except the one which contains x (if there is such). In each $f_n(x)$ is different from 0 only on a subinterval of length 2δ. Since successive fundamental points are separated by a distance $2\pi/(2s+1)$, there are at most $\langle\delta(2s+1)/\pi\rangle+1$ distinct points $x_j(s)$ of the sum $\sum_1$ in this interval, where $\langle y\rangle$ denotes the least integer greater than or equal to y. Since there are not more than r such intervals, we have that

$$k_n(s) \leq \frac{\delta r(2s+1)}{\pi} + 2r$$

and from (3)

$$(4) \qquad |I^{(\alpha)}_{s,1}(x;\, f_n)| \leq \frac{\delta r}{\beta} + \frac{2\pi r}{\beta(2s+1)} .$$

Condition (b) implies that $\delta<\eta/2$ for $r\geq2$, and hence that $\beta>\eta/2$. Thus $\delta r/\beta<2\delta r/\eta<2$, the latter inequality also following from condition (b). For the second quantity on the right side of (4), we have $r/(2s+1)\beta<2r/(2s+1)\eta<1$, the latter inequality being condition (a), which holds since $s>s_1$. Combining these results, we obtain from (4) that

$$(5) \qquad |I^{(\alpha)}_{s,1}(x;\, f_n)| \leq 2+\pi .$$

If x belongs to none of the intervals I_ξ, then the estimate (5) of $I^{(\alpha)}_{s,1}(x;\, f_n)$ will serve also for $I^{(\alpha)}_{s}(x;\, f_n)$. If x does belong to one of the intervals I_ξ, then

$$(6) \qquad I^{(\alpha)}_{s,2}(x;\, f_n) = \pm\frac{1}{2s+1}\sin\,[(s+1/2)(x-\alpha)]$$

$$\cdot \sum\nolimits_2 \lambda_\delta(x_j(s)-\xi)\frac{(-1)^j}{2\sin\,[(x-x_j(s))/2]}$$

where the index j corresponds to the points $x_j(s)$ of the interval I_ξ. The sum in (6) can be written as $\sum_{2,1}+\sum_{2,2}+\sum_{2,3}$ where $\sum_{2,1}$ consists of those terms of $\sum_2$ for which $|x-x_j(s)|\leq 2\pi/(2s+1)$, $\sum_{2,2}$ of the remaining terms for which $\sin[(x-x_j(s))/2]<0$, and $\sum_{2,3}$ of the remaining terms for which $\sin[(x-x_j(s))/2]>0$. Since there are at most three terms of $\sum_{2,1}$,

$$(7)\qquad \left|\frac{1}{2s+1}\sin[(s+1/2)(x-\alpha)]\sum_{2,1}\lambda_\delta(x_j(s)-\xi)\frac{(-1)^j}{2\sin[(x-x_j(s))/2]}\right|\leq 3.$$

The sum of successive terms of $\sum_{2,2}$ is

$$(8)\qquad \frac{\lambda_\delta(x_{j+1}(s)-\xi)}{2\sin[(x-x_{j+1}(s))/2]}-\frac{\lambda_\delta(x_j(s)-\xi)}{2\sin[(x-x_j(s))/2]}.$$

All of the terms, except possibly for two, of $\sum_{2,2}$ can be paired as in (8) so that $\lambda_\delta(x-\xi)$ is linear for $x_j(s)\leq x\leq x_{j+1}(s)$. For these two terms $4s+2$ is a bound. For the remaining, we apply the mean value theorem to obtain that the absolute value of the difference (8) is not greater than

$$\frac{\pi^2}{\delta(2s+1)|x-x_j(s)|}+\frac{\pi^3}{2(2s+1)|x-x_j(s)|^2}$$

and so

$$(9)\qquad \frac{1}{2s+1}\left|\sum_{2,2}\lambda_\delta(x_j(s)-\xi)\frac{(-1)^j}{2\sin[(x-x_j(s))/2]}\right|$$
$$\leq 2+\frac{\pi^2}{\delta(2s+1)^2}\sum_{2,2}\frac{1}{|x-x_j(s)|}+\frac{\pi^3}{2(2s+1)^2}\sum_{2,2}\frac{1}{|x-x_j(s)|^2}.$$

Since the number of terms of $\sum_{2,2}$ is not greater than $2+\delta(2s+1)/\pi$ and the smallest possible value for $|x-x_j(s)|$ is $2\pi/(2s+1)$, the second term on the right side of (9) is not greater than $1/2+\pi/\delta(2s+1)$. If $(2s+1)\delta/\pi<1$, then there is at most one term in the sum of (6) so that one would serve as a bound for $|I^{(\alpha)}_{s,2}(x;\ f_n)|$. Hence, we assume otherwise, and $\pi/\delta(2s+1)\leq 1$. For the third term on the right side of (9), since the smallest possible value for $|x-x_j(s)|$ is $2\pi/(2s+1)$, and since successive terms differ by $2\pi/(2s+1)$, we have

$$\frac{\pi^3}{2(2s+1)}\sum_{2,2}\frac{1}{|x-x_j(s)|^2}\leq\frac{\pi^3}{2(2s+1)^2}\left[\frac{(2s+1)^2}{(2\pi)^2}+\frac{(2s+1)^2}{(4\pi)^2}+\cdots\right]=c,$$

a constant. Hence, collecting these results, we see, using (9), that

$$(10)\qquad \frac{1}{2s+1}\left|\sum_{2,2}\lambda_\delta(x_j(s)-\xi)\frac{(-1)^j}{2\sin[(x-x_j(s))/2]}\right|\leq c+7/2.$$

A similar result holds for $\sum_{2,3}$, so that from (6), (7), and (10), we see that $|I_{s,2}^{(\alpha)}(x;\ f_n)|$ is bounded by a constant A_2 independent of x, s, and n. From (5), $|I_{s,1}^{(\alpha)}(x;\ f_n)|$ is bounded by a constant A_1, independent of x, s, and n. Thus, $|I_s^{(\alpha)}(x;\ f_n)|$ is bounded by a constant $A=A_1+A_2$.

5. The last result shown together with the uniform convergence of $I_s^{(\alpha)}(x;\ f_n)$ gives

$$|I_s^{(\alpha)}(x;\ f)-f(x)|\leq\sum_{i=0}^{N} n_i^{-1/2}|I_s^{(\alpha)}(x;f_{n_i})-f_{n_i}(x)|+\sum_{i=N+1}^{\infty} n_i^{-1/2}|I_s^{(\alpha)}(x;f_{n_i})-f_{n_i}(x)|$$

$$\leq\varepsilon\sum_{i=1}^{N} n_i^{-1/2}+(A+1)\sum_{i=N+1}^{\infty} n_i^{-1/2}$$

for s large enough. Since the right hand side is arbitrarily small with ε and $1/N$, our theorem is proved.

6. With a slight modification of the previous argument, we may establish the following theorem.

THEOREM 2. *There is a continuous function $f(x)$ such that $I_s(x;\ f)$ diverges for every $x\not\equiv 0$ (mod 2π) while for almost every number α, $I_s^{(\alpha)}(x;\ f)$ converges uniformly.*

Our function, $f(x)$, will be of the same form as in Theorem 1, $\sum_{i=1}^{\infty} n_i^{-1/2} f_{n_i}(x)$, where the $f_{n_i}(x)$ are sums of nonoverlapping roof functions. Let I_ξ, η, s_1 be defined as before. Consider

$$\text{(11)}\quad -\frac{2\pi l}{2s+1}+\theta_i^{(p)},\qquad \begin{matrix} l=-2s-1,\cdots,0,1,\cdots,2s;\ s\leq s_1 \\ i=0,1,\cdots,2p;\ p=p_1,p_1+1,\cdots,p_m,p_m+1 \end{matrix},\quad \theta_i^{(p)}=\frac{2\pi i}{2p+1}.$$

Suppose that there are $\tau=\tau_n$ such numbers. Choose symmetric neighbourhoods of length $2/\tau^2$ about each, and denote the set which consists of the sum of these neighbourhoods by R_n'. Let α belong to R_n (complement of R_n'), $0\leq\alpha\leq 2\pi$. Let $\nu(s)$ denote the least integer such that $\alpha+2\pi\nu(s)/(2s+1)\geq 0$. Clearly $-2s-1\leq\nu(s)\leq 0$. Then the numbers

$$\alpha+\left(\frac{2\pi j}{2s+1}\right)+\left(\frac{2\pi\nu(s)}{2s+1}\right),\qquad j=0,\ 1,\cdots,\ 2s$$

belong to the interval $(0,\ 2\pi)$ and so are our $x_j^{(\alpha)}(s)$. Also the numbers $j+\nu(s)$ are included in the numbers l of (11) for $s\leq s_1$ since $-2s-1\leq j+\nu(s)\leq 2s$. We choose δ such that it satisfies $r\delta<\eta$ (Condition (b) of Theorem 1) and in addition (c′) $\delta<1/\tau^2$. We have

$$x_j^{(\alpha)}(s)-\theta_i^{(p)}=\alpha-\left[-\frac{2\pi l}{2s+1}+\theta_i^{(p)}\right]$$

for some l such that $-2s-1\leq l\leq 2s$, $s\leq s_1$. Hence $|x_j^{(\alpha)}(s)-\theta_i^{(p)}|\geq 1/\tau^2>\delta$ by (c′) so that $I_s^{(\alpha)}(x;\ f_n)=0$ for all x, $s\leq s_1$, and all α in R_n. To show that $I_s^{(\alpha)}(x;\ f_n)$ is bounded for all s and all α in R_n, we employ the previous argument, which, beyond this point, used nothing about α.

Considering only the portions of R_n and R_n' in $(0,\ 2\pi)$, we have

$$|R_n'|\leq\frac{2}{\tau_n};\quad \sum_{i=1}^{\infty}|R'_{n_i}|\leq 2\sum_{i=1}^{\infty}\frac{1}{\tau_{n_i}}<\infty$$

for the n_i spread out sufficiently. Hence, except for a set of measure 0, every α belongs to at most a finite number of sets R'_{n_i} and so to every R_{n_i} for i large enough.

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UNIVERSITY OF CONNECTICUT
STORRS, CONNECTICUT

ON GENERATING FUNCTIONS OF THE JACOBI POLYNOMIALS

PETER HENRICI

1. Introduction. The series of Jacobi polynomials

$$(1) \qquad \sum_{n=0}^{\infty} a_n \rho^n P_n^{(\nu,\mu)}(\tau)$$

(a_n independent of ρ and τ) has in the case $a_n=1$ already been evaluated by Jacobi in terms of elementary functions, and there are several other known cases where it can be summed explicity. The sum of (1) is then usually called a generating function of the Jacobi polynomials. On the other hand, according to a particular case of a theorem which we have proved recently, every function of a certain class of regular solutions of the partial differential equation

$$(2) \qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\mu+1}{x}\frac{\partial u}{\partial x}+\frac{2\nu+1}{y}\frac{\partial u}{\partial y}=0$$

can be represented by a series of type (1), where

$$(3) \qquad \rho=x^2+y^2\,, \qquad \tau=\frac{x^2-y^2}{x^2+y^2}\,,$$

and may therefore be considered as a generating function of the Jacobi polynomials in the above sense. This fact is used in the present paper for the construction of an expansion of type (1) which contains several known results of this kind as special cases. As a side result we shall obtain some identities of Cayley-Orr type between the coefficients in the Taylor expansions of certain products of hypergeometric series.

In what follows x and y are considered as independent complex variables. Also the variables

$$(4) \qquad z=x+iy\,, \qquad z^*=x-iy$$

will be used. Our notation of special functions is in accordance with [5].

2. The expansion theorem. The special case $k=0$ of the main theorem of [6] is as follows:

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THEOREM. *Let*

$$\mu+\nu\neq -2,\ -3,\ -4,\ \cdots. \tag{5}$$

Let

$$u(x,\ y)=U(z,\ z^*)$$

be a solution of (2) *regular in the domain* $\mathscr{B}$: $|z|<r,\ |z^*|<r\ (r>0)$ *satisfying the conditions*

$$U(z,\ z^*)=U(-z,\ -z^*)=U(z^*,\ z) \tag{6}$$

and let

$$U(z,\ 0)=\sum_{n=0}^{\infty} a_n z^{2n}. \tag{7}$$

Then $u(x,\ y)$ *has in* $\mathscr{B}$ *the representation*

$$u(x,\ y)=\sum_{n=0}^{\infty} \gamma_n a_n \rho^n P_n^{(\nu,\mu)}(\tau), \tag{8}$$

where ρ *and* τ *are given by* (3) *and*

$$\gamma_n=\frac{2^{2n} n!}{(\mu+\nu+1+n)_n}=\frac{(\mu+\nu+1)_n n!}{\left(\dfrac{\mu+\nu+1}{2}\right)_n\left(\dfrac{\mu+\nu+2}{2}\right)_n} \tag{9}$$

3. A special solution of (2). We substitute in (2) *bipolar coordinates* $(\xi,\ \eta)$ which we define by

$$x=\frac{\sinh\xi}{\cosh\xi+\cos\eta},\qquad y=\frac{\sin\eta}{\cosh\xi+\cos\eta}. \tag{10}$$

They are connected with $(z,\ z^*)$ and $(\rho,\ \tau)$ respectively by the relations

$$\begin{aligned}\cosh\xi&=\frac{1+zz^*}{\tilde\omega}=\frac{1+\rho}{\tilde\omega},\\ \cos\eta&=\frac{1-zz^*}{\tilde\omega}=\frac{1-\rho}{\tilde\omega},\end{aligned} \tag{11}$$

where

$$\tilde\omega=\sqrt{(1-z^2)(1-z^{*2})}=\sqrt{1-2\rho\tau+\rho^2}. \tag{12}$$

(The square roots are positive for $z=z^*=0$, $\rho=0$.) Since (2) may be written in the form

$$\operatorname{div}(x^{2\mu+1}y^{2\nu+1}\operatorname{grad} u)=0 \tag{13}$$

and since the transformation (10) is isothermal[1], we obtain for $\psi(\xi, \eta)=u(x, y)$ the equation

$$\frac{\partial}{\partial\xi}\left(x^{2\mu+1}y^{2\nu+1}\frac{\partial\psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(x^{2\mu+1}y^{2\nu+1}\frac{\partial\psi}{\partial\eta}\right)=0.$$

Setting

$$s=\cosh\xi\,, \qquad t=\cos\eta$$

and

$$\psi(\xi, \eta)=(s+t)^{\mu+\nu+1}S(s)T(t)$$

one finds by the usual separation method that both $S(s)$ and $T(t)$ have to satisfy the differential equation

$$(v^2-1)\frac{d^2V}{dv^2}+2(\lambda+1)v\frac{dV}{dv}+[\lambda(\lambda+1)-\kappa(\kappa+1)]V=0\,, \tag{14}$$

where $v=s$, $\lambda=\mu$, if $V=S$, and $v=t$, $\lambda=\nu$, if $V=T$, κ being a separation parameter. A solution of (14) regular near $v=1$ is the function

$$\begin{aligned} V_\kappa^\lambda(v)&={}_2F_1\left[\begin{matrix}\lambda-\kappa,\ \lambda+\kappa+1; & \dfrac{1-v}{2}\\ \lambda+1 & \end{matrix}\right] \\ &=\Gamma(\lambda+1)(v^2-1)^{-\lambda/2}P_\kappa^{-\lambda}(v)\,. \end{aligned} \tag{15}$$

Here P denotes the Legendre function of the first kind.[2] Tracing back our substitutions and assuming that none of the numbers μ and ν is a negative integer, we may thus define a solution of (2) by

$$\Phi_\kappa^{(\mu,\nu)}(\rho, \tau)=\tilde\omega^{-\mu-\nu-1}V_\kappa^\mu\left(\frac{1+\rho}{\tilde\omega}\right)V_\kappa^\nu\left(\frac{1-\rho}{\tilde\omega}\right)\,. \tag{16}$$

Evidently this function satisfies the functional relations

$$\begin{aligned} \Phi_\kappa^{(\mu,\nu)}(\rho, \tau)&=\Phi_{-\kappa-1}^{(\mu,\nu)}(\rho, \tau) \\ \Phi_\kappa^{(\mu,\nu)}(\rho, \tau)&=\Phi_\kappa^{(\nu,\mu)}(-\rho, -\tau)\,. \end{aligned} \tag{17}$$

Among the many possible representations of $\Phi_\kappa^{(\mu,\nu)}$ in terms of hypergeometric functions we list the following, which is obtained by substituting equation 3.2 (24) of [5] for the Legendre functions involved in (15):

---

[1] Arising from the conformal transformation $z=\tanh\zeta$, $\zeta=\xi+i\eta$.

[2] The functions $V_\kappa{}^\lambda(v)$ could also be expressed in terms of Gegenbauer functions.

$$\Phi_\kappa^{(\mu,\nu)}(\rho,\tau)=\varphi\,{}_2F_1\left[\begin{matrix}\frac{\mu+\kappa+1}{2},\frac{\mu+\kappa+2}{2};X\\ \mu+1\end{matrix}\right] \times {}_2F_1\left[\begin{matrix}\frac{\nu-\kappa}{2},\frac{\nu-\kappa+1}{2};Y\\ \nu+1\end{matrix}\right]. \tag{18}$$

Here we have put

$$\varphi=(1+\rho)^{-\mu-\kappa-1}(1-\rho)^{\kappa-\nu}=(1+zz^*)^{-\mu-\kappa-1}(1-zz^*)^{\kappa-\nu}$$

and

$$X=\frac{2\rho(\tau+1)}{(1+\rho)^2}=\left(\frac{z+z^*}{1+zz^*}\right)^2,$$

$$Y=\frac{2\rho(\tau-1)}{(1-\rho)^2}=\left(\frac{z-z^*}{1-zz^*}\right)^2.$$

It is easy to see from this representation that the function

$$U(z,z^*)=\Phi_\kappa^{(\mu,\nu)}(\rho,\tau)$$

is regular in $|z|<1$, $|z^*|<1$ and that it satisfies the symmetry relations (6). Save for the mentioned exceptional values of the parameters, (15) defines therefore a solution of (2) for which the assumptions of the expansion principle of § 2 are satisfied.

4. The Jacobi expansion of $\Phi_\kappa^{(\mu,\nu)}$. From (18) we have immediately

$$U(z,0)={}_2F_1\left[\begin{matrix}\frac{\mu+\kappa+1}{2},\frac{\mu+\kappa+2}{2};z^2\\ \mu+1\end{matrix}\right]{}_2F_1\left[\begin{matrix}\frac{\nu-\kappa}{2},\frac{\nu-\kappa+1}{2};z^2\\ \nu+1\end{matrix}\right]. \tag{19}$$

If we denote by a_n the coefficient of z^{2n} in the Taylor expansion of the right hand side of (19), we obtain by the expansion principle the series

$$\tilde{\omega}^{-\mu-\nu-1}V_\kappa^\mu\left(\frac{1+\rho}{\tilde{\omega}}\right)V_\kappa^\nu\left(\frac{1-\rho}{\tilde{\omega}}\right)=\sum_{n=0}^\infty \gamma_n a_n\rho^n P_n^{(\nu,\mu)}(\tau), \tag{20}$$

which converges if

$$|z|<1, \qquad |z^*|<1,$$

or, what amounts to the same,

$$|\rho(\tau\pm\sqrt{\tau^2-1})|<1.$$

We note the following representations of a_n in terms of terminating hypergeometric series:

$$(21)\quad a_n=\frac{\left(\frac{\nu-\kappa}{2}\right)_n\left(\frac{\nu-\kappa+1}{2}\right)_n}{(\nu+1)_n n!}\,{}_4F_3\left[\begin{matrix}\frac{\mu+\kappa+1}{2}, & \frac{\mu+\kappa+2}{2}, & -\nu-n, & -n\,; \\ \mu+1, & \frac{2-\nu+\kappa}{2}-n, & \frac{1-\nu+\kappa}{2}-n & \end{matrix}\right],$$

$$(22)\quad a_n=\gamma_n^{-1}\frac{(\nu-\kappa)_n}{(\nu+1)_n}\,{}_3F_2\left[\begin{matrix}\mu+\kappa+1, & \kappa+1, & -n\,; \\ \mu+1, & 1+\kappa-\nu-n & \end{matrix}\right],$$

$$(23)\quad a_n=\gamma_n^{-1}\frac{(\mu+\nu+1)_n}{(\nu+1)_n}\,{}_3F_2\left[\begin{matrix}\mu+\kappa+1, & \mu-\kappa, & -n\,; \\ \mu+1, & \mu+\nu+1 & \end{matrix}\right].$$

Of these, (21) is obtained by straightforward Cauchy multiplication of the two power series on the right of (19). In order to prove (22), we consider (20) for the special value $\tau=1$ (that is, $z=z^*$). This gives on the left, using (17),

$$(24)\quad U(z,z)=(1+z^2)^{-\mu-\kappa-1}(1-z^2)^{\kappa-\nu}\,{}_2F_1\left[\begin{matrix}\frac{\mu+\kappa+1}{2}, & \frac{\mu+\kappa+2}{2}\,; & \frac{4z^2}{(1+z^2)^2} \\ & \mu+1 & \end{matrix}\right].$$

By a quadratic transformation [5, eq. 2.11 (34)] this ${}_2F_1$ can be expressed by one with argument z^2, and in view of

$$P_n^{(\nu,\mu)}(1)=\frac{(\nu+1)_n}{n!}$$

(20) thus becomes

$$(25)\quad (1-z^2)^{\kappa-\nu}\,{}_2F_1\left[\begin{matrix}\mu+\kappa+1, \ \kappa+1\,; \ z^2 \\ \mu+1\end{matrix}\right]=\sum_{n=0}^{\infty}\gamma_n a_n\frac{(\nu+1)_n}{n!}z^{2n}.$$

From this (22) follows again by Cauchy multiplication of the series on the left. Putting $\tau=-1$ (or $z=-z^*$) in (20) leads in a similar way to

$$(26)\quad (1-z^2)^{\kappa-\mu}\,{}_2F_1\left[\begin{matrix}\nu+\kappa+1, \ \kappa+1\,; \ z^2 \\ \nu+1\end{matrix}\right]=\sum_{n=0}^{\infty}\gamma_n a_n\frac{(\mu+1)_n}{n!}z^{2n}.$$

The representation (23) of a_n is remarkable for the fact that only one parameter in the ${}_3F_2$ depends on n. It is obtained by expressing the hypergeometric function on the left of (25) by one with argument $z^2/(z^2-1)$, expanding in terms of this argument, expanding the powers of $z^2/(z^2-1)$ in terms of powers of z^2 and rearranging.

From (19) it is easily seen by applying Euler's linear transformation to the two hypergeometric series simultaneously that $U(z,0)$ and hence a_n is a symmetric function of μ and ν. Therefore in (21), (22) and (23) the variables μ and ν may be interchanged. Furthermore, in view of (17) κ may be replaced everywhere by $-\kappa-1$. Many other representations for the coefficients a_n could be derived from the ones given above by the application of transformations of generalized hypergeometric series of unit argument. One example for this technique will be given at the end of § 6.

5. Special cases. (i) If $\kappa=0$, (23) yields by Vandermonde's theorem

$$a_n=\gamma_n^{-1}\frac{(\mu+\nu+1)_n}{(\nu+1)_n}\,{}_2F_1\begin{bmatrix}\mu, & -n\,; \\ \mu+\nu+1\end{bmatrix}=\gamma_n^{-1}$$

and from (15) we have

$$V_0^\lambda(v)=\left(\frac{1+v}{2}\right)^{-\lambda}.$$

Thus (20) reduces to

$$2^{\mu+\nu}\tilde{\omega}^{-1}(1+\rho+\tilde{\omega})^{-\mu}(1-\rho+\tilde{\omega})^{-\nu}=\sum_{n=0}^{\infty}\rho^n P_n^{(\nu,\mu)}(\tau)\,. \tag{27}$$

This is the classical generating series of Jacobi[3].

(ii) Since

$$V_\lambda^\lambda(v)=1\,, \tag{28}$$

other noteworthy special cases of (20) are to be expected for $\kappa=\mu$ or $\kappa=\nu$. In the first case we have from (23) (using the symmetry with respect to μ and ν)

$$a_n=\gamma_n^{-1}\frac{(\mu+\nu+1)_n}{(\mu+1)_n}\,.$$

Thus (20) yields, if (18) is used on the left,

$$(1+\rho)^{-\mu-\nu-1}\,{}_2F_1\begin{bmatrix}\dfrac{\mu+\nu+1}{2}\,, & \dfrac{\mu+\nu+2}{2}\,; & \dfrac{2\rho(\tau+1)}{(1+\rho)^2} \\ & \mu+1 & \end{bmatrix} \tag{29}$$

$$=\sum_{n=0}^{\infty}\frac{(\mu+\nu+1)_n}{(\mu+1)_n}\rho^n P_n^{(\nu,\mu)}(\tau)\,.$$

An equivalent formula is easily derived from a bilinear generating function due to Watson [10] and has been stated explicitly (but with a slight algebraic error) by Bailey [1, p. 102]. The result is given correctly by Buchholz [3, p. 143. eq. (20)].

The case $\kappa=\nu$ does in view of (17) not lead to something new. A

[3] See [**5**, eq. 10.8 (29)] and, for several direct proofs of the expansion, [**9**, p. 68].

similar, but not equivalent formula can be deduced from (20) by putting $\kappa=\mu+1$ or $\kappa=\nu+1$.

If $\kappa=\mu=\nu$, we obtain from (20) and (29) in virtue of (28) and

$$P_n^{(\mu,\mu)}(\tau)=\frac{(\mu+1)_n}{(2\mu+1)_n}C_n^{\mu+1/2}(\tau)\,,$$

the classical generating series of the Gegenbauer polynomials

$$\tilde{\omega}^{-2\mu-1}=\sum_{n=0}^{\infty}\rho^n C_n^{\mu+1/2}(\tau)\,.$$

(iii) Also in the cases $\mu=\pm\frac{1}{2}$ (or $\nu=\pm\frac{1}{2}$) the Jacobi polynomials reduce to Gegenbauer polynomials. Since $\Phi_\kappa^{(\mu,\nu)}$ likewise may be expressed in terms of Gegenbauer functions, (20) takes then the form of an addition theorem for these functions. This result has been given by us already elsewhere [7].

(iv) Putting $\rho=r/\kappa^2$ and letting $\kappa\rightarrow\infty$, we obtain from (20) and (23), since

$$\lim_{\kappa\rightarrow\infty} V_\kappa^\lambda(1-2w/\kappa^2)={}_0F_1\,[\lambda+1\,;\,w]\,, \tag{30}$$

the well-known formula (see the references to equation (42) of [6])

$$ {}_0F_1\left[\mu+1\,;\,r\,\frac{\tau+1}{2}\right]{}_0F_1\left[\nu+1\,;\,r\,\frac{\tau-1}{2}\right]=\sum_{n=0}^{\infty}\frac{r^n}{(\mu+1)_n(\nu+1)_n}P_n^{(\nu,\mu)}(\tau)\,. \tag{31}$$

With the exception of a result of Brafman [2], the special cases of (20) mentioned above cover to our knowledge all simple (that is, not bilinear) known generating functions of the Jacobi polynomials which are valid for general values of μ and ν.[4]

6. Identities of Cayley-Orr type. The formulae (19), (25) and (26) suggest identities between the coefficients of the expansions of certain hypergeometric products which in a symmetric way may be stated as follows:

Each of the three identities

$$(1-\zeta)^{\kappa-\nu}{}_2F_1\left[\begin{matrix}\mu+\kappa+1,\ \kappa+1\,;\,\zeta\\ \mu+1\end{matrix}\right]=\sum_{n=0}^{\infty}(\nu+1)_n A_n\zeta^n\,, \tag{a}$$

$$(1-\zeta)^{\kappa-\mu}{}_2F_1\left[\begin{matrix}\nu+\kappa+1,\ \kappa+1\,;\,\zeta\\ \nu+1\end{matrix}\right]=\sum_{n=0}^{\infty}(\mu+1)_n A_n\zeta^n\,, \tag{b}$$

[4] Brafman's result, which was originally established as a corollary to Bailey's decomposition formula for Appell's function F_4, has been proved by our method without the use of Bailey's formula in [**6**].

(c) $$ {}_2F_1\left[\begin{matrix}\frac{\mu+\kappa+1}{2}, & \frac{\mu+\kappa+2}{2} & ;\zeta \\ & \mu+1 & \end{matrix}\right] {}_2F_1\left[\begin{matrix}\frac{\nu-\kappa}{2}, & \frac{\nu-\kappa+1}{2} & ;\zeta \\ & \nu+1 & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{\left(\frac{\mu+\nu+1}{2}\right)_n \left(\frac{\mu+\nu+2}{2}\right)_n}{(\mu+\nu+1)_n} A_n \zeta^n $$

implies the other two.

This result is of a type considered first by Cayley and Orr [8]. While (a)$\sim$(b) is a special case of a result by Burchnall and Chaundy (see [4, eq. (13)]), the two equivalencies (a)$\sim$(c) and (b)$\sim$(c) as well as the method of their derivation seem to be new. Identities of this type have been investigated either by a discussion of the ordinary differential equations satisfied by the products of hypergeometric functions (for recent results obtained by this method, see [4]) or by transformations of the generalized hypergeometric series arising in the Cauchy multiplication of the power series under consideration. An account of Bailey's and Whipple's work in this direction can be found in [1]. In order to render our above result independent of the consideration of a special partial differential equation, we sketch a short proof of it by Whipple's method. By reasons of symmetry it suffices to prove (a)$\sim$(c). This amounts to a direct proof of the equality of (21) and (22). We first transform the ${}_3F_2$ in (22) into a saalschützian ${}_4F_3$ by equation 4.5 (1) of [1]. This gives

(32) $$ \gamma_n a_n = \frac{(\nu-\kappa)_n}{(\nu+1)_n} {}_3F_2\left[\begin{matrix}\mu+\kappa+1, & \kappa+1, & -n; \\ \mu+1, & 1+\kappa-\nu-n & \end{matrix}\right] = \frac{(\mu+\nu+1)_n}{(\nu+1)_n} {}_4F_3\left[\begin{matrix}\frac{\mu+\kappa+1}{2}, & \frac{\mu-\kappa}{2}, & \mu+\nu+1+n, & -n; \\ \mu+1, & \frac{\mu+\nu+1}{2}, & \frac{\mu+\nu+2}{2} & \end{matrix}\right]. $$

The desired result is now established by transforming the ${}_4F_3$ according to equation 7.2 (1) of [1]. We emphasize that it is also possible to prove (a)$\sim$(c) by the differential equation method of Burchnall and Chaundy.

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Added in proof: Without giving details we mention an interesting "confluent" case of the generating function (20). This is obtained by setting

$$\tau=\frac{2x}{\nu}-1\,, \qquad k^2=-\nu y$$

(x, y fixed) and letting $\nu\to\infty$. The result is the well-known series (see **[9]**, p. 98)

$$\sum_{n=0}^{\infty}\frac{(-)^n n!}{(\mu+1)_n}L_n^{(\mu)}(x)L_n^{(\mu)}(y)\rho^n=(1+\rho)^{-\mu-1}\exp\frac{\rho(x+y)}{1+\rho}\,{}_0F_1\left[\mu+1;\ -\frac{xy\rho}{(1+\rho)^2}\right],$$

where $L_n^{(\mu)}$ denotes the Laguerre polynomial.

AN ALGEBRA ASSOCIATED WITH A COMPACT GROUP

MEYER JERISON

1. Introduction. This paper deals with a variation on a familiar theme; namely, a proof that a space is determined, in some appropriate sense, by certain properties of a collection of functions on that space. Here, the space in question is a compact abelian group G, and the collection of functions is the set of all continuous functions from G into a commutative, complex Banach algebra R. The relevant properties of the collection of functions make it into a Banach algebra $R(G)$, with addition as well as multiplication by scalars defined in the usual way, that is, pointwise, norm defined by

$$(1) \qquad \|x\| = \sup_{g \in G} \|x(g)\| \qquad x \in R(G),$$

and multiplication of elements in $R(G)$ defined as the convolution

$$(2) \qquad (xy)(g) = \int_G x(gh^{-1})y(h)dh .$$

The integral, like all integrals appearing in this paper, is taken with respect to Haar measure in G, normalized so that the measure of G is 1. The integrand takes on values in the Banach algebra R, and the integral is of the type described in [3]. An alternate approach to this integral is obtained by observing that, as continuous functions on a compact group, the functions with which we deal are almost periodic in the sense of [2], and the integral is the invariant mean whose existence and uniqueness are proved in [2].

We will let $\mathscr{H}$ denote the class of theorems of the type described in the first sentence of the preceding paragraph. Many theorems of this class may be found in the literature; the ones most intimately related to the present investigation appear in [6] and the papers quoted there. I feel, therefore, that some justification is needed for the publication of still another one. Futhermore, there is probably no limit to the number of different kinds of spaces and different sets of functions which might be combined to yield a theorem in $\mathscr{H}$. The choice of the particular set-up that is being studied here was motivated by an attempt to solve a problem in topology proposed by Fox [4].

If X, Y, and Z are topological spaces, and if the cartesian product $X \times Y$ is homeomorphic with $X \times Z$, then it is known that Y and Z need not be homeomorphic. In the simplest example of this phenomenon [4], Y and Z are compact subsets of the plane that are not at all

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pathological, and X is a closed interval. The non-homogeneity of X at the end points seems to be what makes this example work, and Fox raises the question whether homeomorphism of the products will imply homeomorphism of Y and Z if X is a circle. Now, let R_1 and R_2 be the Banach algebras of all continuous complex (real would work just as well) valued functions on the compact spaces Y and Z, respectively, with multiplication as well as addition defined pointwise. Then a well known theorem (in $\mathscr{H}$) asserts that R_1 and R_2 are isomorphic, in symbols[1] $R_1 \approx R_2$, if, and only if, Y and Z are homeomorphic. A function from X into R_1 may be identified in an obvious way with a complex function on $X \times Y$. If one could prove that isomorphism of the algebra of all continuous functions from the circle X into R_1 with the space of functions from X into R_2 implies $R_1 \approx R_2$, and if, moreover, the former isomorphism is a consequence of the homeomorphism of $X \times Y$ with $X \times Z$, then Fox's problem would be solved.

It would be pleasant to be able to report that this has been achieved, especially because the theorems in $\mathscr{H}$ have had no noteworthy applications to problems in topology. Unfortunately, although Theorem 2 does say that $R_1(G) \approx R_2(G)$ implies $R_1 \approx R_2$ for any compact abelian group G, in particular for a circle, the algebraic structure which has been placed on $R(G)$ is of such a nature that I cannot prove that homeomorphism of $G \times Y$ with $G \times Z$ implies $R_1(G) \approx R_2(G)$. Section 4 is devoted to a discussion of some of the reasons for the failure of this approach.

Theorem 2 is probably true without the hypothesis that G is abelian, but I have not been able to prove it. This hypothesis does not influence the applicability of the theorem to Fox's problem. The requirement that R be a commutative Banach algebra whose only idempotent is its unit, is equivalent, if R is the algebra of continuous functions on Y with pointwise multiplication, to the assumption that Y is connected.

2. Complex valued functions.

In this section, we assume that R is the field of complex numbers, and then we no longer need to require that G be abelian.

THEOREM 1. *If D and Δ are the Banach algebras of continuous complex functions on the compact groups G and Γ, respectively (with multiplicatian defined by* (2)*), and $D \approx \Delta$, then G and Γ are isomorphic.*

This theorem can probably be proved by the technique of [6], but

[1] This symbol will be reserved for isomorphism (including preservation of norms) of Banach algebras.

we choose to base our proof on the theory of Banach spaces of continuous functions. If $f: \Delta \to D$ is the isomorphism, then there exists a one-to-one mapping φ of G onto Γ which is a homeomorphism of the underlying topological spaces, and $x_0 \in D$ with the property $|x_0(g)| \equiv 1$, such that [2]

$$f\xi(g) = x_0(g) \cdot \xi(\varphi g) \tag{3}$$

for all $\xi \in \Delta$ and $g \in G$. We propose to show that φ is an isomorphism of the groups.

LEMMA 1. $x_0(g'g'') = x_0(g')x_0(g'')$ for all g', $g'' \in G$.

Proof. If $\xi_0 \in \Delta$ is defined as $\xi_0(\gamma) \equiv 1$, $\gamma \in \Gamma$, then $f\xi_0 = x_0$. Consequently,

$$x_0 \cdot x_0 = f\xi_0 \cdot f\xi_0 = f(\xi_0 \cdot \xi_0) = f(\xi_0) = x_0 ;$$

that is, x_0 is an idempotent in D. Hence,

$$x_0(g) = x_0 x_0(g) = \int x_0(gh^{-1}) x_0(h) dh .$$

Since $|x_0(g)| \equiv 1$, we have

$$1 = \overline{x_0(g)} x_0(g) = \int \overline{x_0(g)} x_0(gh^{-1}) x_0(h) dh .$$

But the absolute value of the integrand is 1 for all g and h, and the measure of G is also 1, so that

$$\overline{x_0(g)} x_0(gh^{-1}) x_0(h) = 1$$

for all g and h (more precisely, for almost all h, but the function is continuous). Setting $g = g'g''$ and $h = g''$, and remembering that $|x_0(g)| = 1$, we obtain $x_0(g'g'') = x_0(g') \cdot x_0(g'')$.

Proof of Theorem 1. It is required only to prove that

$$\varphi(gg') = (\varphi g)(\varphi g') \qquad \text{for all } g, g' \in G.$$

Let Ω be a neighborhood of the identity in Γ, and let ω be a continuous function on Γ which vanishes outside of Ω and such that

$$\int_\Gamma \omega(\gamma^{-1}) \omega(\gamma) d\gamma = 1 .$$

[2] A proof of this assertion for a Banach space of real functions may be found in a number of different places, including [**1**, p. 172]. A generalization which includes the case of complex functions appears in [**5**, Theorem 6.2].

Set

$$\omega_1(\gamma)=\omega((\varphi g)^{-1}\gamma) \quad \text{and} \quad \omega_2(\gamma)=\omega(\gamma\cdot(\varphi g')^{-1}) ;$$

then a straightforward computation yields

$$(4) \qquad \omega_1\omega_2((\varphi g)(\varphi g'))=1 .$$

Using (2), (3), and Lemma 1 in the relation $f(\omega_1\omega_2)=(f\omega_1)\cdot(f\omega_2)$, we find that for any $a\in G$,

$$f(\omega_1\omega_2)(a)=\int_G x_0(ah^{-1})\cdot\omega_1(\varphi(ah^{-1}))\cdot x_0(h)\cdot\omega_2(\varphi h)dh$$
$$=x_0(a)\int_G \omega((\varphi g)^{-1}\varphi(ah^{-1}))\cdot\omega((\varphi h)(\varphi g')^{-1})dh .$$

Since ω vanishes outside of Ω, this implies

$$(5) \qquad f(\omega_1\omega_2)(a)=0 \qquad \text{for } a\notin V$$

where

$$V=\varphi^{-1}[(\varphi g)\Omega]\cdot\varphi^{-1}[\Omega(\varphi g')] .$$

$(\varphi g)\Omega$ is a neighborhood of φg in Γ, so that $\varphi^{-1}[(\varphi g)\Omega]$ is a neighborhood of g in G. Similarly, $\varphi^{-1}[\Omega(\varphi g')]$ is a neighborhood of g', and V is a neighborhood of gg'.

Let a be the (unique) element of G such that $\varphi a=(\varphi g)\cdot(\varphi g')$, and suppose $a\neq gg'$. Since no previous restrictions have been placed upon Ω, we may now choose Ω so that $a\notin V$, that is, so that $f(\omega_1\omega_2)(a)=0$. But,

$$f(\omega_1\omega_2)(a)=x_0(a)\cdot\omega_1\omega_2(\varphi a)=x_0(a)\cdot\omega_1\omega_2(\varphi g\cdot\varphi g')=x_0(a)\neq 0 .$$

This contradicts the assumption that $a\neq gg'$, and therefore $\phi(gg')=(\varphi g)(\varphi g')$.

3. The isomorphism theorem.

THEOREM 2. *Let G_1 and G_2 be compact abelian groups, and R_1 and R_2, commutative Banach algebras whose only idempotents are their respective units. Then $R_1(G_1)\approx R_2(G_2)$ if, and only if, $R_1\approx R_2$ and G_1 is isomorphic to G_2.*

Proof. In one direction, the implication is trivial. To prove the non-trivial half of the theorem, we consider a group G, an algebra R with unit e, and show how R and G may be recovered from $R(G)$, using only the structure of $R(G)$ as a Banach algebra.

The first step is to find D (in the notation of Theorem 1) in $R(G)$.

Specifically, we want to characterize the set De of elements in $R(G)$ of the form $\lambda(g)\cdot e$, where $\lambda(g)$ is a complex function on G.

LEMMA 2. *De is the smallest closed linear subspace containing all of the idempotents of $R(G)$.*

Proof. We review some essential facts concerning Fourier analysis in $R(G)$; the proofs may be found in [2]. Let $\{\chi_\alpha\}$ be the set of all continuous characters of G, that is,

$$|\chi_\alpha(g)|=1 \text{ and } \chi_\alpha(gg')=\chi_\alpha(g)\cdot\chi_\alpha(g')$$

for all $g, g' \in G$. For $x \in R(G)$, define

$$r_\alpha=\int_G \overline{\chi_\alpha(g)}\, x(g)dg\ .$$

This is an element of R. The formal series $\sum_\alpha \chi_\alpha(g)r_\alpha$ represents $x(g)$ in exactly the same way that classical Fourier series represent continuous functions. We write $x\sim\sum\chi_\alpha r_\alpha$. If $x'\sim\sum\chi_\alpha r'_\alpha$ then $xx'\sim\sum\chi_\alpha r_\alpha r'_\alpha$. (This is not proved in [2], but can be done, as in the classical case, simply by evaluating the αth coefficient of xx'.)

Since the formal series representation is unique, x is an idempotent if, and only if, $r_\alpha=0$ or e for all α. Thus, every idempotent of $R(G)$ is in De, and, in fact, is an idempotent of D multiplied by e. Since the idempotents of D span D, the idempotents of $R(G)$ span De. It is obvious, that De is a closed linear subspace of $R(G)$.

Lemma 2 asserts that De is determined by $R(G)$. Since $De\approx D$ (assuming $\|e\|=1$), it follows from Theorem 1 that G is determined by $R(G)$. It remains only to prove that R is determined by $R(G)$, and this will be achieved essentially by fishing the constant functions out of $R(G)$. Specifically, we will find all of the constant functions multiplied by some character of G. It is impossible to distinguish between characters using only their algebraic properties in $R(G)$.

LEMMA 3. *Let x be any irreducible idempotent of $R(G)$, that is, any idempotent which is not the sum of other non-zero idempotents. The principal ideal generated by x is isomorphic with the Banach algebra R.*

Proof. From the discussion of idempotents given earlier, it is clear that $x=\chi e$ for some character χ of G. If $y \in R(G)$, then

$$\begin{aligned} yx(g)&=\int y(gh^{-1})\chi(h)\cdot e\,dh=\int y(k)\chi(k^{-1}g)\cdot e\,dk\\ &=\left[\int y(k)\overline{\chi(k)}dk\right]\chi(g)\cdot e=\chi(g)\cdot r\end{aligned}$$

where r is the "Fourier coefficient" of y with respect to the character χ. Similarly $xy(g)=\chi(g)r$. Consequently, the set of functions in $R(G)$ of the form $\chi(g)r$, $r\in R$, is a two-sided ideal. The correspondence $r\leftrightarrow\chi r$ is the desired isomorphism.

4. Fox's problem. It was remarked earlier that the class of theorems $\mathscr{H}$ has been disappointing as a source of solutions of problems in topology, problems that do not involve the function space directly. The comments which will be made here refer only to the failure to solve Fox's problem, but it seems to me that they lie close to the heart of the difficulties in general.

It is unlikely that Theorem 2 can be used to prove Fox's conjecture because the conclusion of the theorem is so strong. What is needed is a theorem with the statement "G_1 isomorphic to G_2" in the hypothesis rather than in the conclusion. That so much could be proved from the hypothesis $R_1(G_1)\approx R_2(G_2)$ implies that it is a very strong condition and one that will be difficult to verify. Thus, in the application to Fox's problem, we would take R_1 and R_2 as the algebras of continuous functions on Y and Z, respectively, (pointwise multiplication) and we would have to prove that if $G\times Y$ is homeomorphic with $G\times Z$, then $R_1(G)\approx R_2(G)$. One may observe, incidentally, that in the correspondence between these two algebras induced by the homeomorphism of the product spaces, norms are preserved, but the norms do not enter in an essential way into the proof of $R_1\approx R_2$.

Apparently, then, the source of the difficulty is the peculiar definition of multiplication in $R(G)$. I believe, however, that the trouble goes deeper. A theorem in $\mathscr{H}$ generally has a hypothesis which is so strong that to verify it is tantamount to exhibiting a homeomorphism of the topological spaces on which the functions are defined. One manifestation of this is the fact that the hypothesis implies not only a homeomorphism but also an intimate relationship between the homeomorphism of the conclusion and the isomorphism of the hypothesis, as given by formula (3). The presence of such a formula is implicit in all of the techniques for proving theorems in $\mathscr{H}$. It is what requires the strong hypothesis, which, in turn, limits the applicability of the theorem.

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PURDUE UNIVERSITY

INFINITE DETERMINANTS ASSOCIATED WITH HILL'S EQUATION

WILHELM MAGNUS

1. Introduction and Summary. Hill's equation is the differential equation for a one-dimensional linear oscillator with a periodic potential. In most applications, the question of the existence of a periodic solution arises. The main purpose of this investigation is to examine the analytic character of the transcendental function, whose zeros determine the periodic solutions. For the special case of Mathieu's equation the results obtained here have previously been used for solving the inhomogeneous equation, and the cases where Hill's equation has two periodic solutions have been discussed in detail and applied to the construction of "transparent layers" [1].

We consider the differential equation of Hill's type:

$$y''+4(\omega^2+q(x))y=0\ , \tag{1.1}$$

where $q(x)$ is an even function of period π which can be expanded in a Fourier series

$$q(x)=2\sum_{n=1}^{\infty} t_n \cos 2nx\ . \tag{1.2}$$

We shall assume that the constants t_n satisfy

$$\sum_{n=1}^{\infty} |t_n|<\infty\ . \tag{1.3}$$

The most widely investigated problem connected with (1.1) is the question of the existence of solutions with period π or 2π. Let $y_1(x)$, $y_2(x)$ denote the solutions of (1.1) which satisfy the initial conditions

$$y_1(0)=1,\ y_1'(0)=0\ ;\quad y_2(0)=0,\ y_2'(1)=1. \tag{1.4}$$

Then the following elementary statements hold (see for instance Schaefke [5]: Equation (1.1) has

(α) an even solution of period π if and only if $y_1'(\pi/2)=0$
(α') an odd solution of period π if and only if $y_2(\pi/2)=0$
(β) an even solution of period 2π if and only if $y_1(\pi/2)=0$
(β') an odd solution of period 2π if and only if $y_2'(\pi/2)=0$.

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The conditions (α), (α') and (β), (β') can be reduced to two single ones because

$$y_1(\pi)-1=2y_1'(\pi/2)y_2(\pi/2)\ , \tag{1.5}$$

$$y_1(\pi)+1=2y_1(\pi/2)y_2'(\pi/2)\ . \tag{1.6}$$

In order to find directly a solution of (1.1) which has a period π, we put

$$y=\sum_{n=-\infty}^{\infty} c_n \exp(2nxi)\ , \tag{1.7}$$

where

$$\bar{c}_n=c_{-n} \tag{1.8}$$

for a real function $y(x)$. (As usual, a bar denotes the conjugate complex quantity). By substituting (1.7) into (1.2) we obtain an infinite system of homogeneous linear equations for the c_n. The determinant of this system can be written in the form

$$\sin^2 \pi\omega\, D_0(\omega) \tag{1.9}$$

where $D_0(\omega)$ is an infinite determinant of the type

$$D_0(\omega)=|d_{n,\,m}|, \qquad n,\ m=0,\ \pm 1,\ \pm 2, \cdots . \tag{1.10}$$

Here

$$d_{n,\,m}=\delta_{n,\,m}+\left(\frac{t_{n-m}}{\omega^2-n^2}\right)\ , \tag{1.11}$$

$$t_{n-m}=t_{m-n}=t_{|n-m|}\ , \quad t_0=0\ . \tag{1.12}$$

As usual, $\delta_{n,\,m}=1$ if $n=m$ and $\delta_{n,\,m}=0$ if $n\neq m$.

The vanishing of the expression (1.9) is a necessary and sufficient condition for (1.1) to have a solution with period π. According to Whittaker and Watson [7]

$$y_1(\pi)-1=-2D_0(\omega)\sin^2 \pi\omega\ . \tag{1.13}$$

This shows that the vanishing of (1.5) is an immediate consequence of the vanishing of the term (1.9) and vice versa. Also, it provides two alternative ways of approximating the eigenvalues ω for which $y_1(\pi)=1$. If we compute $y_1(\pi)$ approximately by applying the Picard iteration to (1.1), we arrive at trigonometric polynomials or series. If we use the principal minors of D_0, we obtain algebraic equations for the approximate values of ω which will be particularly suitable for large ω.

To obtain even or odd solutions of (1.1) which are of period π we may put

$$y=\left(\frac{c_0}{\sqrt{2}}\right)+\sum_{n=1}^{\infty} c_n \cos 2nx \tag{1.14}$$

or

$$y=\sum_{n=1}^{\infty} c_n \sin 2nx \tag{1.15}$$

respectively. By substituting (1.14) or (1.15) into (1.1) we obtain an infinite system of homogeneous linear equations for the c_n. After an appropriate normalization of these equations, we can write the determinants of the resulting systems in the form $\omega \sin(\pi\omega)C_+$ and $\omega^{-1} \sin(\pi\omega)S_+$, where the infinite determinants C_+ and S_+ can be defined as follows: Let

$$\varepsilon_m=2 \text{ for } m=\pm 1, \pm 2, \pm 3, \cdots; \ \varepsilon_0=1 \tag{1.16}$$

$$\begin{cases} \operatorname{sgn} m=1 \text{ for } m=1, 2, 3, \cdots; \ \operatorname{sgn} 0=0 \\ \operatorname{sgn} m=-1 \text{ for } m=-1, -2, -3, \cdots. \end{cases} \tag{1.17}$$

Let the t_n be defined by (1.2) and (1.12). Then

$$C_+=|(\varepsilon_n\varepsilon_m)^{-1/2}(1+\operatorname{sgn} n \operatorname{sgn} m)[\delta_{n,m}+(t_{n-m}+t_{n+m})(\omega^2-n^2)^{-1}]| \quad (n, m=0, 1, 2, \cdots), \tag{1.18}$$

$$S_+=|\delta_{n,m}+(t_{n-m}-t_{n+m})(\omega^2-n^2)^{-1}| \qquad (n, m=1, 2, 3, \cdots), \tag{1.19}$$

where n denotes the rows and m denotes the columns of the infinite determinants C_+ and S_+.

We shall prove the following extension of Equation (1.13):

THEOREM 1. *The infinite determinants C_+ and S_+ can be expressed in terms of $y_1'(\pi/2)$ and $y_2(\pi/2)$ as*

$$2\omega \sin(\pi\omega)C_+=-y_1'(\pi/2), \tag{1.20}$$

$$\omega^{-1} \sin(\pi\omega)S_+=2y_2(\pi/2). \tag{1.21}$$

They are related to the infinite determinant D_0 by

$$D_0=C_+S_+. \tag{1.22}$$

A similar factorization theorem can be proved for the infinite determinant arising in the problem of determing whether (1.1) has a solution of period 2π.

Equations (1.19) and (1.21) show that S_+ and $y_2(\pi/2)$ depend in a special way on ω. We shall write $S_+(\omega)$ for S_+ and $y_2(\pi/2, \omega)$ for $y_2(\pi/2)$ if we wish to emphasize the dependency on ω. $S_+(\omega)$ has poles of the first order (at most) at $\omega=\pm 1, \pm 2, \cdots$. Since the individual terms in the determinant $S_+(\omega)$ tend to $\delta_{n,m}$ as $|\omega|\to\infty$, we may expect that $S_+(\omega)\to 1$ as $|\omega|\to\infty$. Therefore we may expect that (1.21) will lead to a formula of the type

$$y_2(\pi/2, \omega)=\sum_{n=0}^{\infty} g_n \frac{\sin \pi\omega}{\omega-n}, \tag{1.23}$$

where g_n are constant coefficients. Now the form of the infinite series on the right-hand side of (1.23) suggests that it can also be written as

$$\int_{-\pi/2}^{\pi/2} G(\theta) \exp(2i\omega\theta)\, d\theta, \tag{1.24}$$

which would imply the existence of a formula of the type

$$\int_{-\infty}^{\infty} y_2(\pi/2, \omega) \exp(-2i\omega\theta)\, d\omega=0 \quad \text{for} \quad |\theta|>\frac{\pi}{2}. \tag{1.25}$$

Actually, a result more general than (1.25) is true. We shall prove the following formula for the Fourier transformation with respect to ω.

THEOREM 2. *Let the t_n in* (1.12) *be real constants satisfying*

$$\sum_{n=1}^{\infty} n^2|t_n|<\infty,$$

and let $y(x, \omega)$ be the solution of (1.1) *for a real value of ω which satisfies the initial conditions*

$$y(0, \omega)=a, \quad y'(0, \omega)=b. \tag{1.26}$$

Then there exists a function $G(x, \theta)$ of the real variables x and θ which is defined in the region $-|x|\leq\theta\leq|x|$ such that

$$y(x, \omega)=a \cos 2\omega x+\int_{-x}^{x} G(x, \theta)e^{2i\omega\theta}d\theta, \tag{1.27}$$

$$\frac{\partial^2 G}{\partial x^2}-\frac{\partial^2 G}{\partial\theta^2}+4q(x)G=0, \tag{1.28}$$

$$G(x, x)=G(x, -x)=\frac{b}{2}-a\sum_{n=1}^{\infty}\frac{t_n}{n}\sin 2\pi x, \tag{1.29}$$

$$(1.30)\qquad G_\theta(x,\ x)=-G_\theta(x,\ -x)=2\sum_{n=1}^{\infty}t_n\sin nx\left\{a\sin nx+\frac{b}{n}\cos nx\right\}$$
$$+a\sum_{n,\ m=1}^{\infty}\frac{t_nt_m}{nm}\sin 2nx\sin 2mx\ .$$

Here G_θ stands for $\partial G/\partial\theta$.

2. Proof of Theorem 1. Since Theorem 1 involves the determinants of infinite matrices, it is important to know something about their finite "sections". We shall define these sections as follows: Let N be a nonnegative integer, and let (M) be an infinite matrix. If the rows and columns of (M) are labeled by subscripts running from one to infinity, we denote by (M_N) the square matrix of order N which results if we let the subscripts in (M) run from one to N only. If the rows and columns in (M) are labeled by the subscripts $0,\ 1,\ 2,\cdots$, we define (M_N) by the rows and columns of (M) for which the subscripts run from zero to N. Finally, if the subscripts in (M) run from $-\infty$ to $+\infty$, then in (M_N) we let them run from $-N$ to $+N$ only. In each case, (M_N) is called the Nth section of (M). The determinant of (M) is defined as the limit of the determinants of (M_N) as $N\to\infty$.

We shall denote by (D), (C), (S) the matrices whose elements are given respectively by the elements of the infinite determinants D_0, C_+, and S_+. In addition, we shall introduce the matrix (T) with the general element $\tau_{n,\ m}(n,\ m=0,\ \pm1,\ \pm2,\cdots)$, where

$$(2.1)\qquad \tau_{n,\ m}=(\delta_{n,\ m}+\operatorname{sgn} n\delta_{-n,\ m})(\varepsilon_n)^{1/2}.$$

As usual the first subscript n in $\tau_{n,\ m}$ denotes the rows of (T) and the second subscript denotes the columns. The matrix (T) has a formal inverse (T^{-1}), whose general element is given by

$$(2.2)\qquad (\delta_{n,\ m}+\operatorname{sgn} m\delta_{-n,\ m})(\varepsilon_m)^{-1/2}.$$

In fact it follows from an easy computation that the general element of $(T)(T^{-1})$ is

$$(2.3)\qquad \{\delta_{n,\ m}(1+\operatorname{sgn} n\operatorname{sgn} m)+\delta_{-n,\ m}(\operatorname{sgn} n+\operatorname{sgn} m)\}(\varepsilon_n\varepsilon_m)^{-1/2}=\delta_{n,\ m}\ .$$

It is important to observe that the Nth section (T_N^{-1}) of (T^{-1}) is the inverse of the Nth section (T_N) of T .

Now we shall compute, in a purely formal way, the elements of the matrix

$$(2.4)\qquad (D^*)=(T)(D)(T^{-1})\ .$$

By a simple computation we find from (1.11), (1.12), (2.1) and (2.2) that the general element $d^*_{n,m}$ of (D^*) is given by

$$(2.5)\quad (\varepsilon_n\varepsilon_m)^{1/2}\, d^*_{n,m}=\delta_{n,m}(1+\operatorname{sgn} n \operatorname{sgn} m)+\delta_{n,-m}(\operatorname{sgn} n+\operatorname{sgn} m)$$
$$+\frac{t_{n-m}}{\omega^2-n^2}(1+\operatorname{sgn} n \operatorname{sgn} m)+\frac{t_{n+m}}{\omega^2-n^2}(\operatorname{sgn} n+\operatorname{sgn} m)\,.$$

Equation (2.5) shows that $d^*_{n,m}=0$ if n and m are both different from zero and of different sign. It also shows that for $n, m=0, 1, 2, 3, \cdots$ the elements of (D^*) are exactly those of (C). In fact, for $n\geq 0$, $m\geq 0$, we always have $\delta_{n,-m}(\operatorname{sgn} n+\operatorname{sgn} m)=0$, and $\operatorname{sgn} n+\operatorname{sgn} m=1+\operatorname{sgn} n \operatorname{sgn} m$, unless $n=m=0$. But in this case, $t_{n-m}=t_{n+m}=0$, and again $d^*_{n,m}$ is equal to the corresponding element of C_+ in (1.18). Similarly, we find that for $n, m=-1, -2, -3, \cdots$, the elements of (D^*) are exactly those of (S) if we "invert" the labeling of the elements of (S) by substituting for every subscript its opposite (negative) value.

Therefore (1.22) would be proven if we could deal with infinite determinants in the same way as with finite ones. In the particular problem under consideration this is actually the case. If we form the matrix $(T_N)(D_N)(T_N^{-1})$ we obtain (D^*_N) for all N and we find that its determinant actually equals the product of the determinants of (S_N) and (C_N) because its elements are those of (S_N) and (C_N) respectively. Equation (1.22), namely $D_0=C_+S_+$, follows if we simply let N tend towards infinity.

Next we must prove equations (1.20) and (1.21). It suffices to do this for arbitrary but fixed real values of $t_1, t_2, t_3, \cdots$. Indeed, it is not difficult to show that both sides in (1.20) and (1.21) depend analytically on any particular parameter t_ν $(\nu=1, 2, \cdots)$. Then the only variable which matters is ω. As mentioned above, we shall write $y_2(\pi/2, \omega)$ and $y_1'(\pi/2, \omega)$ for $y_2(\pi/2)$ and $y_1'(\pi/2)$ whenever we wish to exhibit the dependency on ω of these quantities; similarly, we shall write $C_+(\omega)$ and $S_+(\omega)$ for C_+ and S_+. It is easily seen that both sides in (1.20) and (1.21) are entire functions of ω and also entire functions of $\lambda=\omega^2$.

Now we can prove (1.20) and (1.21) by proving the following lemmas:

LEMMA 1. *The quotients*

$$(2.6)\qquad \frac{2\omega \sin \pi\omega\, C_+(\omega)}{y_1'(\pi/2,\ \omega)}\,,\qquad \frac{\omega^{-1}\sin \pi\omega S(\omega)}{2y_2(\pi/2,\ \omega)}$$

are entire functions of $\omega^2=\lambda$.

Proof. It has been mentioned in the introduction that the numerator and denominator of (2.6) vanish for the same values of $\lambda=\omega^2$. It remains merely to be shown that the denominators have simple zeros only. We observe first that these zeros are real, because any solution or derivative of a solution of (1.1) that vanishes at $x=0$ and $x=\pi/2$ is a solution of a Sturm-Liouville problem. Since

$$\frac{\partial}{\partial\lambda} y_2(\pi/2)=4\{y_2'(\pi/2)\}^{-1}\int_0^{\pi/2}\{y_2(x)\}^2dx \tag{2.7}$$

$$\frac{\partial}{\partial\lambda} y_1'(\pi/2)=-4\{y_1(\pi/2)\}^{-1}\int_0^{\pi/2}\{y_1(x)\}^2dx\,, \tag{2.8}$$

the right-hand sides of (2.7) and (2.8) are different from zero and therefore the denominator in (2.6) has simple zeros. This completes the proof of Lemma 1.

LEMMA 2. *The quotients* (2.6) *are entire functions without zeros.*

Proof. From (1.5), (1.13), (1.22) we see that the product of the quotients (2.6) equals -1.

LEMMA 3. *The quotients* (2.6) *are independent of* $\lambda=\omega^2$.

Proof. This lemma follows from the fact that for both the numerators and the denominators of the quotients (2.6) the order of growth with respect to λ does not exceed 1/2. For $y_2(\pi/2,\ \omega)$ we can show this by solving (1.1) with the help of Picard's iteration method. Putting

$$u_0(x,\ \omega)=(\sin 2\omega x)/(2\omega)\,, \tag{2.9}$$

$$u_n(x,\ \omega)=-\frac{2}{\omega}\int_0^x \sin 2\omega(x-\xi)q(\xi)u_{n-1}(\xi,\ \omega)d\xi\,, \quad (n=1,2,\cdots), \tag{2.10}$$

we have

$$y_2(x,\ \omega)=\sum_{n=0}^{\infty} u_n(x,\ \omega)\,. \tag{2.11}$$

In order to estimate $|y_2|$ for large values of $|\omega|$, let Q be a positive constant such that

$$|q(\xi)|\leq Q \tag{2.12}$$

for all real values of ξ. Let $|\omega|\geq 2$. Then obviously $|u_0|\leq \exp(2|\omega|x)$ for real positive x. From this it follows by induction and by using

(2.10) that for real positive values of x

$$|u_n(x,\ \omega)|\leq x^n Q^n e^{2|\omega|x}(n!)^{-1}(\omega/2)^{-n-1}. \tag{2.13}$$

Therefore we have from (2.11) for $|\omega|\geq 2$:

$$|y_2(\pi/2,\ \omega)|\leq \exp(\pi|\omega|+Q\pi/2)\ . \tag{2.14}$$

A similar estimate can be derived for $y_1'(\pi/2,\ \omega)$. Since the right-hand side of (2.14) is of order of growth unity with respect to ω, its order of growth with respect to λ is 1/2.

The corresponding statement for the numerators in (2.6) can be derived from Hadamard's inequality for determinants. If we write

$$(\pi/2)\sum_{n=1}^{\infty}\left(1-\frac{\omega^2}{n^2}\right)\ , \tag{2.15}$$

for $(\sin\pi\omega)/(2\omega)$, and if we multiply each row of S_+ by the corresponding factor of (2.15), the numerator involving S_+ in (2.6) becomes a determinant for which the sum of the squares of the absolute values of the nth row is at most σ_n, where

$$\sigma_n=\{1+(|\omega|^2+|t_{2n}|)n^{-2}\}^2+\sum_{m=1}^{\infty}(|t_{n-m}|+|t_{n+m}|)^2 n^{-4}. \tag{2.16}$$

We have from Hadamard's inequality

$$|(2\omega)^{-1}(\sin\pi\omega)S_+(\omega)|\leq 2\pi^{-1}\prod_{n=1}^{\infty}\{\sigma_n\}^{1/2}. \tag{2.17}$$

Now we wish to estimate $|\sigma_n|$. From (1.2) we find that there exists a constant M such that for all $n=1,\ 2,\ 3,\ \cdots$

$$|t_{2n}|\leq 2M,\ \sum_{m=1}^{\infty}(|t_{n-m}|+|t_{n+m}|)^2\leq M^2. \tag{2.18}$$

Therefore

$$|\sigma_n|\leq\{1+(|\omega|^2+M)n^{-2}\}^2 \tag{2.19}$$

and

$$\prod_{n=1}^{\infty}\{\sigma_n\}^{1/2}\leq\{\sinh\pi(|\omega^2|+M)^{1/2}\}\pi^{-1/2}(|\omega|^2+M)^{-1/2}. \tag{2.20}$$

Together with (2.17), this shows that the left-hand side of (2.17) is of order of growth $\leq 1/2$ with respect to $\lambda=\omega^2$. An analogous proof can be given for $|2\omega\sin\pi\omega\, C_+(\omega)|$.

Now we can prove Lemma 3 by using a known theorem about factorization of functions of an order of growth <1 (See Nevanlinna

[2, pp. 205–213] or Titchmarsh [6, Chap. VIII]. According to this theorem we have for both the numerators and the denominators of the quotients (2.6) a representation of the form

$$A\,\omega^{2a}\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{a_n}\right),$$

where the a_n are the simple roots common to the numerator and denominator if both are considered as functions of $\lambda=\omega^2$. Therefore, the quotients in (2.6) are independent of ω, as stated in Lemma 3.

Now we can prove (1.20) and (1.21) by computing the value of the quotients in (2.6) for $\omega\to i\infty$. It is easily seen that for $\omega\to i\infty$ both S_+ and C_+ tend toward unity. From (2.9), (2.10) and (2.11) we can show that $y_2(\pi/2,\ \omega)/u_0(\omega)$ tends also towards unity as $\omega\to i\infty$, regardless of the particular nature of $q(x)$. The behavior of $y_1'(\pi/2,\ \omega)/(2\omega \sin \pi\omega)$ can be described in a similar manner, and this completes the proof of Theorem 1.

3. **Proof of Theorem 2.** In this section, we shall use a theorem given by Paley and Wiener [3, Theorem X, p. 13]. According to this theorem, *the following two classes of functions are identical*:

(I) *The class of all entire functions* $F(\omega)$ *satisfying*

$$|F(\omega)|=o(e^{2A|\omega|}) \qquad (|\omega|\to\infty) \tag{3.1}$$

for a positive real value of A; *and*

(II) *The class of all entire functions of the form*

$$F(\omega)=\int_{-A}^{A} f(\theta)e^{2i\omega\theta}\,d\theta\,, \tag{3.2}$$

where $f(\theta)$ *belongs to* L_2 *over* $(-A,\ A)$.

In proving Theorem 2 we shall confine ourselves to the case where $a=0$, $y=y_2(x,\ \omega)$. If we construct y_2 in the manner described by (2.9), (2.10), (2.11), we find from (2.13) that for $x>0$ and $|\omega|\to\infty$:

$$\Big|y_2(x,\ \omega)-\{u_0(x,\ \omega)+\cdots+u_n(x,\ \omega)\}\Big|=O(|\omega|^{-n-2}e^{2|\omega|x}) \tag{3.3}$$

and

$$|u_n(x,\ \omega)|=O(|\omega|^{-n-1}e^{2|\omega|x}). \tag{3.4}$$

Now it follows from an application of Paley and Wiener's theorem that

$$y_2(x,\ \omega)=\int_{-x}^{x} e^{2i\omega\theta}G(x,\ \theta)\,d\theta, \tag{3.5}$$

where

(3.6) $$G(x,\ \theta)=\sum_{n=0}^{\infty} g_n(x,\ \theta)\ ,$$

(3.7) $$g_n(x,\ \theta)=\pi^{-1}\int_{-\infty}^{\infty} e^{-2i\omega\theta}u_n(x,\ \omega)\, d\omega\ .$$

It follows from (3.4) that for $n>0$, $g_n(x,\ \theta)$ is $(n-1)$ times differentiable with respect to θ, with a continuous $(n-1)^{\text{st}}$ derivative. Outside the interval $-x\leq\theta\leq x$, all of the $g_n(x,\ \theta)$ vanish identically. Therefore at $\theta=\pm x$ only $g_0(x,\ \theta)$ and $g_1(x,\ \theta)$ contribute to the value of $G(x,\ \theta)$ and to its first derivative with respect to θ. These contributions can be found by a direct computation. In the same way, it can be verified that $g_0,\ g_1,\ g_2$ are twice differentiable within the region $-x<\theta<x$, having one-sided continuous derivatives at $\theta=\pm x$, provided that $\sum_{n=1}^{\infty} n^2|t_n|<\infty$.

The only part of Theorem 2 that now remains to be proved is equation (1.28). If we substitute the expression (3.6) for G into (1.28), it will suffice to prove that for $n=1,\ 2,\ 3,\cdots$,

(3.8) $$\frac{\partial^2 g_n}{\partial x^2}-\frac{\partial^2 g_n}{\partial \theta^2}+4q(x)g_{n-1}=0$$

and for $n=0$

(3.9) $$\frac{\partial^2 g_0}{\partial x^2}-\frac{\partial^2 g_0}{\partial \theta^2}=0\ .$$

Since $g_0=1/2$ for $-x<\theta<x$, it is trivial to show that (3.9) holds. Equation (3.8) may be verified for $n=1$ directly by observing that

(3.10) $$g_1(x,\ \theta)=\sum_{n=1}^{\infty}\frac{2t_n}{n^2}\cos nx\ (\cos nx-\cos n\theta)\ .$$

For $n\geq 2$ we may proceed as follows. It suffices to prove, instead of (3.8), that

(3.11) $$\int_{-x}^{x}\left(\frac{\partial^2 g_n}{\partial x^2}-\frac{\partial^2 g_n}{\partial \theta^2}+q(x)g_{n-1}\right)e^{2i\omega\theta}d\theta=0$$

for all values of ω. Since the left-hand side of (3.11) is an analytic function of ω, it suffices to show that it vanishes for all real values of ω. We shall prove this by expressing the left-hand side of (3.11) in terms of the $u_n(x,\ \omega)$ which satisfy the recurrence relations

(3.12) $$\frac{\partial^2 u_n}{\partial x^2}+4\omega^2 u_n+4q(x)u_{n-1}=0.$$

((3.12) can be derived easily from (2.9) and (2.10)). It follows from (3.5) and (3.7) that

$$u_n(x,\ \omega)=\int_{-x}^{x} g_n(x,\ \theta)e^{2i\omega\theta}\,d\theta\,. \tag{3.13}$$

Therefore we have for $n\geq 2$:

$$\frac{\partial^2 u_n}{\partial x^2}=\int_{-x}^{x} \frac{\partial^2 g_n}{\partial x^2} e^{2i\omega\theta}\,d\theta\,, \tag{3.14}$$

since any term derived by differentiating the integral in (3.13) with respect to its limits vanishes for $n\geq 2$. For the same reason we find from an integration by parts that

$$-\int_{-x}^{x} \frac{\partial^2 g_n}{\partial x^2} \exp(2i\omega\theta)\,d\theta=4\omega^2 u_n(x,\ \theta)\,. \tag{3.15}$$

Equations (3.15), (3.13), (3.12) show that (3.11) and (3.12) are equivalent. Since (3.12) is true, the proof of Theorem 2 has been completed.

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New York University

THE SLOW STEADY MOTION OF LIQUID PAST A SEMI-ELLIPTICAL BOSS

G. POWER AND D. L. SCOTT-HUTTON

1. **Introduction.** In this problem of two-dimensional viscous flow, liquid is supposed to have a rigid boundary represented by *ABCDE* in Figure 1 and, apart from the disturbance caused by the presence of the elliptical boss *BCD*, is assumed to be in uniform shearing motion. The stream function is thus a biharmonic function vanishing together with its normal derivative at all points of the boundary, and proportional to y^2 at a great distance from the boss. A series of functions is found, each of which satisfies all the boundary conditions save one. A linear combination of these functions will also satisfy the boundary conditions with this one exception, and by a particular choice of the arbitrary constants which it contains, the remaining condition can be satisfied at as many points as desired. Special cases are discussed, and a process of approximation is outlined which yields the most accurate results at C, and also gives a convenient function for determining at any point of the boundary the magnitude of the error in the unsatisfied boundary condition. A special case of this problem has previously been considered [1].

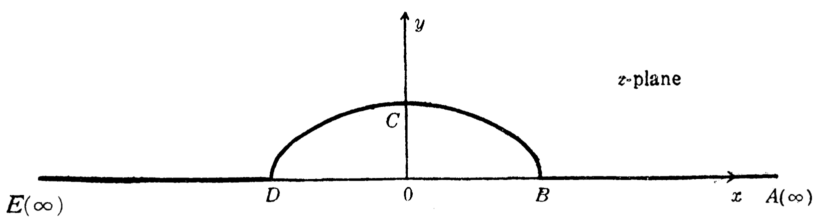

Figure 1.

2. **The stream function.** We take the equation of the boundary *BCD* to be $x^2/a^2+y^2/b^2=1$, and note that the region occupied by the fluid, for which y is never negative, is transformed into the interior of the semi-circle of unit radius shown in Figure 2 by

$$-2z=(a-b)w+(a+b)/w\,. \tag{1}$$

The stream function ψ is biharmonic, that is to say it must satisfy $\nabla^4\psi=0$, and a satisfactory solution to the problem is

$$\psi=y^2+U+yV\,, \tag{2}$$

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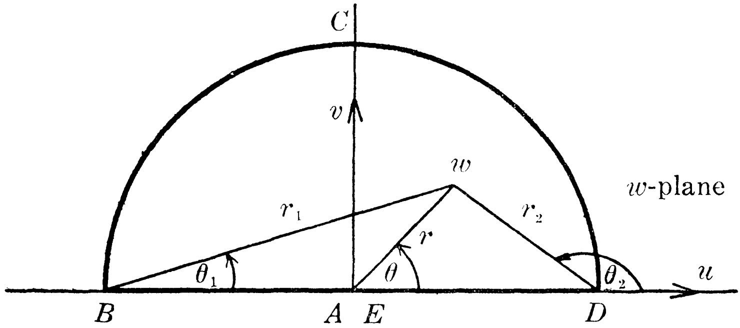

Figure 2.

provided U and V are harmonic functions which are chosen so that $U+yV$ does not tends to infinity as z tends to infinity.

The boundary conditions to be satisfied are $\psi=0$ and $\partial\psi/\partial\nu=0$ along the boundary, where $\delta\nu$ denotes an element of normal to the boundary. From (2), we see that four conditions are required, namely

$$
(3)\begin{cases}
\text{(a)}\quad U=0 & \text{along } AB \text{ and } DE, \text{ that is when } y=0,\\
\text{(b)}\quad V+\partial U/\partial y=0 & \text{along } AB \text{ and } DE, \text{ that is when } y=0,\\
\text{(c)}\quad y^2+U+yV=0 & \text{along } BCD, \text{ that is when } x^2/a^2+y^2/b^2=1,\\
\text{(d)}\quad (\partial/\partial\nu)(y^2+U+yV)=0 & \text{along } BCD, \text{ that is when } x^2/a^2+y^2/b^2=1.
\end{cases}
$$

Writing $w=re^{i\theta}=u+iv$, and using the tranformation (1), we see that the boundary conditions (3) become, after a little reduction,

$$
(4)\begin{cases}
\text{(a)}\quad U=0, & \text{when } v=0\,,\\
\text{(b)}\quad (b-a)V'+\dfrac{2u^2}{(c+u^2)}\cdot\dfrac{\partial U}{\partial v}=0\,, & \text{when } v=0\,,\\
\text{(c)}\quad U+b\sin\theta V'=0\,, & \text{when } r=1\,,\\
\text{(d)}\quad b(a+b)\sin\theta=\dfrac{1}{\sin\theta}\dfrac{\partial U}{\partial r}+b\dfrac{\partial V'}{\partial r}-aV', & \text{when } r=1\,,
\end{cases}
$$

where $V'=V+br\sin\theta$, and $c=(b+a)/(b-a)$.

We will proceed to find pairs of harmonic functions U_{2n-1}, V'_{2n-1}, such that each pair will satisfy exactly the first three of the above boundary conditions. Any linear combination of these functions will also satisfy exactly these three conditions, and by giving special values to the arbitrary constants in this linear combination we can satisfy approximately the fourth equation. Physically, this means that in the fluid motion represented by our solution there will be a small velocity of slip along the boss BCD, which can be calculated from the error involved in the last boundary condition.

If we take

(5) $$U=\sum_{n=1}^{q} a_{2n-1}U_{2n-1}\ ,$$

where

(6) $$U_{2n-1}=\mathscr{I}\left\{\frac{A_{2n-1}}{2n+1}\,w^{2n+1}+\frac{B_{2n-1}}{2n-1}\,w^{2n-1}+\frac{C_{2n-1}}{2n-3}\,w^{2n-3}\right\},$$

then U is harmonic, and 4 (a) is satisfied. Moreover, the consideration of symmetry shows that even powers of w are not required. Now we have

(7) $$\left(\frac{\partial U_{2n-1}}{\partial v}\right)_{v=0}=(u^2+c)\left(A_{2n-1}u^{2n-2}+\frac{C_{2n-1}}{c}u^{2n-4}\right),$$

provided $cA_{2n-1}-B_{2n-1}+C_{2n-1}/c=0$,

and

(8) $$(U_{2n-1})_{r=1}=\sin\theta\left(\frac{2A_{2n-1}}{2n+1}\cos 2n\theta-\frac{2C_{2n-1}}{2n-3}\cos(2n-2)\theta\right),$$

provided $\dfrac{A_{2n-1}}{2n+1}+\dfrac{B_{2n-1}}{2n-1}+\dfrac{C_{2n-1}}{2n-3}=0$.

From (7), (8), we see that we can take

(9) $$\begin{cases} A_{2n-1}=\dfrac{2(2n+1)}{b-a}\{2(n-1)b+a\}p_n\ , \\ B_{2n-1}=\dfrac{4(2n-1)}{(b-a)^2}\{b^2+(2n-1)ab+a^2\}p_n\ , \\ C_{2n-1}=-\dfrac{2(2n-3)(b+a)}{(b-a)^2}\{2nb+a\}p_n\ , \end{cases}$$

where the unknown p_n has yet to be determined. By setting

$$V'_{2n-1}=W_{2n-1}+\frac{2C_{2n-1}r^{2n-2}\cos(2n-2)\theta}{(2n-3)b}-\frac{2A_{2n-1}r^{2n}\cos 2n\theta}{(2n+1)b}$$

we see that 4 (c) yields

(10) $$W_{2n-1}=0 \quad \text{when} \quad r=1\ .$$

If further we take

(11) $$p_n=\frac{b(b-a)^2}{4(2nb+a)\{2(n-1)b+a\}}$$

then condition 4 (b) gives

$$W_{2n-1}=u^{2n-2}-u^{2n} \quad \text{when} \quad v=0\,. \tag{12}$$

Equations (9), (11) now give

$$\begin{cases} A_{2n-1}=\dfrac{(2n+1)b(b-a)}{2(2nb+a)}\,, \\[2ex] B_{2n-1}=\dfrac{(2n-1)b\{b^2+(2n-1)ab+a^2\}}{(2nb+a)\{2(n-1)b+a\}}\,, \\[2ex] C_{2n-1}=-\dfrac{(2n-3)b(b+a)}{2\{2(n-1)b+a\}}\,. \end{cases} \tag{13}$$

To find the function W_{2n-1} satisfying (10), (12), we consider $\psi_{2n}=\mathscr{R}\{\chi_{2n}(w)\}$, where

$$\begin{cases} \chi_{2n}(w)=-w^{-2n}+\dfrac{i}{\pi}\left\{w^{2n}+\dfrac{1}{w^{2n}}\right\}\{\log(w+1)-\log(w-1)\} \\[2ex] \quad -\dfrac{2i}{\pi}\left\{\left(w^{2n-1}+\dfrac{1}{w^{2n-1}}\right)+\dfrac{1}{3}\left(w^{2n-3}+\dfrac{1}{w^{2n-3}}\right)+\cdots+\dfrac{1}{2n-1}\left(w+\dfrac{1}{w}\right)\right\}, \\[2ex] \chi_0(w)=-1+\dfrac{2i}{\pi}\{\log(w+1)-\log(w-1)\}\,. \end{cases} \tag{14}$$

It is easy to verify that $\mathscr{R}\{\chi_{2n}(w)\}=0$ when $r=1$, and that $\mathscr{R}\{\chi_{2n}(w)\}=u^{2n}$ when $v=0$, since from Figure 2

$$\log(w-1)-\log(w+1)=\log\frac{r_2}{r_1}+i(\theta_2-\theta_1)\,.$$

The function W_{2n-1} is thus given by

$$W_{2n-1}=\phi_{2n-2}-\phi_{2n}\,. \tag{15}$$

Finally, we see that the required stream function ψ is given by equation (2), where

$$\begin{cases} U=\mathscr{I}\displaystyle\sum_{n=1}^{q}a_{2n-1}\left\{\frac{A_{2n-1}}{2n+1}w^{2n+1}+\frac{B_{2n-1}}{2n-1}w^{2n-1}+\frac{C_{2n-1}}{2n-3}w^{2n-3}\right\}, \\[2ex] V=\mathscr{R}\left\{ibw+\displaystyle\sum_{n=1}^{q}a_{2n-1}\left\{\frac{2C_{2n-1}}{(2n-3)b}w^{2n-2}-\frac{2A_{2n-1}}{(2n+1)b}w^{2n}\right.\right. \\[2ex] \qquad\qquad\qquad\qquad \left.\left.+\chi_{2n-2}(w)-\chi_{2n}(w)\right\}\right\}, \end{cases} \tag{16}$$

the constants A_{2n-1}, B_{2n-1}, C_{2n-1} being determined from (13), $\chi_{2n}(w)$ from (14). It is quite easy to verify that $\psi\to y^2$ as $z\to\infty$, that is as $w\to 0$, since the most significant terms in U and yV are respectively

$2a_1C_1y/(a+b)$ and $a_1y(-2C_1/b+1)$, the sum of these being clearly zero from (13).

3. The fourth boundary condition. It is now necessary to consider the boundary condition as yet unsatisfied, given by equation 4 (d), in the form

$$b(a+b)\sin\theta=\sum_{n=1}^{q}a_{2n-1}\lambda_{2n-1}(\theta)\qquad \text{when } r=1\,, \tag{17}$$

where

$$\lambda_{2n-1}(\theta)=\frac{1}{\sin\theta}\,\frac{\partial U_{2n-1}}{\partial r}+b\frac{\partial V'_{2n-1}}{\partial r}-aV'_{2n-1}\,.$$

Theoretically, the constants a_{2n-1} must be chosen so that (17) is satisfied for all values of θ lying between 0 and π, and this would require an infinite number of terms. Clearly, therefore, some form of approximation must be applied. Suppose that the constants a_{2n-1} are chosen so that

$$\sum_{n=1}^{q}a_{2n-1}\lambda_{2n-1}=b(a+b)\sin\theta+F(\theta)\,,$$

then $\sin\theta F(\theta)$ is the error involved in the boundary derivative $(\partial\psi/\partial r)_{r=1}$, and the actual velocity of slip on the boundary BCD in the z-plane is

$$\left\{\left(\frac{\partial\psi}{\partial r}\right)\left|\frac{dw}{dz}\right|\right\}_{r=1}=\frac{\sin\theta\,F(\theta)}{(b^2\cos^2\theta+a^2\sin^2\theta)^{1/2}}\,.$$

This becomes infinite at $\theta=\pi/2$ in the degenerate case $a=0$, unless $F(\pi/2)$ happens to be zero. Therefore we must consider a method of approximation which gives no error at all when $\theta=\pi/2$. The coefficients a_{2n-1} will be chosen so that the expressions on each side of (17) have the same values and the same differential coefficients with regard to θ when $\theta=\pi/2$, the number of differential coefficients that can be equated depending on the value of the integer q. From (16), we see that when $r=1$,

$$\begin{aligned}\lambda_{2n-1}(\theta)=&\frac{1}{\sin\theta}\{A_{2n-1}\sin(2n+1)\theta+B_{2n-1}\sin(2n-1)\theta+C_{2n-1}\sin(2n-3)\theta\}\\&+2\left\{\frac{2(n-1)b-a}{(2n-3)b}\right\}C_{2n-1}\cos(2n-2)\theta-2\left\{\frac{2nb-a}{(2n+1)b}\right\}A_{2n-1}\cos 2n\theta\\&+b(f_{2n-2}(\theta)-f_{2n}(\theta))\,,\end{aligned}\tag{18}$$

where

$$f_{2n}(\theta)=\left(\frac{\partial\phi_{2n}}{\partial r}\right)_{r=1}=\mathscr{R}\left\{w\frac{d\chi_{2n}(w)}{dw}\right\}_{r=1}$$

$$=\frac{8n}{\pi}\left\{\sin(2n-1)\theta+\frac{1}{3}\sin(2n-3)\theta+\cdots+\frac{1}{2n-1}\sin\theta\right\}$$

$$+2n\cos 2n\theta+\frac{4n}{\pi}\sin 2n\theta\log\tan\frac{\theta}{2}-\frac{2}{\pi}\csc\theta\,,$$

$$f_0(\theta)=-\frac{2}{\pi}\csc\theta\,.$$

It is to be noted, that although $f_{2n}(\theta)$ is infinite at $\theta=0,\ \pi$, the expression $f_{2n-2}(\theta)-f_{2n}(\theta)$, which occurs in $\lambda_{2n-1}(\theta)$ is finite at these points. Equation (17) is satisfied exactly when $\theta=0,\ \pi$, and putting $\theta=\pi/2$, we have

$$b(a+b)=\sum_{n=1}^{q}a_{2n-1}\lambda_{2n-1}(\pi/2)\,, \tag{19}$$

and by differentiation we are led to

(19′)
$$\left\{\begin{aligned}-b(a+b)&=\sum_{n=1}^{q}a_{2n-1}\lambda''_{2n-1}(\pi/2)\,,\\ b(a+b)&=\sum_{n=1}^{q}a_{2n-1}\lambda^{IV}_{2n-1}(\pi/2)\,,\end{aligned}\right.$$

and so forth.

It is from this set of equations that the constants a_{2n-1} are to be calculated.

4. **Special cases.** The two special cases of the semi-circular boss and projection will now be discussed.

(α) *semi-circular boss.*

Setting $a=b=1$, equation (16) yields

$$U=\sum_{n=1}^{q}\frac{a_{2n-1}}{2n-1}\{r^{2n-1}\sin(2n-1)\theta-r^{2n-3}\sin(2n-3)\theta\}\,,$$

$$V=-r\sin\theta+\sum_{n=1}^{q}a_{2n-1}(\phi_{2n-2}-\phi_{2n})-\sum_{n=1}^{q}\frac{2a_{2n-1}}{2n-1}r^{2n-2}\cos(2n-2)\theta\,,$$

and from (18), we get when $r=1$

$$\lambda_{2n-1}(\theta)=\frac{1}{\sin\theta}\left\{\frac{2}{2n-1}\sin(2n-1)\theta\right\}+f_{2n-2}(\theta)-f_{2n}(\theta)\,.$$

As an example, let us take $q=3$, so that from (19) we are required to solve the equations

$$2=a_1\lambda_1(\pi/2)+a_3\lambda_3(\pi/2)+a_5\lambda_5(\pi/2)\ ,$$
$$-2=a_1\lambda_1''(\pi/2)+a_3\lambda_3''(\pi/2)+a_5\lambda_5''(\pi/2)\ ,$$
$$2=a_1\lambda_1^{IV}(\pi/2)+a_3\lambda_3^{IV}(\pi/2)+a_5\lambda_5^{IV}(\pi/2)\ .$$

By substitution and straightforward calculation we obtain the following table :-

	$n=1$	$n=2$	$n=3$
$\lambda_{2n-1}(\pi/2)$	+1.45352	−0.72488	+0.38385
$\lambda''_{2n-1}(\pi/2)$	−0.36056	+5.18309	−9.65706
$\lambda^{IV}_{2n-2}(\pi/2)$	−1.10423	−22.24210	+191.53800

which leads directly to $a_1=+1.21058$, $a_3=-0.34379$, $a_5=-0.02299$.

A more accurate result can be obtained by taking more terms in the linear expression for ψ, and it is found that the coefficients a_{2n-1} decrease rapidly in magnitude, but the numerical work involved soon becomes exceedingly heavy.

This choice of approximation method is seen to advantage if calculating the error function

$$F(\theta)=\sum_{n=1}^{3}a_{2n-1}\lambda_{2n-1}(\theta)-2\sin\theta$$

at any point by means of the Taylor expansion about $\theta=\pi/2$, several of the significant differential coefficients being zero by definition. The following table gives the value of $F(\theta)$ for various values of θ, and Figure 3 shows the graph of $F(\theta)$ plotted against values of θ lying between 0 and $\pi/2$. The graph for $\pi/2\leq\theta\leq\pi$ will, of course, be similar, since $F(\theta)$ is symmetrical about $\theta=\pi/2$.

θ	0	$\pi/45$	$\pi/15$	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
$F(\theta)$	0	0.05918	0.08225	0.05838	0.00915	0.00041	0

It will be noticed that the results are most accurate in the vicinity of $\theta=\pi/2$, as might be expected from the method of approximation. Although the value of $F(\theta)$ becomes greater than 0.08 for a certain (small) values of θ, the velocity of slip, given by $\sin\theta\cdot F(\theta)$, is really very small at these points.

(β) *degenerate boss.*

If $a=0$, the semi-elliptical boss degenerates to a projection into

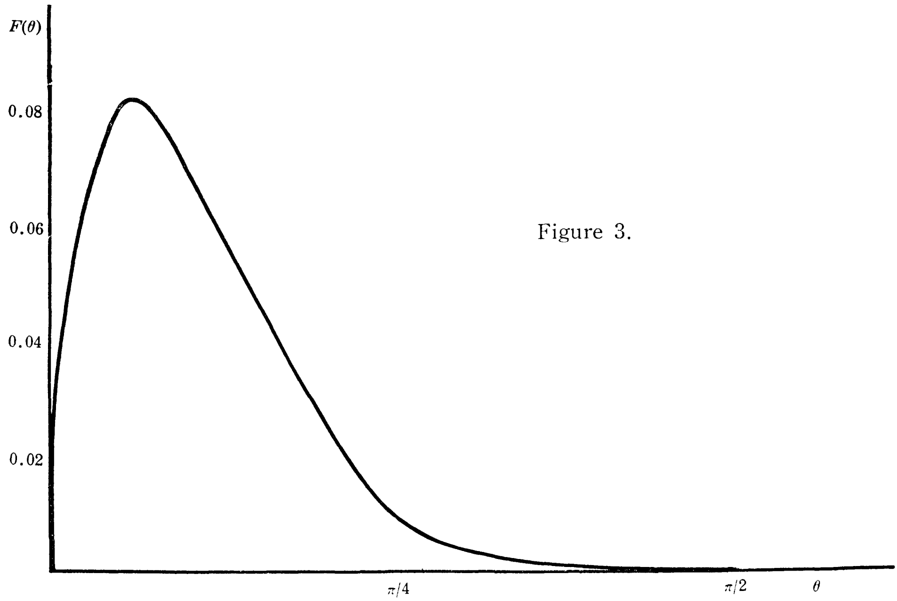

Figure 3.

the moving liquid, and the formulae (13) become

$$A_{2n-1}=\frac{(2n+1)b}{4n}\,, \qquad B_{2n-1}=\frac{(2n-1)b}{4n(n-1)}\,, \qquad C_{2n-1}=-\frac{(2n-3)b}{4(n-1)}\,.$$

These values will hold except for $n=1$. For this case we will follow W. R. Dean [1], and will find a pair of solutions U_1, V_1' which satisfy equations 4 (a), (b), (c). The procedure outlined in § 2 is again followed, and omitting details we are led to the two pairs of solutions

$$U_1=\mathscr{I}bw\,, \qquad V_1'=-\mathscr{R}\left\{\frac{2w}{i+w}\right\},$$

and

$$\mathscr{I}\frac{b}{4}\{w^3+3w\}\,, \quad U_1=\frac{w^3}{4}+w\,, \quad V_1'=\mathscr{R}\left\{-1-\frac{1}{2}w^2+\chi_0(w)-\chi_2(w)\right\}.$$

The final solution is thus

$$(20)\quad \begin{cases} U=\mathscr{I}\left\{a_0bw+\dfrac{a_1b}{4}(w^3+3w)+\displaystyle\sum_{n=2}^{q}a_{2n-1}b\left(\dfrac{w^{2n+1}}{4n}+\dfrac{w^{2n-1}}{4n(n-1)}-\dfrac{w^{2n-3}}{4(n-1)}\right)\right\}, \\ V=\mathscr{R}\left\{ibw+a_1\left(-1-\dfrac{1}{2}w^2\right)+\displaystyle\sum_{n=2}^{q}a_{2n-1}\left(\dfrac{w^{2n-2}}{2(n-1)}+\dfrac{w^{2n}}{2n}\right)\right\} \\ \qquad +\mathscr{R}\left\{-\dfrac{2a_0w}{i+w}+\displaystyle\sum_{n=1}^{q}a_{2n-1}(\chi_{2n-2}(w)-\chi_{2n}(w))\right\}. \end{cases}$$

Again we note that $\psi \to y^2$ as $z \to \infty$.

5. The pressure equation. The pressure p is determined from the equations of motion in the form

$$\frac{\partial p}{\partial x} = -\mu \frac{\partial}{\partial y} \nabla^2 \psi , \qquad \frac{\partial p}{\partial y} = \mu \frac{\partial}{\partial x} \nabla^2 \psi ,$$

where μ is the coefficient of viscosity. Now $\psi = y^2 + U + yV$, where U and V are harmonic, so that $\nabla^2 \psi = 2 + 2(\partial V/\partial y)$, and hence

$$\frac{\partial p}{\partial x} = -2\mu \frac{\partial^2 V}{\partial y^2} = 2\mu \frac{\partial^2 V}{\partial x^2} , \qquad \frac{\partial p}{\partial y} = 2\mu \frac{\partial^2 V}{\partial x \partial y} .$$

Ignoring an arbitrary constant, we have therefore

$$p = 2\mu \frac{\partial V}{\partial x} = 2\mu \mathscr{R} \left\{ \frac{dV(w)}{dz} \right\} = 2\mu \mathscr{R} \left\{ \frac{dw}{dz} \cdot \frac{dV(w)}{dw} \right\} ,$$

where $V = \mathscr{R}\{V(w)\}$. From equation (16) we see that

$$V(w) = ibw + \sum_{n=1}^{q} a_{2n-1} \left\{ \frac{2C_{2n-1} w^{2n-2}}{(2n-3)b} - \frac{2A_{2n-1} w^{2n}}{(2n+1)b} + \chi_{2n-2}(w) - \chi_{2n}(w) \right\} ,$$

and so

$$p = 2\mu \mathscr{R} \left\{ \left[\frac{2w^2}{(b+a)+(b-a)w^2} \right] \left[ib + \sum_{n=1}^{q} a_{2n-1} \left\{ \frac{4(n-1)C_{2n-1} w^{2n-3}}{(2n-3)b} - \frac{4nA_{2n-1} w^{2n-1}}{(2n+1)b} + \frac{d\chi_{2n-2}(w)}{dw} - \frac{d\chi_{2n}(w)}{dw} \right\} \right] \right\} . \tag{21}$$

Equation (21) gives the pressure distribution, and on the plane boundary, where $v=0$, this becomes

$$p = \left\{ \frac{4\mu u^2}{(b+a)+(b-a)u^2} \right\} \left\{ \sum_{n=1}^{q} a_{2n-1} \left\{ \frac{4(n-1)C_{2n-1} u^{2n-3}}{(2n-3)b} - \frac{4nA_{2n-1} u^{2n-1}}{(2n+1)b} + 2(n-1)u^{2n-3} - 2nu^{2n-1} \right\} \right\} . \tag{22}$$

In particular the pressure at B exceeds that at D by

$$p_{\text{diff}} = \frac{8\mu}{b} \sum_{n=1}^{q} a_{2n-1} \left\{ -\frac{2(n-1)C_{2n-1}}{(2n-3)b} + \frac{2nA_{2n-1}}{(2n+1)b} + 1 \right\} . \tag{23}$$

For the special case $a=b=1$ discussed in § 4 (α), this expression is

$$8\mu \sum_{n=1}^{q} \frac{(4n-3)a_{2n-1}}{2n-1}$$

and with the values for a_1, a_3, a_5 substituted we obtain a difference of $4.78\,\mu$.

For the degenerate case $a=0$, $b=1$, Dean [1] obtains a difference of approximately $5.80\,\mu$ between the pressures at B and D.

References

1. W. R. Dean, *Note on the slow motion of fluid.* Proc. Cambridge Philos. Soc., **32** (1936), 598.

The University of Nottingham

Added in Proof: The equations governing the slow steady flow of a viscous incompressible fluid are the same as those characterizing an equilibrium state of an incompressible elastic solid, if one simply replaces velocity and coefficient of viscosity by displacement and shear modulus. Thus the results here obtained can be used as the solution for the tension of a semi-infinite plane whose edge is indented and traction free.

AN ALGEBRAIC CHARACTERIZATION OF FIXED IDEALS IN CERTAIN FUNCTION RINGS

LYLE E. PURSELL

1. Introduction. In this paper an algebraic characterization of the fixed ideals in a certain class of function rings is given (an ideal in a function ring is fixed if there is a point at which all functions in the ideal vanish). This class of function rings includes the rings of all real-, complex-, or quaternion-valued continuous functions on a normal Hausdorff space whose points are G-delta sets and the ring of r-fold differentiable functions on an r-differentiable manifold whose coordinate covering is neighborhood finite. For these rings of functions we construct the underlying space from the fixed ideals in the same way that Gelfand and Kolmogoroff [3] have constructed a compact space from the non-unit ideals in its ring of all real-valued continuous functions.

We also show the existence of certain homomorphisms from the automorphism groups of these function rings into the group of homeomorphisms of the underlying space onto itself. In § 5 we find that an isomorphism between the rings of all r-differentiable functions on two r-differentiable manifolds can be extended to an isomorphism between the rings of all continuous functions on these manifolds and that the homeomorphism determined by this isomorphism is differentiable.

2. The general case.

(2.1) *By $\mathscr{R}$ we mean a ring of functions from a regular Hausdorff space X to a division ring D having the following properties:*

P_1. *If f is in $\mathscr{R}$, then the set of zeros of f, which we denote by $Z(f)$, is closed.*

P_2. *If x is not in a closed set F, then there is a function f in $\mathscr{R}$ such that $Z(f)$ contains a neighborhood of F but does not contain x.*

P_3. *If f in $\mathscr{R}$ does not vanish at any point of a closed set F, then there is a function g in $\mathscr{R}$ such that fg (and also gf) has the value 1 at every point of F.*

P_4. *For each x in X there is a function f_x in $\mathscr{R}$ which vanishes at x and only at x.*

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(2.2) *By the "support of a function f" in $\mathscr{R}$, which we denote by* $\mathrm{Sp}(f)$, *we mean the set* $\mathrm{Cl}(X-Z(f))$, From the properties of closure we have:

$$\text{(i)}\quad \mathrm{Sp}(f)=X-\mathrm{Int}\,Z(f)\,,$$
$$\text{(ii)}\quad \mathrm{Int}\,\mathrm{Sp}(f)=X-\mathrm{Cl}\,\mathrm{Int}\,Z(f),$$
$$\text{(iii)}\quad \mathrm{Sp}(f)=\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Sp}(f)\,.$$

By the "annihilator of a function f" in $\mathscr{R}$, which we denote by $A(f)$, *we mean the set of all g in $\mathscr{R}$ such that $fg=0$ (and hence $gf=0$).* For any ring of functions with values in a division ring the annihilator of an element is a two-sided ideal. In addition we have

$$A(f)=\{g\in\mathscr{R}\ |Z(g)\supset\mathrm{Sp}\,(f)\}\,,$$

and $A(f)=\mathscr{R}$ if an only if $f=0$.

(2.3) LEMMA. *If f and g are in $\mathscr{R}$ and $g\neq 0$, then $Z(f)$ and* $\mathrm{Sp}\,(g)$ *are disjoint if and only if $f-A(g)$ has an inverse in the residue class ring $\mathscr{R}-A(g)$.*

Proof. Since $g\neq 0$, then $Z(g)\neq X$ and $\mathrm{Sp}\,(g)$ is not empty. If f does not vanish at any point of $\mathrm{Sp}\,(g)$, then there is a function h in $\mathscr{R}$ such that fh and hf have the value 1 at every point of $\mathrm{Sp}\,(g)$, that is, $fh\equiv 1 \pmod{A(g)}$ and $hf\equiv 1 \pmod{A(g)}$. Hence f-$A(g)$ has an inverse in $\mathscr{R}-A(g)$. If $f-A(g)$ has an inverse in $\mathscr{R}-A(g)$, then there is a function h in $\mathscr{R}$ such that $(fh-1)$ is in $A(g)$, that is $(fh-1)$ vanishes at every point of $\mathrm{Sp}\,(g)$. Hence f does not vanish at any point of $\mathrm{Sp}\,(g)$.

(2.4) *If f is in $\mathscr{R}$, let $H(f)$ be the set of all nonzero g in $\mathscr{R}$ such that $f-A(g)$ has an inverse in the ring $\mathscr{R}-A(g)$. An ideal I in $\mathscr{R}$ is "bounded" if there is a function f in $\mathscr{R}$ without an inverse such that $H(f)$ contains $H(g)$ for every g in I. We say that "I is bounded by f". An ideal which is maximal in the set of all bounded ideals is called a "maximal bounded ideal". The set of all maximal bounded ideals is denoted by $M[\mathscr{R}]$. We observe that an ideal contained in a bounded ideal is bounded.*

(2.5) LEMMA. *For f and g in $\mathscr{R}$, $Z(g)$ contains $Z(f)$ if and only if $H(f)$ contains $H(g)$.*

Proof. From (2.3) and (2.4) $H(f)$ is the set of all functions h in $\mathscr{R}$ such that $Z(f)$ and $\mathrm{Sp}\,(h)$ are disjoint. Hence if $Z(f)$ is a subset of $Z(g)$, then $H(f)$ contains $H(g)$. Suppose there is a point x in $Z(f)$

but not in $Z(g)$, then by P_2 there is a function h in $\mathscr{R}$ which is different from zero on a neighborhood of x but vanishes on a neighborhood of $Z(g)$. For this function h, $Z(f)$ meets $\mathrm{Sp}(h)$ but $Z(g)$ does not. Hence if $Z(f)$ is not a subset of $Z(g)$, then $H(f)$ does not contain $H(g)$.

(2.6) THEOREM. *If $\mathscr{R}$ is a ring of functions from a regular Hausdorff space X to a division ring D which satisfies P_1, P_2, P_3, and P_4 of* (2.1), *then an ideal I in $\mathscr{R}$ is a fixed ideal if and only if it is a bounded ideal.*

Proof. If I is a fixed ideal, then there is a point x at which all elements of I vanish. From P_4 there is a function f_x in $\mathscr{R}$ which vanishes at x and only at x. For every g in I, $Z(g)$ contains $Z(f_x)$, that is $H(f_x)$ contains $H(g)$. Since f_x has no inverse, I is bounded. If I is bounded by a function f in $\mathscr{R}$ without an inverse, then $Z(f)$ is a subset of $Z(g)$ for every g in I. Since $Z(f)$ is not empty, I is fixed.

(2.7) *For x in X, $I(x)$ means the fixed ideal* $\{f\in\mathscr{R} \mid f(x)=0\}$. From (2.6) an ideal is a maximal bounded ideal if and only if it is of this form.

(2.8) *Let A be a subset of $M[\mathscr{R}]$. If we define*

$$J\in \mathrm{Cl}(A) \quad \text{if and only if} \quad J\supset \textstyle\bigcap_{I\in A} I$$

for A nonempty and $\mathrm{Cl}(A)=A$ *for A empty, then $M[\mathscr{R}]$ is said to have the "Stone topology". We denote the set $M[\mathscr{R}]$ with the Stone topology by X^*.*

(2.9) THEOREM. *If $\mathscr{R}$ is a ring of functions from a regular Hausdorff space X to a division ring D satisfying P_1, P_2, P_3, and P_4 of* (2.1), *then X is homeomorphic to X^*.*

Proof. From (2.7) the mapping $x\rightarrow I(x)$ is a one-to-one mapping of X onto X^*. Let $a\in \mathrm{Cl}(A)$, $A\subset X$, and let A^* be the image of A under the mapping $x\rightarrow I(x)$, then every function in $\mathscr{R}$ vanishing on A (that is, every function in $\bigcap_{x\in A} I(x)=\bigcap_{I\in A^*} I$) also vanishes at a (that is, is in $I(a)$) and $I(a)$ is in $\mathrm{Cl}(A^*)$. If, however, a is not in $\mathrm{Cl}(A)$, then there is a function f in $\mathscr{R}$ vanishing on A but not at a. Then f is in $\bigcap_{I\in A^*} I$ but not in $I(a)$, and $I(a)$ is not in $\mathrm{Cl}(A^*)$. Hence the correspondence $x\rightarrow I(x)$ is a homeomorphism of X onto X^*.

(2.10) COROLLARY. *If the rings $\mathscr{R}$ and $\mathscr{R}'$ of functions from the regular Hausdorff spaces X and X' to the division rings D and D',* *respectively, satisfy* P_1, P_2, P_3, *and* P_4 *of* (2.1) *and are isomorphic, then the spaces* X *and* X' *are homeomorphic.*

(2.11) Let $i: \mathscr{R} \to \mathscr{R}'$ be the isomorphism referred to in the preceding paragraph. With the point x in X we associate the x' (which exists uniquely) in X' which is the common zero of all functions in the isomorphic image of the ideal consisting of all functions in $\mathscr{R}$ which vanish at x, that is,

$$x' = \bigcap_{f \in i(I(x))} Z(f) .$$

One can show that the correspondence $x \to x'$ is a homeomorphism from X onto X'. *We will denote this homeomorphism by* $\phi(i)$ *and refer to it as "the homeomorphism from X onto X′ corresponding to (or determined by) the isomorphism i from* $\mathscr{R}$ *onto* $\mathscr{R}'$"

(2.12) *By* $\mathscr{A}(\mathscr{R})$ *we mean the automorphism group of* $\mathscr{R}$. *By* $\mathscr{H}(X)$ *we mean the homeomorphism group of X, that is the group of all homeomorphisms of X onto itself.* If i_1 and i_2 are in $\mathscr{A}(\mathscr{R})$, then it follows from (2.11) that $\phi(i_1 i_2) = \phi(i_1)\phi(i_2)$. Hence we have the theorem of the following paragraph.

(2.13) THEOREM. *The mapping* $\phi: \mathscr{A}(\mathscr{R}) \to \mathscr{H}(X)$ *is a homomorphism from the automorphism group of* $\mathscr{R}$ *into the homeomorphism group of X.*

(2.14) *For x in X we denote the set of values* $\{f(x) | f \in \mathscr{R}\}$ *by* $V(x)$. From P_2 and P_3 the set $V(x)$ is a subdivision ring of D. The correspondence $f \to f(x)$ is a homomorphism from $\mathscr{R}$ onto $V(x)$ with kernel $I(x)$, hence the correspondence $f - I(x) \to f(x)$ is an isomorphism from the residue class ring $\mathscr{R} - I(x)$ onto $V(x)$. Since $\mathscr{R} - I(x)$ is, therefore, a division ring, $I(x)$ is a maximal ideal, that is, every maximal bounded ideal is a maximal ideal.

(2.15) LEMMA. *If* $f \to f'$ *is an isomorphism from* $\mathscr{R}$ *onto* $\mathscr{R}'$ *and* $x \to x'$ *is the corresponding homeomorphism, then the correspondence* $f(x) \to f'(x')$ *is an isomorphism from* $V(x)$ *onto* $V'(x')$.

Proof. Since $I'(x')$ is the isomorphic image of $I(x)$, the correspondence $f - I(x) \to f' - I'(x')$ is an isomorphism. Since $f(x) \to f - I(x)$ and $f' - I'(x') \to f'(x')$ are isomorphisms, $f(x) \to f'(x')$ is an isomorphism from $V(x)$ onto $V'(x')$.

3. Rings of continuous functions.

(3.1) Čech [2] has shown that a subset of a normal Hausdorff space is the zero set of some real-valued continuous function if and only if it is a closed G_δ set. Using his result and Urysohn's lemma concerning real-valued continuous functions on a normal space, one may show that the rings — $C(X, R)$ of all real-valued continuous functions on X, $C(X, K)$ of all complex-valued continuous functions on X, and $C(X, Q)$ of all quaternion-valued continuous functions on X—satisfy P_1, P_2, P_3, and P_4 of (2.1) if X is a normal Hausdorff space all of whose points are G_δ sets. Hence we have the following.

(3.2) THEOREM. *Let X and X' be normal Hausdorff spaces all of whose points are G_δ sets and let F denote either the real field, the complex field, or the quaternion ring. If $C(X, F)$ and $C(X', F)$ are isomorphic, then X and X' are homeomorphic.*

(3.3) According to results obtained by Gelfand and Kolmogoroff [3], Hewitt [6], and Gillman, Henriksen, and Jerison [4], Theorem (3.2) holds for completely regular spaces satisfying the first axiom of countability. There are, however, normal spaces all of whose points are G_δ sets which do not satisfy the first axiom of countability (cf. Bing [1, p. 180, Example C]).

(3.4) For the rings $C(X, F)$ it can be established that the homomorphism $\phi : \mathscr{A}(C(X, F)) \rightarrow \mathscr{H}(X)$ of (2.11) and (2.13) is a homomorphism onto $\mathscr{H}(X)$.

4. Rings of real-valued functions.

(4.1) If $\mathscr{R}$ is a ring of real-valued functions on X satisfying P_1, P_2, P_3, and P_4, then for each x the set of values $V(x) = \{f(x) | f \in R\}$ is a subfield of the real field R. We now introduce an additional property for the ring $\mathscr{R}$:

P_5. *For each x in X the set of values $V(x)$ is a subfield of the real field R which has only one isomorphism into R, the identity isomorphism.*

Property P_5 holds if $V(x) = R$; hence $C(X, R)$ satisfies P_5. There are rings of real-valued functions satisfying P_1, P_2, P_3, P_4, and P_5 which contain discontinuous functions as is shown by the example of the following paragraph given to the author by D. W. Dubois.

(4.2) EXAMPLE. Let X be the closed interval $[0, 1]$, α be a finite subdivision $\{0 = x_0, x_1, \cdots, x_{n-1}, x_n = 1\}$ of X, and A be the set of all α. Let $\theta(x) = \exp(x)$ for $x \neq 0$ and $\theta(0) = 0$. Let $B(\alpha)$ be the set of all real-valued functions f on X such that

$$f(x)=\frac{p_i(x,\theta(x))}{q_i(x,\theta(x))}, \qquad x_{i-1}\leq x\leq x_i,\ i=1,2,\cdots,n,$$

where $p_i(x,\theta)$ and $q_i(x,\theta)$ are polynomials in x and $\theta(x)$ such that $f(x)$ is continuous at $x_1, x_2, \cdots, x_{n-2}$, and x_{n-1} and $q_i(x,\theta(x))$ does not vanish for $x_{i-1}\leq x\leq x_i$ for any i. If $\mathscr{R}=\bigcup_{\alpha\in A}B(\alpha)$, then $\mathscr{R}$ is a ring of real-valued functions which satisfy P_1, P_2, P_3, P_4, and P_5 but some of which are discontinuous.

(4.3) Theorem (4.4) and (4.5) may be established by using P_5 and the results of § 2.

(4.4) THEOREM. *If $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic rings of real valued functions on regular Hausdorff spaces X and X' satisfying P_1, P_2, P_3, P_4, and P_5, i is the isomorphism from $\mathscr{R}$ onto $\mathscr{R}'$, and h is the corresponding homeomorphism from X onto X', then:*

(i) *$f(x)=(if)(h(x))$ for all f in R and x in X. Hence f is bounded above (below) if and only if (if) is bounded above (below)*; $\operatorname{lub} f=\operatorname{lub} f'$, $\operatorname{glb} f=\operatorname{glb} f'$; *and the subrings of all bounded functions in R and R' are isomorphic.*

(ii) *There is an isomorphism i^* from $C(X,R)$ onto $C(X',R)$ such that $i(f)=i^*(f)$ for all f in $C(X,R)\cap\mathscr{R}$.*

(4.5) THEOREM. *If $\mathscr{R}$ is a ring of real-valued functions on a regular Hausdorff space satisfying P_1, P_2, P_3, P_4, and P_5, then the homomorphism ϕ of* (2.11) *and* (2.13) *is an isomorphism of $\mathscr{A}(R)$ into $\mathscr{H}(X)$.*

From (3.4) and (4.5) we have the following.

(4.6) THEOREM. *The groups $\mathscr{A}(C(X,R))$ and $\mathscr{H}(X)$ are isomorphic.*

5. Rings of continuously differentiable functions.

(5.1) If $C^r(M)$ is the ring of r-fold continuously differentiable functions on an r-differentiable manifold M with a neighborhood-finite covering of coordinate neighborhoods (r may be either a positive integer or the symbol ∞), then $C^r(M)$ satisfies P_1, P_2, P_3, P_4, and P_5. The theorem of the following paragraph may be obtained.

(5.2) THEOREM. *If $C^r(M)$ and $C^{r'}(M')$ are isomorphic, then M and M' are homeomorphic. The homeomorphism h determined by the isomorphism is differentiable (that is, $f(h)$ is in $C^r(M)$ if f is in $C^{r'}(M')$) and the isomorphism can be extended to an isomorphism from $C(M,R)$ onto $C(M',R)$.*

6. Additional remarks. Since the above was written the author has observed that P_4 may be replaced by the weaker hypothesis:

P_4^*. *For each x in X there is a pair of functions g and h in $\mathscr{R}$ such that $x = Z(g) - Z(h)$.*

If $\mathscr{R}$ satisfies P_1, P_2, P_3, and P_4^*, one can show that an ideal I is fixed if and only if there is a pair of functions g and h in $\mathscr{R}$ such that $H(g)$ does not contain $H(h)$ but $H(gh)$ does contain $H(fh)$ for every f in the ideal I. (Lemma (2.5) holds as before.) The results of (2.9) through (2.15) may then be established if X^* is defined to be the set of maximal fixed ideals with the Stone topology.

If X is a completely regular, locally-compact space all of whose points are G_δ sets, then the rings $C_0(X, R)$, $C_0(X, K)$, and $C_0(X, Q)$ of all real-, complex-, or quaternion-valued continuous functions with compact supports satisfy P_1, P_2, P_3, and P_4^*. Hence it follows that they determine X. (This result for $C_0(X, R)$ has already been established by Shanks [7] without assuming that points are G_δ sets). One can also show that the automorphism group $\mathscr{A}(C_0(X, R))$ is isomorphic to $\mathscr{A}(C(X, R))$ and $\mathscr{H}(X)$.

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GRINNELL COLLEGE

ADDITIONAL NOTE ON SOME TAUBERIAN THEOREMS OF O. SZÁSZ

C. T. RAJAGOPAL

1. An additional theorem. In the note [3] to which this is an addition, Theorem II is exhibited as a generalization of Theorem I and an appeal is made to Szász [6] to indicate the transition from Theorem II to the final result stated as Corollary III′. However, in view of the formal simplicity of Corollary III′ and the wide generality (reflected in its apparent complexity) of Theorem II, it seems worth while to adopt the opposite point of view and record a method, based on the following result, of deducing Theorem II and all related theorems (which cover Szász's) from Corollary III′ [3, p. 384].

THEOREM IV. *If a (real) series* $\sum_{n=1}^{\infty} a_n$ *is* (Φ, λ)*-summable to* s, *where* λ *denotes the strictly positive increasing divergent sequence* $\{\lambda_n\}$ *subject to the additional condition* $\lambda_{n+1}/\lambda_n \to 1$, *and if the series satisfies the Tauberian condition*:

$$\liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{\nu=n+1}^{m} \lambda_\nu a_\nu \geq 0\,, \qquad m>n,\ \frac{\lambda_m}{\lambda_n} \to 1\,, \tag{1}$$

then $\sum_{n=1}^{\infty} a_n$ *is convergent to* s. (Amnon Jakimovski [1, Theorem 1] gives the case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

Proof. We have, by Abel's partial-summation lemma,

$$\sum_{\nu=n+1}^{m} a_\nu = \sum_{\nu=n+1}^{m} \frac{\lambda_\nu a_\nu}{\lambda_\nu} \geq \frac{\lambda_n}{\lambda_{n+1}} \cdot \frac{1}{\lambda_n} \min_{n+1\leq k\leq m} \sum_{\nu=n+1}^{k} \lambda_\nu a_\nu\,.$$

Hence, by (1),

$$\liminf_{n\to\infty} \sum_{\nu=n+1}^{m} a_\nu \geq 0, \qquad m>n, \quad \frac{\lambda_m}{\lambda_n} \to 1.$$

It is well-known [2, p. 33] that the above Schmidt condition is equivalent to the second alternative of hypothesis (12) of Corollary III′ [3, p. 384]. Therefore this corollary establishes that $\sum_{n=1}^{\infty} a_n = s$.

2. Deductions from Theorem IV.

COROLLARY IV.1. *In Theorem* IV, (1) *is implied by, and so can be replaced by,* ONE *of the following conditions*:

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$$\left.\begin{aligned} &(2) \qquad \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{\nu=n+1}^{m}\lambda_\nu(|a_\nu|-a_\nu)=0\,, \\ &(3) \qquad \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{\nu=n+1}^{m}\lambda_\nu|a_\nu|=0\,, \end{aligned}\right\} \qquad m>n,\quad \frac{\lambda_m}{\lambda_n}\to 1.$$

(Szász [6, Theorem 3] gives the case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

COROLLARY IV.2. *In Corollary* IV.1, (2) *can be replaced by the condition*:

$$(4) \qquad \begin{aligned} &U_n\equiv\sum_{\nu=1}^{n}\lambda_\nu(|a_\nu|-a_\nu)=O(\lambda_n), && n\to\infty, \\ &\lim_{n\to\infty}\left(\frac{U_m}{\lambda_m}-\frac{U_n}{\lambda_n}\right)=0\,. && m>n,\ \frac{\lambda_m}{\lambda_n}\to 1\,. \end{aligned}$$

(Szász [6, Theorem 2] gives the case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

The above corollary is the same as Theorem II of my note [3]. We can deduce it from the preceding corollary merely by noting that (4) implies (2)[1] as a result of letting $n\to\infty$, $\lambda_m/\lambda_n\to 1$ in the identity:

$$\frac{U_m-U_n}{\lambda_n}=\left(\frac{U_m}{\lambda_m}-\frac{U_n}{\lambda_n}\right)\frac{\lambda_m}{\lambda_n}+\frac{U_n}{\lambda_n}\left(\frac{\lambda_m}{\lambda_n}-1\right), \qquad m>n\,.$$

COROLLARY IV.3. *In Corollary* IV.1, (3) *can be replaced by the hypothesis*:

$$(5) \qquad \begin{aligned} &V_n\equiv\sum_{\nu=1}^{n}\lambda_\nu|a_\nu|=O(\lambda_n)\,, && n\to\infty\,, \\ &\lim_{n\to\infty}\left(\frac{V_m}{\lambda_m}-\frac{V_n}{\lambda_n}\right)=0\,, && m>n,\ \frac{\lambda_m}{\lambda_n}\to 1\,, \end{aligned}$$

which implies (3) *exactly as* (4) *implies* (2).

Plainly the last hypothesis (5) *can assume the special form*:

$$\lim_{n\to\infty}\frac{V_n}{\lambda_n}=l \qquad l<\infty\,.$$

(Szász and Rényi [6, Theorems 1 and B] give the particular case $\phi(u)=e^{-u}$, $\lambda_n=n$.)

3. **A second additional theorem.** Theorem IV is a deduction from Corollary III′ [3] and so ultimately from Theorem A [3, p. 378]. The following is another deduction from Theorem A deserving of mention.

[1] In fact (4) is equivalent to (2) as (2) implies (4) by an argument exactly like Szász's in the case $\lambda_n=n$ [**6**, Lemma 2].

THEOREM B. *Let $\phi(u)$ fulfill the conditions* C(i)–(v) *of the Introduction* [3, p. 377].[2] *Suppose that $A(u)$ is a (real) function of bounded variation in every finite interval of* $(0, \infty)$, $A(0)=0$. *If*

$$\frac{1}{u}\int_0^u x\, d\{A(x)\} \tag{6}$$

is slowly decreasing, that is,

$$\liminf_{u\to\infty}\left(\frac{1}{v}\int_0^v x\, d\{A(x)\}-\frac{1}{u}\int_0^u x\, d\{A(x)\}\right)\geq 0\,, \qquad v>u\,,\ \frac{v}{u}\to 1\,,$$

and if $A(u)$ is Φ-summable to s, that is, if

$$\Phi(t)=\int_0^\infty \phi(ut)\, d\{A(u)\} \tag{7}$$

exists for $t>0$ and tends to s as $t\to+0$, then $A(u)\to s$ as $u\to\infty$.

Proof. We write as before [3, pp. 377–378]:

$$A_1(u)=\int_0^u A(x)dx\,, \qquad \phi(u)=\int_u^\infty \psi(x)dx\,.$$

Then (7) gives successively [4, pp. 346–347], as $t\to+0$,

$$\Phi(t)=t\int_0^\infty \psi(ut)A(u)du\to s\,, \qquad \Phi_1(t)=t\int_0^\infty \psi(ut)\frac{A_1(u)}{u}\, du\to s\,.$$

$$\Phi(t)-\Phi_1(t)=t\int_0^\infty \psi(ut)\{A(u)-u^{-1}A_1(u)\}du\to 0\,.$$

Thus $A(u)-u^{-1}A_1(u)$ is Φ-summable to 0 and satisfies the Tauberian condition in (6). Hence, by a known result [4, Corollary 2.2] following from Theorem A [3], $A(u)-u^{-1}A_1(u)$ tends to 0 as $u\to\infty$. Consequently, by Theorem A [3], $u^{-1}A_1(u)$, and hence also $A(u)$, tends to s as $u\to\infty$.

4. **Remarks.** (i) Amnon Jakimovski [1, Theorem 1] has dealt with the case of Theorem B in which $\phi(u)=e^{-u}$ and

$$A(u)=\begin{cases} a_1+a_2+\cdots+a_n & \text{for } n\leq u<n+1,\ n\geq 1\,,\\ 0 & \text{for } 0\leq u<1\,,\end{cases}$$

showing, by a modification of the method used above to prove Theorem B, that we may in this case replace (6) by

$$\liminf_{n\to\infty}\left(\frac{U_m^*}{m}-\frac{U_n^*}{n}\right)\geq 0\,, \qquad m>n,\ \frac{m}{n}\to 1\,, \tag{6*}$$

[2] These conditions can be slightly relaxed (for example, [5, Theorem A]).

where $U_n^* = \sum_{\nu=1}^{n} \nu a_\nu$, leaving the statement of Theorem B otherwise unaltered. He also observes that (6*) includes (or generalizes) the second half of (4) with $\lambda_n = n$, implying that, in Szász's result cited under Corollary IV.2, the first half of (4) is superfluous. This observation is, however, incorrect as shown by the following example.

EXAMPLE 1. Let a_n be defined so that

$$\left.\begin{aligned} na_n &= \nu \quad\quad \text{for} \quad 4^\nu \leq n < 2\cdot 4^\nu, \\ na_n &= -n^{-2} \ \text{for} \ 2\cdot 4^\nu \leq n < 4^{\nu+1}, \end{aligned}\right\} \nu = 0, 1, 2, \cdots .$$

Then it is easily verified that (4) with $\lambda_n = n$ holds because

$$\sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu) = o(n), \qquad n \to \infty,$$

but that (6*) does not hold since

$$\text{if } n = 2\cdot 4^\nu,\ \nu \to \infty, \text{ then } \frac{U_n^*}{n} = \frac{\sum_{k=0}^{\nu} k\cdot 4^k + O(1)}{2\cdot 4^\nu} \sim \frac{2\nu}{3},$$

$$\text{if } m = \text{the integral part of } 2\cdot 4^\nu \frac{\nu}{\nu - \sqrt{\nu}}, \text{ then } \frac{U_m^*}{m} = \frac{U_n^* + o(1)}{n}\,\frac{n}{m}$$

where $(n/m - 1) \sim -\nu^{-1/2}$, so that

$$\liminf_{n\to\infty} \left(\frac{U_m^*}{m} - \frac{U_n^*}{n}\right) = -\infty, \qquad m > n,\ \frac{m}{n} \to 1.$$

While the above example shows that (4) *with* $\lambda_n = n$ *does not in general imply* (6*), the one which follows makes it clear that *neither does* (6*) *necessarily imply* (4) *with* $\lambda_n = n$.

EXAMPLE 2. Let a_n be defined so that

$$\left.\begin{aligned} (-1)^n na_n &= \nu \ \text{for} \quad 4^\nu \leq n < 2\cdot 4^\nu, \\ a_n &= 0 \ \text{for} \ 2\cdot 4^\nu \leq n < 4^{\nu+1}, \end{aligned}\right\} \nu = 0, 1, 2, \cdots .$$

Then (6*) holds since $U_n^*/n \to 0$ as $n \to \infty$. However, (4) with $\lambda_n = n$ does not hold since now

$$U_n = \sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu)$$

and we have:

$$\text{if } n = 2\cdot 4^\nu,\ \nu \to \infty, \text{ then } \frac{U_n}{2n} = \frac{\sum_{k=1}^{\nu} k\cdot 4^k/2}{2.4^\nu} \sim \frac{\nu}{3},$$

$$\text{if } m = \text{the integral part of } 2 \cdot 4^{\nu} \frac{\nu}{\nu - \sqrt{\nu}}, \text{ then } \frac{U_m}{2m} = \frac{U_n}{2n} \frac{n}{m}$$

where $(n/m - 1) \sim -\nu^{-1/2}$, with the result that

$$\liminf_{n\to\infty} \left(\frac{U_m}{m} - \frac{U_n}{n} \right) = -\infty, \qquad m > n, \ \frac{m}{n} \to 1.$$

(ii) In the definition of Φ-summability of $A(u)$, set forth in (7) and assumed in both Theorem A [3] and Theorem B, the integral $\Phi(t)$ is to be interpreted as a Lebesgue-Stieltjes integral (absolutely) convergent for $t > 0$ unless further considerations, as in the case $\phi(u) = e^{-u}$, permit us to view it as a (non-absolutely) convergent Riemann-Stieltjes integral (cf. [5, p. 103, Note]).

(iii) In Theorem III [3, p. 383] the condition $\lambda_{n+1}/\lambda_n \to \infty$ of hypothesis (11) is a misprint for $\lambda_{n+1}/\lambda_n \to 1$.

In conclusion I wish to thank Dr. T. Vijayaraghavan[3] and the referee for helpful suggestions.

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RAMANUJAN INSTITUTE OF MATHEMATICS
MADRAS, INDIA

[3] Who passed into the beyond on April 20, 1955.

ERROR BOUNDS FOR ITERATIVE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

L. B. RALL

1. Introduction. A number of iterative procedures for obtaining the solution $x(s)$ of the integral equation of Fredholm type and second kind,

$$y(s)=x(s)-\lambda\int_a^b K(s,\ t)x(t)\,dt, \qquad a\leq s\leq b, \tag{1.1}$$

have been developed, notably by G. Wiarda [10, pp. 119–128], Hans Bückner [2, pp. 68–71], Carl Wagner [8], and P.A. Samuelson [7]. These methods are generalizations of the one due to Neumann [3, pp. 119–120] in the sense that they converge where the Neumann process fails, or else offer the possibility of more rapid convergence. The purpose of this paper is to obtain estimates for the error resulting from the use of a finite number of steps of these interative processes in forms suitable for numerical computation.

The author wishes to thank Professor A.T. Lonseth for many enlightening discussions concerning the material presented here, and the Reviewer for his helpful remarks.

2. The solution of linear equations. Methods for the approximate solution of Fredholm integral equations such as (1.1) and error estimates for these methods may be obtained directly from known results concerning the solution of linear equations in certain abstract spaces; it will be convenient to summarize some of these results here.

A set $X=\{x\}$ of elements is called a *linear space* if $x\in X$ implies $(\theta x)\in X$, where θ is any real number, and a binary operation $+$ is defined in X, with respect to which X is an Abelian group. The identity element of X for the operation $+$ will be denoted by 0. In order to discuss convergence and error estimation, with each $x\in X$ associate a finite, non-negative real number $\|x\|$, called the *norm* of x, which satisfies the following conditions:

1°. $\|x\|>0$ if $x\neq 0$, $\|0\|=0$;

2°. $\|\theta x\|=|\theta|\cdot\|x\|$ for any real number θ;

3°. $\|x+y\|\leq\|x\|+\|y\|$ for all $x,\ y\in X$.

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The space X is now said to be a *normed* linear space, and all spaces considered subsequently will be of this type.

A sequence $\{x_n\}$ in X is said to *converge* to the element $x \in X$, in symbols, $x_n \to x$ as $n \to \infty$, if $\|x - x_n\| \to 0$ as $n \to \infty$. A normed linear space X is called *complete* if for every sequence $\{x_n\}$ in X such that $\|x_n - x_{n+p}\| \to 0$ as $n \to \infty$ for all positive integers p, there exists an $x \in X$ such that $x_n \to x$ as $n \to \infty$.

A transformation which carries each $x \in X$ into a $y \in X$ is symbolized by $Tx = y$, where T is called an *operator* in X. T is *additive* if $T(x+y) = Tx + Ty$ for all $x, y \in X$, and *continuous* if $x_n \to x$ as $n \to \infty$ implies that $Tx_n \to Tx$ as $n \to \infty$. An additive and continuous T is said to be *linear*; for such a T, the nonnegative real numbers

$$M(T) = \text{l.u.b.}\ (\|Tx\|/\|x\|), \qquad \|x\| \neq 0, \tag{2.1}$$

$$m(T) = \text{g.l.b.}\ (\|Tx\|/\|x\|), \qquad \|x\| \neq 0, \tag{2.2}$$

exist and are finite [1, p. 54]. A linear T is *homogeneous*, that is $T(\theta x) = \theta(Tx)$ for any real θ [1, p. 36]. The *sum* $T+U$ and *product* TU of two linear operators T and U in X are defined respectively by the relations $(T+U)x = Tx + Ux$ and $(TU)x = T(Ux)$ for all $x \in X$. Furthermore,

$$M(T+U) \leq M(T) + M(U), \tag{2.3}$$

$$M(TU) \leq M(T)M(U), \tag{2.4}$$

[6]. The operator I such that $Ix = x$ for all $x \in X$ is defined to be the *identity operator* in X. The nth *power* T^n of an operator T in X is defined by $T^n = TT^{n-1}$ for all positive integers n, with $T^0 = I$ by definition. The *inverse* of an operator T in X is the operator T^{-1} such that $T^{-1}T = TT^{-1} = I$ if such exists. If T is linear and T^{-1} exists, T^{-1} is likewise linear; moreover, if $m(T) > 0$,

$$m(T)M(T^{-1}) = 1 \tag{2.5}$$

[6]. If T is a linear operator in a complete space X and $M(T) < 1$,

$$(I-T)^{-1} = \sum_{j=0}^{\infty} T^j, \tag{2.6}$$

[6, 9]. This result in combination with (2.3, 4, 5) gives

$$1 - M(T) \leq m(I-T) \leq M(I-T) \leq 1 + M(T) \tag{2.7}$$

for $M(T) < 1$. An operator T in a normed linear space X (not necessarily complete) is called *completely continuous* if for every bounded set $B = \{x\colon \|x\| \leq \theta\}$ for θ finite, in the set $TB = \{Tx\colon x \in B\}$ every infinite sequence converges to an element of X. In a general normed linear space X, (2.6) and (2.7) hold with the additional assumption that T is completely continuous [6]. These results furnish the following theorems:

THEOREM 1. *If F is a given linear operator in a complete normed linear space X, then the linear equation*

$$Fx=y \tag{1}$$

has a unique solution $x \in X$ for every $y \in X$ if and only if there exists a linear operator P in X such that P^{-1} exists, and

$$M(I-PF)<1 . \tag{2}$$

The solution x of (1) *in this case is given by*

$$x=\sum_{j=0}^{\infty}(I-PF)^{j}Py . \tag{3}$$

Proof: To prove the sufficiency of Theorem 1, assume that a linear operator P having the desired properties exists. The series

$$\sum_{j=0}^{\infty}(I-PF)^{j}\,Py$$

thus converges to an element, say z, of X; furthermore, $(I-PF)z=z-Py$, so $PFz=Py$. The application of P^{-1} yields $Fz=y$, and thus z satisfies (1). If $Fz_1=y$ and $Fz_2=y$, then $F(z_1-z_2)=0$, so that $(I-PF)(z_1-z_2)=z_1-z_2$, and if $z_1\neq z_2$, $M(I-PF)\geq 1$, contrary to assumption; hence $x=z$ is the unique solution of (1), and is given by (3). The necessity of Theorem 1 results from the fact that if there is a unique solution x of (1) for every $y\in X$, F^{-1} exists. Taking $P=F^{-1}$, P^{-1} exists and $M(I-PF)=M(I-F^{-1}F)=M(0)=0<1$, which completes the proof of the theorem.

COROLLARY 1. *Subject to the conditions of Theorem* 1, *F has the unique inverse*

$$F^{-1}=\sum_{j=0}^{\infty}(I-PF)^{j}P. \tag{4}$$

These results hold in a general normed linear space X subject only to the additional condition that $(I-PF)$ be completely continuous.

THEOREM 2. *If a unique solution $x \in X$ of* (1) *exists for every $y \in X$, X a normed linear space, and an operator P on X exists such that*

(2) *is satisfied, then the iterative process*

$$x_n=(I-PF)x_{n-1}+Py \tag{5}$$

is totally convergent (Bückner) *to the solution x of* (1), *that is, for all $x_0 \in X$, $x_n \to x$ as $n \to \infty$, and its error is bounded by*

$$\|x-x_n\| \leq [M(I-PF)]^n \|x-x_0\| \tag{6}$$

and

$$\|x-x_n\| \leq \frac{M(I-PF)}{m(PF)} \|x_n-x_{n-1}\|. \tag{7}$$

Proof. Following [9], note that, from (1) and (5),

$$x-x_n=(I-PF)^n(x-x_0)\ ,$$

from which (6) follows at once from (2.4). Condition (2) evidently insures that $x_n \to x$ as $n \to \infty$, whatever x_0. From (5),

$$x_n-x_{n-1}=PF(x-x_{n-1}), \text{ and } x-x_n=(I-PF)(x-x_{n-1}),$$

from which (7) is obtained by (2.1) and (2.2). Condition (2) insures that $m(PF)>0$, for, if $PFz=0$ for any $z \neq 0$, then $(I-PF)z=z$, and thus $M(I-PF) \geq 1$, contrary to assumption.

For the purposes of practical computation, it may prove expedient to calculate only one of the bounds $M(I-PF)$, $m(PF)$. By (2.7), the quantities $m(PF)$ and $1-M(I-PF)$ may be interchanged in (6) and (7); in what follows, the symbol μ will be used to denote either of these quantities. These results have been obtained on the assumption that all operations have been carried out exactly, which is frequently not possible in practice. Set $z_0=x_0$, and let z_n denote the results obtained from (5) by the use of some method of approximate evaluation. If Δ_j is the difference of the exact and the approximate evaluation of $(I-PF)z_{j-1}+Py$, from (5),

$$x-z_n=x-x_n+\sum_{j=0}^{n-1}(I-PF)^j\Delta_{n-j}\ . \tag{2.8}$$

Thus,

$$\|x-z_n\| \leq \|x-x_n\| + \sum_{j=0}^{n-1}(1-\mu)^j\|\Delta_{n-j}\|\ , \tag{2.9}$$

and as $0<1-\mu<1$, for $\delta=\max\|\Delta_{n-j}\|$, $(j=0, \cdots, n-1)$,

$$\|x-z_n\| \leq \|x-x_n\|+\delta/\mu\ , \tag{8}$$

where the estimate for $\|x-x_n\|$ is obtained from the error bounds previously derived.

3. Application to integral equations. The space C of functions $x=x(s)$ which are real, single-valued, and continuous on the interval $a\leq s\leq b$ is an example of a linear space. For the purpose of error estimation, useful definitions of the norm of an element $x\in C$ are:

(i) $\|x\|=\max\limits_{[a,\,b]}|x(s)|$, (iii) $\|x\|=\left[\int_a^b x^2(s)\,ds\right]^{1/2}$,

(ii) $\|x\|=\int_a^b |x(s)|\,ds$, (iv) $\|x\|=\left[\int_a^b |x(s)|^\rho\,ds\right]^{1/\rho}$, $\rho\geq 1$;

all of these definitions are obtainable from (iv), (i) being the limit of (iv) as $\rho\rightarrow\infty$, [5, pp. 134–150]. The *inner product* (x, y) of two elements $x, y\in C$ is the real number

$$(x,\ y)=\int_a^b x(s)y(s)\,ds\ . \tag{3.1}$$

An operator Q in C is said to be *positive definite* if $(Qx, x)>0$ for all $x\neq 0$ in C, and to be *positive semi-definite* if $(Qx, x)\geq 0$ for all $x\in C$. If Q is positive definite and $M(Q)<1$, then $M(I-Q)<1$ [11, p. 213], a fact which will be useful in establishing the convergence of iterative processes of the form (5). If $K(s, t)$ is real, single-valued, and continuous on the square $a\leq s$, $t\leq b$, the *integral transform* K defined by

$$Kx=\int_a^b K(s,\ t)\,x(t)\,dt \tag{3.2}$$

is a completely continuous linear operator in C, so that the results of § 2 apply at once to the equation (1.1) with $F=(I-\lambda K)$. A number λ is called a *characteristic value* of an integral transform K if $m(I-\lambda K)=0$; Fredholm's general theorem [4] states that (1.1) has a unique solution $x(s)$ in C for every $y(s)$ in C provided that λ is not a characteristic value of K. If λ is a characteristic value of K, is follows at once from Theorem 1 that (1.1) cannot have a unique solution, and thus it will be assumed throughout that λ is not a characteristic value of K, unless the contrary is explicitly stated.

The error bounds (6) and (7) for the iterative method (5) as applied to (1.1) may be put in the following convenient forms:

$$\|x-x_n\|\leq(1-\mu)^n\|x-x_0\|\ ; \tag{E1}$$

$$\|x-x_n\|\leq\frac{1-\mu}{\mu}\|x_n-x_{n-1}\|\ ; \tag{E2}$$

for k a nonnegative integer,

$$\|x-x_{n+k}\| \leq \frac{(1-\mu)^{k+1}}{\mu}\|x_n-x_{n-1}\| ; \tag{E3}$$

while for $x_0=y$,

$$\|x-x_n\| \leq \frac{(1-\mu)^n}{\mu} M(P)M(\lambda K)\|y\| . \tag{E4}$$

As before, $\mu=m[P(I-\lambda K)]$ or $\mu=1-M[I-P(I-\lambda K)]$. These bounds depend on the values of μ and $M(P)$. The operator P will now be specified to obtain several iterative methods of practical importance, for which explicit bounds for μ and $M(P)$ will be calculated.

Method I (Neumann):

$$x_n=y+\lambda K x_{n-1} . \tag{3.3}$$

This process is (5) with $P=I$, and thus $(I-PF)=\lambda K$. It follows from Theorem 2 that (3.3) is totally convergent provided that $M(\lambda K)<1$. If this is the case, explicit error estimates are obtained from the general expressions by setting $\mu=1-M(\lambda K)$ and noting that $M(P)=M(I)=1$. Usually $M(\lambda K)$ is not known exactly, but estimates for $M(\lambda K)$ are obtainable for various definitions of $\|x\|$ from known inequalities [5, loc. cit.; 6; 9].

Method II (Wiarda):

$$x_n=(1-\theta)x_{n-1}+\theta\lambda K x_{n-1}+\theta y , \qquad 0<\theta<1 . \tag{3.4}$$

This method is (5) with $P=\theta I$. Sufficient conditions for (3.4) to be totally convergent are that $-\lambda K$ is positive semi-definite and

$$0<\theta<\frac{1}{1+M(\lambda K)} . \tag{3.5}$$

These conditions insure that $PF=\theta(I-\lambda K)$ is positive definite and that $M(PF)<1$; the total convergence of Method II is a consequence of Theorem 2 in this case. As $-\lambda K$ is positive semi-definite, $m(PF)=m[\theta(I-\lambda K)]\geq\theta$, and as $0<\theta<1$, explicit error bounds for Method II may be obtained from the general expressions by the substitution $\mu=\theta$, and noting that $M(P)=\theta$.

Method III (Bückner):

$$x_n=(1+\theta)v_{n-1}-\theta\lambda K v_{n-1}-\theta y , \tag{3.6}$$

where

$$v_{n-1}=(1-\theta)x_{n-1}+\theta\lambda Kx_{n-1}+\theta y\,. \tag{3.7}$$

This process is totally convergent provided that θ satisfies (3.5) and the *kernel* $K(s,\ t)$ of K is *symmetric*, that is, $K(s,t)=K(t,s)$, $a\leq s,t\leq b$. From (3.6) and (3.7),

$$x_n=x_{n-1}-\theta^2(I-\lambda K)^2x_{n-1}+\theta^2(I-\lambda K)y\,. \tag{3.8}$$

This is (5) with $P=\theta^2(I-\lambda K)$. If the kernel $K(s,\ t)$ of K is symmetric, direct calculation from (3.1) verifies that

$$([I-\lambda K]^2x,\ x)=([I-\lambda K]x,\ [I-\lambda K]x), \tag{3.9}$$

which is positive for all $x\neq 0$ in C as λ is not a characteristic value of K. Thus $PF=\theta^2(I-\lambda K)^2$ is positive definite, and if θ satisfies (3.5), $M(PF)<1$. By Theorem 2, Method III is totally convergent. If $\{\lambda_m\}$ denotes the set of characteristic values of K, for the norm defined by (iii),

$$\mu=m(PF)=\theta^2\cdot\min_{(m)}\,[1-\lambda/\lambda_m]^2, \tag{3.10}$$

[2, pp. 10–11; 3, pp. 112–113]. This, together with the fact that $M(P)<\theta$, as $M[\theta(I-\lambda K)]<1$ from (3.5), allows the explicit evaluation of the general error estimates for Method III for the norm (iii).

Method IV (Wagner):

$$x_n=x_{n-1}-(1/g)(I-\lambda K)x_{n-1}+(1/g)y\,, \tag{3.11}$$

where

$$g=g(s)=1-\lambda\int_a^b K(s,\ t)\,dt\,,\qquad a\leq s\leq b\,; \tag{3.12}$$

here it is assumed throughout that $g(s)\neq 0$, $a\leq s\leq b$. If $K(s,\ t)$ has a high maximum for $s=t$ and is nearly zero elsewhere, then $(I-\lambda K)x\approx gx$ for all $x\in C$. Define the function $\phi(s;\ x)$ by

$$\phi(s;\ x)=(1/g)(I-\lambda K)x \tag{3.13}$$

for all $x\in C$. If

$$\omega=\max_{\substack{a\leq s\leq b\\ x\in C}}|1-\phi(s;\ x)|<1\,, \tag{3.14}$$

then Method IV is totally convergent, as it is (5) with $P=(1/g)I$, and (3.14) gives $M(I-PF)\leq\omega<1$. Explicit error bounds are obtained from the general expressions by the substitution $\mu=1-\omega$ and the fact that

$$M(P)=M[(1/g)I]\leq[\min_{a\leq s\leq b}|g(s)|]^{-1}. \tag{3.15}$$

For kernels of the type considered, it may be true that $\omega\ll 1$, in which case Method IV will converge rapidly.

Method V (Samuelson):

$$x_n=x_{n-1}-(I+J)(I-\lambda K)x_{n-1}+(I+J)y \tag{3.16}$$

is totally convergent, provided that

$$M(G-J)\leq\frac{1}{1+M(\lambda K)}\ , \tag{3.17}$$

where G is the *resolvent operator* for λK which gives the solution x of (1.1) as

$$x=(I+G)y, \tag{3.18}$$

This follows at once from Theorem 2, as (3.16) is (5) with $P=I+J$. Hence,

$$PF=[(I+G)-(G-J)](I-\lambda K)=I-(G-J)(I-\lambda K),$$

as $(I+G)$ is the inverse of $(I-\lambda K)$. Thus $(I-PF)=(G-J)(I-\lambda K)$, and (3.17) insures that $M(I-PF)<1$. Explicit error estimates for Method V are obtained by setting $\mu=1-M(G-J)[1+M(\lambda K)]$ and from $M(P)=M(I+J)\leq 1+M(J)$. In case that $M(G-J)$ is very small, Method V converges rapidly.

4. Numerical example. To illustrate the application of some of the methods and error bounds given, an approximate solution of the integral equation

$$s^2=x(s)-\lambda\int_0^1 K(s,\ t)x(t)\,dt\ , \qquad 0\leq s\leq 1\ , \tag{4.1}$$

where

$$K(s,\,t)=\begin{cases} s(1-t), & 0\leq s\leq t\leq 1\ ,\\ t(1-s), & 0\leq t\leq s\leq 1\ ,\end{cases} \tag{4.2}$$

will be sought for various values of λ. An approximation $x_n(s)$ to $x(s)$ will be considered to be satisfactory if $\|x-x_n\|<0.01$ with the norm defined by (iii). The characteristic values of K are known to be $\lambda_m=$

$n^2\pi^2$ and $M(K)=1/\pi^2$.

For $\lambda=-1$, $M(\lambda K)=1/\pi^2<1$, and thus Method I will be used. For $x_0(s)=s^2$, as $\|s^2\|=5^{-1/2}$, from (E4) the number of iterations required will not exceed one, so that

$$x_1(s)=s^2-s(1-s^3)/12 \tag{4.3}$$

is a satisfactory approximation to $x(s)$ on $[0, 1]$.

For $\lambda=-10$, $M(\lambda K)=10/\pi^2>1$, and the condition for the total convergence of Method I is not satisfied. However, $-\lambda K$ is positive definite, so that Method II is applicable. Take $x_0(s)=s^2$ and

$$\theta=0.49650<1/(1+10/\pi^2). \tag{4.4}$$

From (E4), the number of iterations will not exceed six. Successive iterations yield

$$x_1(s)=s^2-(0.41375)s(1-s^3), \tag{4.5}$$

$$x_2(s)=s^2+(0.34238)s(1-s^2)-(0.62207)s(1-s^3)-(0.06848)s(1-s^5), \tag{4.6}$$

with

$$\|x_2-x_1\|=0.01140\ , \tag{4.7}$$

and thus from (E3),

$$\|x-x_3\|\leq 0.006\ . \tag{4.8}$$

It follows that

$$\begin{aligned} x_3(s)=s^2&+(0.46050)s(1-s^2)-(0.72960)s(1-s^3) \\ &+(0.08500)s(1-s^4)-(0.13542)s(1-s^5)-(0.00607)s(1-s^7) \end{aligned} \tag{4.9}$$

is a satisfactory approximation to $x(s)$ on $[0, 1]$.

For $\lambda=25$, $M(\lambda K)=25/\pi^2>1$, so that Method I is not applicable. As $(-\lambda Ks,\ s)=-5/9$, $-\lambda K$ is not positive semi-definite, and Method II also fails. However, $K(s,\ t)$ is symmetric, and 25 is not a characteristic value of K, so Method III is totally convergent in this case. Choose

$$\theta=0.28394<1/(1+25/\pi^2) \tag{4.10}$$

and $x_0(s)=s^2$. From (3.10),

$$\mu=0.01084\ . \tag{4.11}$$

The upper bound for the number of iterations necessary is calculated from (E4) to be 727. The slowness of convergence in this case excludes manual methods of computation, but would be of little concern if a high-speed computing machine is available.

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Added in proof: Error bounds for Methods II and III are also contained in:

Mario Schönberg, *Sur la methode d'iteration de Wiarda et de Bückner pour la résolution de Fredholm*, Acad. Roy. Belgique, Bull. Cl. Sci., **37** (1951), 1141-1156, and **38** (1952), 154-167.

PSEUDO-ANALYTIC VECTORS ON PSEUDO-KÄHLERIAN MANIFOLDS

SHIGEO SASAKI AND KENTARO YANO

1. Introduction. A pseudo-Kählerian manifold is by definition a Riemannian manifold M^{2n} of class C^r $(r\geq 2)$ which has a skew-symmetric tensor field I_{AB}[1] of class C^{r-1} with non-vanishing determinant satisfying following two conditions:

$$(1) \qquad I^A{}_B I^B{}_C = -\delta^A_C\,, \qquad (I^{AB} I_{BC} = -\delta^A_C)$$

$$(2) \qquad I_{AB,C} = 0\,,$$

where

$$(3) \qquad I^A{}_B = g^{AE} I_{EB}\,, \qquad I^{AB} = g^{AE} g^{BF} I_{EF}\,,$$

and a comma denotes the covariant differentiation with respect to g_{AB}. It is known that the real representation of a Kählerian manifold of complex dimension n is a pseudo-Kählerian manifold of dimension $2n$ and of class C^ω and the converse is also true. However, the problem whether a pseudo-Kählerian manifold M^{2n} of class C^r $(r\neq\omega)$ can be regarded, by introducing suitable complex coordinate systems on M^{2n}, as a real representation of a (complex) Kählerian manifold or not is, as far as we know, still an open problem. In this paper we shall generalize some theorems which concern analytic vectors on Kählerian manifolds to pseudo-Kählerian manifolds.

2. Definitions of pseudo-analyticity.

DEFINITION 1. A set of *functions* (ϕ, ψ) defined over a pseudo-Kählerian manifold M^{2n} is said to be *pseudo-analytic* if

$$(4) \qquad I^A{}_B \phi_{,A} = \psi_{,B}\,.$$

If (ϕ, ψ) is pseudo-analytic, then $(-\psi, \phi)$ is pseudo-analytic too.

DEFINITION 2. A *contravariant vector field* u^A defined over M^{2n} is said to be *pseudo-analytic* if

$$(5) \qquad I^A{}_B u^B{}_{,C} = u^A{}_{,B} I^B{}_C\,.$$

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[1] We assume that the indices run as follows:

$$\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, n,$$
$$A, B, C, \cdots = 1, 2, \cdots, n, n+1, \cdots, 2n.$$

DEFINITION 3. A *covariant vector field* u_A defined over M^{2n} is said to be *pseudo-analytic* if

$$I^A{}_B u_{A,C} = I^A{}_C u_{B,A} . \tag{6}$$

If we denote a complex coordinate system of a Kählerian manifold K^n by z^α $(=x^\alpha + iy^\alpha)$ and take (x^α, y^α) as coordinates of the real representation of K^n, then (cf. [3])

$$I^\alpha{}_\beta = I^{n+\alpha}{}_{n+\beta} = 0 , \qquad I^\alpha{}_{n+\beta} = -I^{n+\alpha}{}_\beta = \delta^\alpha_\beta . \tag{7}$$

In this case, (4), (5) and (6) are nothing but Cauchy-Riemann equations for a complex analytic function $\phi + i\psi$, for a self-adjoint complex analytic contravariant vector $u^\alpha + iu^{n+\alpha}$ and for a self-adjoint complex analytic covariant vector $u_\alpha + iu_{n+\alpha}$. (We must take account of the fact that the real representation of a contravariant vector $u^\alpha + iu^{n+\alpha}$ is $(u^\alpha, u^{n+\alpha})$ and that of a covariant vector $u_\alpha + iu_{n+\alpha}$ is $(2u_\alpha, -2u_{n+\alpha})$). Hence, the Definitions 1, 2 and 3 are appropriate.

When $I^A{}_B$ takes the value (7), (5) means that $u^A{}_{,B}$ is a matrix which is the real representation of a unitary $(n \times n)$ matrix. Hence we may say that $u^A{}_{,B}$ is pseudo-unitary.

THEOREM 1. *If a set of functions* (ϕ, ψ) *is pseudo-analytic, then* ϕ *and* ψ *are both harmonic functions on our pseudo-Kählerian manifold.*

Proof. By hypothesis

$$\psi_{,B} = I^A{}_B \phi_{,A} ,$$

hence we get

$$\Delta\psi = \psi_{,BC} g^{BC} = I^{AC} \phi_{,AC} = 0 .$$

As (4) can be written also in the form

$$\phi_{,B} = -I^A{}_B \psi_{,A} ,$$

we get in the same way $\Delta\phi = 0$.

THEOREM 2. *If a contravariant vector field* u^A *and its associated covariant vector* u_A *are both pseudo-analytic, then* u^A *is a parallel vector field.*

Proof. If we use covariant components of u^A, then (5) can be written as

$$-I^A{}_B u_{A,C} = I^A{}_C u_{B,A} .$$

Comparing the last equation with (6), we can immediately see that our assertion is true.

THEOREM 3. *If u^A is a pseudo-analytic contravariant vector field, then $I^A{}_B u^B$ is also pseudo-analytic.*

Proof.
$$(I^A{}_B u^B)_{,C} I^C{}_D = I^A{}_B u^B{}_{,C} I^C{}_D$$
$$= I^A{}_B I^B{}_C u^C{}_{,D} = I^A{}_B (I^B{}_C u^C)_{,D} .$$

We shall remark that equation (6) can be written also as

$$XI^A{}_C = 0 \tag{8}$$

where $X = u^A(\partial/\partial x^A)$ and $XI^A{}_C$ is the Lie derivative of $I^A{}_C$ (cf. [5]). We put further $Y = v^A(\partial/\partial x^A)$ and

$$[uv]^A = u^B(v^A{}_{,B}) - v^B(u^A{}_{,B})$$

and similarly define $[Iu, v]^A$, $[u, Iv]^A$, $[Iu, Iv]^A$. Then we get the following

THEOREM 4. *Let u^A and v^A be two pseudo-analytic contravariant vector fields, then $[uv]^A$, $[Iu, v]^A$, $[u, Iv]^A$ and $[Iu, Iv]^A$ are pseudo-analytic too.*

Proof. It is sufficient to prove the pseudo-analyticity of $[uv]^A$. As

$$(XY - YX)f = [uv]^A \frac{\partial f}{\partial x^A} ,$$

it is sufficient to show that

$$(XY - YX)I^A{}_B = 0.$$

However, this follows immediately from the assumption that u^A and v^A are pseudo-analytic.

3. Curvature tensors.

THEOREM 5. (cf. [3])

$$\text{(9)} \qquad \begin{aligned} &\text{(i)} \qquad R^A{}_{ECD} I^E{}_B = I^A{}_E R^E{}_{BCD} , \\ &\text{(ii)} \qquad R_{ABCD} = I^E{}_A I^F{}_B R_{EFCD} . \end{aligned}$$

Proof. From (1) we get

$$0 = I^A{}_{B,CD} - I^A{}_{B,DC} = R^A{}_{ECD} I^E{}_B - R^E{}_{BCD} I^A{}_E .$$

Equation (9ii) follows immediately from (9i). The curvature tensor $R^A{}_{BCD}$ is pseudo-unitary with respect to the first two indices.

THEOREM 6.

$$(10)\qquad \begin{aligned} &\text{(i)} && R_{BC}=I^E{}_B I^F{}_C R_{EF}\,,\\ &\text{(ii)} && R^A{}_B=-I^A{}_E I^F{}_B R^E{}_F\,,\\ &\text{(iii)} && I^A{}_B R^B{}_C=R^A{}_B I^B{}_C\,,\\ &\text{(iv)} && I^E{}_A R_{EB}=-I^E{}_B R_{EA}\,. \end{aligned}$$

$R^A{}_B$ is pseudo-unitary too.

THEOREM 7.

$$(11)\qquad R_{ABCD}I^{AB}=2I^E{}_C R_{ED}\ (=-2I^E{}_D R_{EC})\,.$$

Let us prove Theorems 6 and 7 at the same time. First

$$R_{ABCD}I^{AB}=-I^A{}_E R^E{}_{ACD}=I^A{}_E(R^E{}_{CDA}+R^E{}_{DAC})=I^E{}_C R^A{}_{EDA}+I^E{}_D R^A{}_{EA}$$

Hence we get

$$(12)\qquad R_{ABCD}I^{AB}=I^E{}_C R_{ED}-I^E{}_D R_{EC}\,.$$

Now, from (9) we see that

$$R_{BC}=I^E{}_A I^F{}_B R_{EFCD}g^{AD}=-I^{ED}I^F{}_B(R_{FCED}+R_{CEFD})\,.$$

By virtue of (12), the first term of the right hand side of equation becomes $R_{BC}-R_{EF}I^E{}_B I^F{}_C$ and the second term can be ea: to be $-R_{BC}$. Hence we get

$$R_{BC}=R_{EF}I^E{}_B I^F{}_C\,.$$

(10 ii, iii, iv) can be immediately seen to be equivalent to (10 i). use (10 iii), then (12) reduces to (11).

4. Pseudo-analytic vector fields.

THEOREM 8. *Let u_A be a pseudo-analytic covariant vector fi pseudo-Kählerian manifold M^{2n}, then it satisfies the relation*

$$(13)\qquad u_{A,BC}g^{BC}-R^B{}_A u_B=0\,.$$

Especially, if M^{2n} is compact, then u_A is a harmonic vector. (cf. |

Proof. By hypothesis

$$I^A{}_E u_{A,B}=I^A{}_B u_{E,A}\,,$$
$$I^A{}_E u_{A,BC}g^{BC}=I^A{}_B u_{E,AC}g^{BC}=I^{AC}u_{E,AC}=-\tfrac{1}{2}R^F{}_{EAC}u_F I^{AC}\,.$$

The last equation can be transformed by (12) into

$$I^A{}_E u_{A,BC}g^{BC}=-I^H{}_F R_{HE}g^{FB}u_B=I^A{}_E R_A{}^B u_B\,.$$

As det $(I^A{}_E)\neq 0$, we see that (13) is true. Especially, if M^{2n} is compact and orientable, then by de Rham's theorem [2], u_A is a harmonic vector.

THEOREM 9. *Let u^A be a pseudo-analytic contravariant vector field over a pseudo-Kählerian manifold M^{2n}, then*

$$u^A{}_{,BC}g^{BC}+R^A{}_B u^B=0 . \tag{14}$$

Especially, if $u^A{}_{,A}=0$ and M^{2n} is compact, then u^A is a Killing vector. (cf. [4], [6]).

The proof is quite similar to that of Theorem 8. Instead of de Rham's theorem we use a theorem due to one of the authors (cf. [4], [6]).

THEOREM 10. *Suppose that u^A and v_A are contravariant and covariant pseudo-analytic vector field over a compact pseudo-Kählerian manifold M^{2n}. Then $u^A v_A$ is a constant over the manifold M^{2n}.*

Proof. It is sufficient to show that $u^A v_A$ is harmonic. We put

$$\phi=u^A v_A .$$

Then we get

$$\Delta\phi=(u^A{}_{,BC}v_A+2u^A{}_{,B}v_{A,C}+u^A v_{A,BC})g^{BC} .$$

Putting (13) and (14) into the right hand side of the last equation we get

$$\Delta\phi=2v_{A,C}u^A{}_{,B}g^{BC} .$$

However, the right hand side can be transformed as follows:

$$\begin{aligned}2v_{A,C}u^A{}_{,B}g^{BC}&=-2(u^A{}_{,E}I^E{}_D I^D{}_B)g^{BC}v_{A,C}=-2(I^A{}_E u^E{}_{,D}I^D{}_B)g^{BC}v_{A,C}=-2u^E{}_{,D}I^{DC}I^F{}_C v_{E,F}\\ &=-2u^E{}_{,D}(-I^F{}_C I^C{}_K g^{KD})v_{E,F}=-2u^E{}_{,D}v_{E,F}g^{FD}=-\Delta\phi .\end{aligned}$$

Hence $\Delta\phi=0$, and ϕ is a harmonic function.

THEOREM 11. *Suppose that M^{2n} is a compact pseudo-Kählerian manifold. If the Ricci tensor R_{AB} is positive definite, then there exists no pseudo-analytic covariant tensors other than the zero vector. (If R_{AB} is positive semi-definite, then the covariant derivative of any pseudo-analytic covariant vector field vanishes).* (cf. [1], [6]).

Proof. We put

$$\phi=g^{AB}u_A u_B ,$$

then we get

$$\Delta\phi=(g^{AB}u_Au_B)_{,CD}g^{CD}=2g^{AB}(u_{A,C}u_{B,D}+u_{A,CD}u_B)g^{CD} .$$

If we substitute (13) in the second term of the last equation we get

$$\Delta\phi=2g^{AB}g^{CD}u_{A,C}u_{B,D}+2R^{AB}u_Au_B .$$

Hence, by virtue of Bochner's lemma (cf. [1], [6]), we can see immediately that our assertion is true.

THEOREM 12. *Suppose that M^{2n} is a compact pseudo-Kählerian manifold. If the Ricci tensor R_{AB} is negative definite, then there exists no pseudo-analytic contravariant vector field other than the zero vector. (If R_{AB} is negative semi-definite, then the covariant derivative of any pseudo-analytic contravariant vector field vanishes).* (cf [1], [6]).

The proof is quite similar to that of Theorem 11.

THEOREM 13. *Suppose that M^{2n} is a compact pseudo-Kählerian manifold and u_B is a covariant vector field over M^{2n} such that u_B is expressible in a neighborhood of each point of M^{2n} as $\phi_{,B}+I^A{}_B\psi_{,A}$ where ϕ and ψ are harmonic functions in such neighborhood with respect to the pseudo-Kählerian metric. If the Ricci tensor R_{AB} is positive definite, then $u_B\equiv 0$, that is, the set of functions (ϕ, ψ) is pseudo-analytic. (If R_{AB} is positive semi-definite, then the covariant derivative of u_B vanishes).* (cf. [1], [6]).

Proof. We put

$$T=g^{AB}u_Au_B , \qquad u_A=\phi_{,A}+I^E{}_A\psi_{,E} ,$$

then we get

$$\Delta T=(2g^{AB}u_{A,C}u_{B,D}+2g^{AB}u_{A,CD}u_B)g^{CD} .$$

Now, the second term of the right hand side of the last equation is transformed in the following way:

$$\begin{aligned} II&=2g^{AB}(\phi_{,CAD}+I^E{}_A\psi_{,CED})(\phi_{,B}+I^F{}_B\psi_{,F})g^{CD} \\ &=2g^{AB}(-R^H{}_{CAD}\phi_{,H}-I^E{}_AR^H{}_{CED}\psi_{,H})(\phi_{,B}+I^F{}_B\psi_{,F})g^{CD} \\ &=2g^{AB}(R^H{}_A\phi_{,H}+I^E{}_AR^H{}_E\psi_{,H})(\phi_{,B}+I^F{}_B\psi_{,F})g^{CD} \\ &=2R^{HB}u_Hu_B . \end{aligned}$$

In the process of the transformation we used (10 ii) and the fact that ϕ and ψ are harmonic functions. Hence

$$\Delta T=2g^{AB}g^{CD}u_{A,C}u_{B,D}+2R^{AB}u_Au_B ,$$

so $\Delta T\geq 0$. Accordingly, by virtue of Bochner's lemma (cf. [1], [6]), we see that the theorem is true.

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STITUTE FOR ADVANCED STUDY, PRINCETON AND
TITUTO NAZIONALE DI ALTA MATEMATICA, ROME

ON THE TOWER THEOREM FOR FINITE GROUPS

EUGENE SCHENKMAN

Wielandt [2] has given a very ingenious proof of the fact that the tower of automorphisms of a finite group without center ends after a finite number of steps. Using his work as a model a proof of a similar tower theorem for Lie algebras was given in [1]. This depends on the following three facts:

(a) If A (with no center) is a member of the tower of derivation algebras of a Lie algebra L then the centralizer of L in A is (0).

(b) If L is a subinvariant Lie algebra of A and if the centralizer of L in A is (0) then the centralizer of L^{ω} in A is contained in A.

(c) If L is subinvariant in A then L^{ω} is normal in A.

In view of the much sharper estimate obtained in the theorem on Lie algebras it seemed to be of interest to attempt to improve on the results of Wielandt using the method of [1]. The group theory analogue of (a) is to be found in Wielandt's work. I shall prove here the analogue to (b) and then show by a counter-example that the method is not applicable to get the tower theorem even for solvable groups since the analogue to (c) does not hold for groups even under the additional hypothesis of (b).

THEOREM. *If G is a subinvariant subgroup of the finite group A and if the centralizer of G in A is the identity, then the centralizer of G^{ω} in A is contained in G^{ω}. It follows that if N is normal in G such that G/N is nilpotent then $N \supset G^{\omega}$ and the centralizer of N is contained in N.*

Here

$$G^{\omega} = \bigcap_{k=1}^{\infty} G^k$$

where $G^k = [G^{k-1}, G]$ is the subgroup generated by commutators of the form $[h, g] = hgh^{-1}g^{-1}$, $h \in G^{k-1}$, $g \in G$.

The proof of the Theorem depends on two lemmas.

LEMMA 1. *If G is a finite group then $G = G^{\omega}H$ where H is a nilpotent subgroup of G.*

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LEMMA 2. *Let G be a group with center E; then the centralizer W of G^ω in G is contained in G^ω. It follows that if N is normal in G such that G/N is nilpotent then $N \supset G^\omega$ and the centralizer of N is contained in N.*

I shall also give an example to show that under the hypotheses of the Theorem G^ω need not be normal in A, even with the added restriction that G be solvable.

Proof of Lemma 1. The proof is based on the fact that a group is nilpotent if and only if its Φ-subgroup contains the commutator subgroup G^2 [3, p. 114]. If G is nilpotent the theorem is trivially true since $G^\omega = E$ and $G = GE$. If G is not nilpotent then the Φ-subgroup does not contain G^2. Accordingly we can pick a minimal set of generators $g_1, \cdots, g_k$ of G where at least one of the generators, g_k for definiteness, is in G^2. Then $g_1, \cdots, g_{k-1}$ generate a proper subgroup K of G; and $G = KG^2$. On the other hand G/G^ω is nilpotent and hence the Φ-subgroup of G/G^ω contains the commutator subgroup of G/G^ω. Accordingly g_kG^ω is not essential as a generator of G/G^ω and therefore $g_1G^\omega, \cdots, g_{k-1}G^\omega$ generate G/G^ω. It follows that $G = G^\omega K$.

Now we proceed by induction on the order of the group. Since K is a proper subgroup of G, its order is less than that of G and we can assume that $K = K^\omega H$ where H is a nilpotent group. Then

$$G = G^\omega K = G^\omega K^\omega H = G^\omega H$$

since K^ω is contained in G^ω and the lemma is proved.

Proof of Lemma 2. G^ω is normal in G and hence so also is W. By Lemma 1, $G = G^\omega H$ where H is a nilpotent group. Let $G_1 = WH$. Then G_1 is a group since W is normal and H is a group. Also $G_1 = G_1^\omega H_1$ where H_1 is nilpotent. But $G_1^\omega = (WH)^\omega$ is contained in W since H is nilpotent and W is normal. This can be seen by showing inductively that $(WH)^k \subseteq WH^k$. For let x and y be elements of W, h be in H, and k be in H^k. Then if xh is in WH and if yk is in WH^k

$$[xh, yk] = \underline{x}\,\underline{hyh^{-1}}\,\underline{hkh^{-1}x^{-1}hk^{-1}h^{-1}}\,\underline{hkh^{-1}k^{-1}y^{-1}[h, k]^{-1}}[h, k]$$

which is an element of WH^{k+1}, since the four underlined expressions are in W and $[h, k] \in H^{k+1}$.

Now if W is contained in G_1^ω then $W \subseteq G^\omega$ since $G_1^\omega \subseteq G^\omega$. Hence the lemma is false only if $W \not\subseteq G_1^\omega$. We need only consider therefore if there is an element w in W, w not in G_1^ω. We shall write $w = gh$ where g is in G_1^ω and h in H_1. Of course $h \neq e$ since then w would be in G_1^ω. It follows that $h = g^{-1}w$ is in W, since $W \supseteq G_1^\omega$ and therefore $H_1 \cap W \neq E$. But $H_1 \cap W$ is normal in H_1 since W is normal in G. Thus $H_1 \cap W$

has intersection $P \neq E$ with the center of H_1. But this will imply that P is in the center of G. For $G = G^{\omega}H = G^{\omega}H_1$ since $H \subseteq G_1^{\omega}H_1$; and P is in the centralizer of G^{ω} and in the center of H_1. We have shown that if the centralizer of G^{ω} is not contained in G^{ω} then G has center not equal to E contradicting the hypothesis of the lemma.

Proof of the Theorem. By Lemma 2 we know that if Z is the centralizer of G^{ω} in A then $Z \cap G \subseteq G^{\omega}$ since otherwise G would have a non-trivial center. Now if Z is not contained in G^{ω} let K be the group generated by G and Z. G^{ω} is normal in K since

$$[G^{\omega}, G] \subseteq G^{\omega} \quad \text{and} \quad [G^{\omega}, Z] = E \subseteq G^{\omega}.$$

It follows that Z is normal in K and hence $K = ZG$. But G is subinvariant in A, and hence in K. That is, G is a proper normal subgroup of G_1, G_1 contained in K. Pick g_1 in G_1 but not in G. Since $K = ZG$, $g_1 = gz$ where g is in G, and z in Z. Furthermore $z = g^{-1}g_1$ is in G_1 and not in G. Now G and z generate a group $L = G(z)$ since G is normal in G_1. Also $L^{\omega} = G^{\omega}$; for

$$L/G^{\omega} = G/G^{\omega} \times (z)G^{\omega}/G^{\omega}$$

and hence is nilpotent.

Now since $z \notin G^{\omega} = L^{\omega}$ it follows by Lemma 2 that L has a non-trivial center; but this is a contradiction of the fact that G has centralizer E in A. This completes the proof of the Theorem.

The counter-example mentioned earlier is as follows. Let H be the non-Abelian group of order 27 all of whose elements are of order 3; and let a and b be generators of H. Let σ and τ be automorphisms of H defined by $a^{\sigma} = a^2$, $b^{\sigma} = b$; and $a^{\tau} = a$, $b^{\tau} = b^2$. Let B be the holomorph of H with σ and τ and let G be the subgroup of B containing σ, a, and $[a, b]$. Then G is invariant in the subgroup containing G and b which subgroup in turn is invariant in B; and it is easy to check that G^{ω} is the group generated by a and $[a, b]$.

Now let ρ be the automorphism of order 2 of B defined by $b^{\rho} = a$, $a^{\rho} = b$, $\sigma^{\rho} = \tau$, $\tau^{\rho} = \sigma$, and let A be the holomorph of B and ρ. Then G is subinvariant in A; the centralizer of G in A is the identity, but clearly G^{ω} is not normal in A since $a^{\rho} = b$.

Omitting the hypothesis of solvability Professor Zassenhaus kindly furnished me with a similar example; in fact, an example of a group $G = G^2$ with trivial center such that the group of automorphisms of G is not complete.

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LOUISIANA STATE UNIVERSITY AND
INSTITUTE FOR ADVANCED STUDY

ON THE NUMERICAL SOLUTION OF POISSON'S EQUATION OVER A RECTANGLE

P. STEIN AND J. E. L. PECK

Introduction. We consider the equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=f(x, y)$$

over the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, with given boundary values for z. Following the usual procedure (see for example Hyman [1]) we approximate the solution by solving a set of mn simultaneous equations, arising from the corresponding difference equation. If we write

$$a=(n+1)\Delta x, \quad b=(m+1)\Delta y, \quad \rho=\Delta y/\Delta x, \quad a_{i,j}=-f(j\Delta x, i\Delta y)\Delta y^2$$

and $z_{i,j}=z(j\Delta x, i\Delta y)$, the mn equations are of the form

$$(1) \qquad 2(1+\rho^2)z_{i,j}=\rho^2(z_{i,j+1}+z_{i,j-1})+z_{i+1,j}+z_{i-1,j}+a_{i,j}\,, \qquad i=1,\cdots,m, \quad j=1,\cdots,n\,.$$

A solution of this set of equations is given by Hyman [1]. In the case where the boundary values are zero, the solution takes the form $Z=C\omega D$ [1, p. 340] where C and D are matrices which depend on n and m and may be written down without any calculations, and ω is a matrix depending on m, n, ρ and the values of $f(x, y)$ at the lattice points. The matrix ω requires somewhat elaborate calculations. To obtain the solution with given boundary values, he adds to the matrix $C\omega D$ the value of u as a matrix obtained from the solution of the equation $\Delta^2 u/\Delta x^2+\Delta^2 u/\Delta y^2=0$ with the given boundary values. He obtains for u the matrix value $U=C\phi$ [1, p. 329], where C is the matrix mentioned above and ϕ is a matrix depending on n, m, ρ and the boundary values and requires to be recalculated for every set of boundary values.

In this paper the solutions of equations (1) are obtained, column by column, in the form $Z_j=\sum_k M_{j,k}B_k$, where the $M_{j,k}$ are matrices depending on m, n, and ρ and which require somewhat elaborate calculations, and the B_k are vectors depending on m, n, ρ, the values of $f(x, y)$ at the lattice points and the boundary values and can be written down without calculation. We may regard this solution as giving an explicit formula for the values of z at the lattice points.

The principal work in the calculation of Z_j is the calculation of the matrices $M_{j,k}$. It will be shown that it is sufficient to calculate a

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selection of columns of Z, as the method lends itself to a stepping off process; also that all the matrices used can be written down easily from a knowledge of their top rows. The calculation is simplest when $\rho=1$. Further, the case when $\rho=1$ or is nearly 1 is the most accurate [**1**, p. 332]. It will be shown that when $|\rho^2-1|<1$, Z may be obtained by successive approximations with the help of the matrices calculated for $\rho=1$. It appears to the authors that if a not very elaborate set of tables were to be prepared for selected values of j, m and n with $\rho=1$, the calculation of Z would be much simplified. Further, if such a set of tables were available, it might be of assistance in the iterative method of the solution of these simultaneous equations when the boundary is not a rectangle.

In §1 we develop the method of solution. In §§2 and 3 we give methods by which the required matrices may be evaluated. Section 4 deals with the iterative process when ρ is nearly 1, and this is amplified in §§5 and 6.

1. We write the mn equations (1) in n sets each consisting of m equations. A typical set is

$$\begin{aligned} 2(1+\rho^2)z_{1,j}-z_{2,j} &= \rho^2(z_{1,j+1}+z_{1,j-1})+a_{1,j}+z_{0,j} \\ -z_{1,j}+2(1+\rho^2)z_{2,j}-z_{3,j} &= \rho^2(z_{2,j+1}+z_{2,j-1})+a_{2,j} \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ -z_{m-1,j}+2(1+\rho^2)z_{m,j} &= \rho^2(z_{m,j+1}+z_{m,j-1})+a_{m,j}+z_{m+1,j}\,. \end{aligned} \tag{2}$$

We write Z_j for the vector $(z_{1,j}, z_{2,j}, \cdots, z_{m,j})$, A_j for the vector $(a_{1,j}, a_{2,j}, \cdots, a_{m,j})$, Z_j' for the vector $(z_{0,j}, 0, 0, \cdots, 0, z_{m+1,j})$ and $M_m(\alpha)$ for the $m\times m$ matrix

$$\begin{pmatrix} \alpha, & -1, & 0, & \cdot \\ -1, & \alpha, & -1, & \cdot \\ 0, & -1, & \alpha, & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{3}$$

The equations (2) then take the form

$$M_m(2+2\rho^2)Z_j=\rho^2(Z_{j+1}+Z_{j-1})+A_j+Z_j', \qquad j=1,\cdots,n,$$

or

$$\begin{aligned} \rho^{-2}M_m(2+2\rho^2)Z_1-Z_2 &= \rho^{-2}(A_1+Z_1')+Z_0 &&= B_1 \\ -Z_1+\rho^{-2}M_m(2+2\rho^2)Z_2-Z_3 &= \rho^{-2}(A_2+Z_2') &&= B_2 \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ -Z_{n-1}+\rho^{-2}M_m(2+2\rho^2)Z_n &= \rho^{-2}(A_n+Z_n')+Z_{n+1} &&= B_n\,. \end{aligned} \tag{4}$$

These equations can be solved by iteration. See for example Todd [3].

The class of all ordered sets of m real numbers is a vector space over the ring of polynomials in the matrix $M_m(2+2\rho^2)$. Interpreting equations (4) in this way, we may obtain their solution from Cramer's rule in the form

$$\mathscr{D} Z_j = \sum_{k=1}^{n} \mathscr{M}_{j,k} B_k \tag{5}$$

where $\mathscr{D}$ is the determinant of the matrix of matrix coefficients on the left of (4) and the $\mathscr{M}_{j,k}$ are cofactors of $\mathscr{D}$. One may readily prove that $\mathscr{M}_{j,k} = \mathscr{M}_{k,j}$ and that when $j \leq k$

$$\mathscr{M}_{j,k} = D_{j-1}(\rho^{-2} M_m(2+2\rho^2)) D_{n-k}(\rho^{-2} M_m(2+2\rho^2)) \tag{6}$$

where D_n is the polynomial defined by the nth order determinant

$$D_n(x) = \begin{vmatrix} x, & -1, & 0, & \cdot \\ -1, & x, & -1, & \cdot \\ 0, & -1, & x, & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

and $D_0(x) = 1$.

One may regard (5) as expressing z in terms of the given values for $f(x, y)$ and the boundary values. In particular when $j=1$, we have from (5) and (6)

$$Z_1 = D_n^{-1}(\rho^{-2} M_m(2+2\rho^2)) \sum_{k=1}^{n} D_{n-k}(\rho^{-2} M_m(2+2\rho^2)) B_k . \tag{7}$$

As was pointed out by Hyman [1, p. 331] it is unnecessary to calculate the remaining values of z by the use of (5). It is sufficient to use (7). Knowing Z_0 and Z_1 we may "step off" using (4) to determine Z_2 and then use it again to get Z_3 from Z_2 and Z_1.

2. In this section we obtain some properties of the polynomial D_n and of the matrices $D_n(\beta M_m(\alpha))$.

THEOREM 1.

$$D_n(x) = x D_{n-1}(x) - D_{n-2}(x) \tag{8}$$

$$D_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} x^{n-2r} \tag{9}$$

$$D_n(x) = \frac{a^{n+1} - b^{n+1}}{2^n (a-b)}, \qquad a = x + \sqrt{x^2-4}, \qquad b = x - \sqrt{x^2-4} \tag{10}$$

(11) $$D_n(x)=2^{-n}\sum_{r=0}^{[n/2]}\binom{n+1}{2r+1}x^{n-2r}y^{2r}\,, \qquad y=\sqrt{x^2-4}$$

(12) $$D_n(x)=\frac{\sinh\,(n+1)\phi}{\sinh\phi}\,, \qquad x=2\cosh\phi$$

(13) $$D_n(x)=\frac{\sin\,(n+1)\,\theta}{\sin\theta}\,, \qquad x=2\cos\theta$$

(14) $$D_n(x)=\prod_{r=1}^{n}\left(x-2\cos\frac{r\pi}{n+1}\right).$$

Formulae (8), (13) and (14) are known ([2] and [4]). Formula (8) follows immediately from the definition and (9) may be proved by induction using (8). Formula (10) also follows from (8) by induction. Formula (11) comes from (10) on writing $a=x+y$, $b=x-y$. The equation $x=2\cosh\phi$ means that $a=2e^{\phi}$, $b=2e^{-\phi}$ whence (10) gives (12). Formula (13) is proved similarly. By (13) the roots of the equation $D_n(x)=0$ are $2\cos(r\pi/(n+1))$, $(r=1,\cdots,n)$ giving (14).

COROLLARY. *If M is a square matrix and I is the corresponding identity matrix:—*

(15) $$D_n(M)=MD_{n-1}(M)-D_{n-2}(M)$$

(16) $$D_n(M)=\sum_{r=0}^{[n/2]}(-1)^r\binom{n-r}{r}M^{n-2r}$$

(17) $$D_n(M)=2^{-n}\sum_{r=0}^{[n/2]}\binom{n+1}{2r+1}M^{n-2r}(M^2-4I)^r$$

(18) $$D_n(M)=\prod_{r=1}^{n}\left(M-2\cos\frac{r\pi}{n+1}I\right).$$

THEOREM 2. *If P is any polynomial, then $P(M_m(\alpha))$ is an $m\times m$ matrix which is symmetric about both diagonals.*

If two matrices which commute are symmetric about both diagonals, then so is their sum, product and any scalar multiple. This theorem therefore proves that the matrices $D_n(\beta M_m(\alpha))$ are symmetric about both diagonals.

THEOREM 3. *Let P be any polynomial and let $a_{i,j}$ be the elements of $P(M_m(\alpha))$. Then if we interpret $a_{i,j}=0$ whenever i or j is <1 or $>m$, we have, for $1\leq i\leq m$, $1\leq j\leq m$ and $i+j\leq m+2$,*

(19) $$a_{i,j}=a_{i-1,\,j-1}+a_{1,i+j-1}\,,$$

and for $1 \leq i \leq j \leq m$ *and* $i+j \leq m+1$,

$$a_{i,j} = a_{1,-i+j+1} + a_{1,-i+j+3} + \cdots + a_{1,i+j-1} . \tag{20}$$

Theorems 2 and 3 enable us to write down all the elements of $P(M_m(\alpha))$ from a knowledge of the elements in the first row.

Proof of Theorem 3. We observe that (19) is invariant under addition and scalar multiplication. We observe also that in the case $i=1$, (19) reduces to a triviality. If $j=1$ it becomes $a_{i,1}=a_{1,i}$ which is true by symmetry about the main diagonal, and if $i+j=m+2$, it becomes $a_{i,j}=a_{i-1,j-1}$, which is true because of symmetry about the other diagonal (Theorem 2). Formula (19) will therefore be established if we can show that it is true for $M_m^r(\alpha)$, where r is a nonnegative integer, and when $2 \leq i \leq m$, $2 \leq j \leq m$ and $i+j \leq m+1$.

By inspection it is true when $r=0$, 1. Let $a_{i,j}^r$ denote the i, jth element of $M_m^r(\alpha)$. Then $a_{i,j}^r = -a_{i,j-1}^{r-1} + \alpha a_{i,j}^{r-1} - a_{i,j+1}^{r-1}$ for $1 \leq i \leq m$, $1 \leq j \leq m$. If we assume that it is true for $r-1$ we have for $2 \leq i \leq m$, $2 \leq j \leq m$, $i+j \leq m+1$, that

$$\begin{aligned} a_{i,j}^r &= -a_{i-1,j-2}^{r-1} + \alpha a_{i-1,j-1}^{r-1} - a_{i-1,j}^{r-1} \\ &\quad - a_{1,i+j-2}^{r-1} + \alpha a_{1,i+j-1}^{r-1} - a_{1,i+j}^{r-1} = a_{i-1,j-1}^r + a_{1,i+j-1}^r \end{aligned}$$

which completes the proof of (19).

Formula (20) follows from a repeated use of (19).

THEOREM 4. *If we denote the* i, j*th element of* $D_n(\beta M_m(\alpha))$ *by* $a_{i,j}^{m,n}$ *then*

$$a_{1,j}^{k,n} = a_{1,j}^{m,n}, \qquad 1 \leq j \leq k, \qquad n \leq k \leq m . \tag{21}$$

From (15) we have

$$a_{1,j}^{m,n} = -\beta a_{1,j-1}^{m,n-1} + \alpha\beta a_{1,j}^{m,n-1} - \beta a_{1,j+1}^{m,n-1} - a_{1,j}^{m,n-2} \tag{22}$$

where $a_{1,0}^{m,n} = a_{1,m+1}^{m,n-1} = 0$ and $1 \leq j \leq m$. From (22) we have by induction on n that $a_{1,j}^{m,n}=0$ if $j \geq n+2$, which means that $a_{1,k+1}^{m,n-1}=0$. Since we must write $a_{1,0}^{m,n-1} = a_{1,0}^{k,n-1} = a_{1,k+1}^{k,n-1} = 0$, this allows the following induction on n

$$\begin{aligned} a_{1,j}^{m,n} &= -\beta a_{1,j-1}^{m,n-1} + \alpha\beta a_{1,j}^{m,n-1} - \beta a_{1,j+1}^{m,n-1} - a_{1,j}^{m,n-2} \\ &= -\beta a_{1,j-1}^{k,n-1} + \alpha\beta a_{1,j}^{k,n-1} - \beta a_{1,j+1}^{k,n-1} - a_{1,j}^{k,n-2} = a_{1,j}^{k,n} . \end{aligned}$$

However the theorem is true for $n=1$ and $n=2$.

Formula (21) shows that the top row of the matrix $D_n(\beta M_m(\alpha))$ is essentially the same for all useful values of n, while (22) gives a recursive method of computing this top row. Theorem 3 and the remark after Theorem 2 show how the remainder of the matrix can be filled in from the top row. Thus the computation of the matrices $D_n(\beta M_m(\alpha))$

for $n \leq m$ is simplified, and the matrices lend themselves to easy tabulation.

3. **In this section** we give some results which are useful in the calculation of $D_n^{-1}(\beta M_m(\alpha)) = [D_n(\beta M_m(\alpha))]^{-1}$.

Since the inverse of any matrix is a polynomial in that matrix, we have $D_n^{-1}(\beta M_m(\alpha))$ a polynomial in $D_n(\beta M_m(\alpha))$ and therefore a polynomial in $M_m(\alpha)$. Theorems 2 and 3 therefore apply. It is thus sufficient to compute only its first row. From the first row we may obtain its elements $a_{i,j}$ for $1 \leq i \leq j \leq m$ and $i+j \leq m+1$ by (19) or (20) and then the other elements can be filled in by symmetry.

THEOREM 5. *If the element in the ith row and jth column of* $M_m^{-1}(\alpha)$ *is* $a_{i,j}$ *and if* $\alpha = 2\cosh\phi$, *then*

$$a_{i,j} = \frac{\sinh i\phi \sinh(m+1-j)\phi}{\sinh\phi \sinh(m+1)\phi}, \qquad i \leq j. \tag{23}$$

For proof we have first that

$$a_{i,j} = \frac{D_{i-1}(\alpha) D_{m-j}(\alpha)}{D_m(\alpha)}, \qquad i \leq j. \tag{24}$$

The result then follows from (12).

THEOREM 6. *If*

$$\alpha_r = \alpha - \frac{2}{\beta}\cos\frac{r\pi}{n+1},$$

then

$$D_n^{-1}(\beta M_m(\alpha)) = \beta^{-n} \prod_{r=1}^{n} M_m^{-1}(\alpha_r). \tag{25}$$

From (18) we have

$$\begin{aligned} D_n(\beta M_m(\alpha)) &= \prod_{r=1}^{n} \left(\beta M_m(\alpha) - 2\cos\frac{r\pi}{n+1} I \right) \\ &= \beta^n \prod_{r=1}^{n} M_m\left(\alpha - \frac{2}{\beta}\cos\frac{r\pi}{n+1} \right) \\ &= \beta^n \prod_{r=1}^{n} M_m(\alpha_r). \end{aligned} \tag{26}$$

The result follows immediately.

A result which may be easier from the computational point of view is to express $D_n^{-1}(\beta M_m(\alpha))$ as a sum of matrices. This is done in the following theorem.

THEOREM 7. *If*

$$\alpha_r=\alpha-\frac{2}{\beta}\cos\frac{r\pi}{n+1},$$

then

(27) $$D_n^{-1}(\beta M_m(\alpha))=\frac{2}{\beta(n+1)}\sum_{r=1}^{n}(-1)^{r+1}\sin^2\frac{r\pi}{n+1}M_m^{-1}(\alpha_r).$$

From (26) we have that

$$D_n(\beta M_m(\alpha))=\prod_{r=1}^{n}\beta M_m(\alpha_r).$$

Therefore

$$D_n^{-1}(\beta M_m(\alpha))=\sum_{r=1}^{n}c_r\{\beta M_m(\alpha_r)\}^{-1}$$

where the c_r's are suitably chosen scalars.

If $f(x)=\Pi_{r=1}^{n}(x-\gamma_r)$, $\gamma_r\neq\gamma_s$ when $r\neq s$, then

$$f(x)^{-1}=\sum_{r=1}^{n}f'(\gamma_r)^{-1}(x-\gamma_r)^{-1}.$$

To obtain the values of the scalars c_r, we put $f=D_n$. From (13) we have

$$D_n'(2\cos\theta)=\frac{(n+1)\cos(n+1)\theta\sin\theta-\sin(n+1)\theta\cos\theta}{\sin^2\theta}\left(\frac{-1}{2\sin\theta}\right).$$

This gives

$$D_n'\left(2\cos\frac{r\pi}{n+1}\right)=\frac{(-1)^{r+1}(n+1)}{2\sin^2\frac{r\pi}{n+1}},$$

and therefore

$$c_r=(-1)^{r+1}2(n+1)^{-1}\sin^2\frac{r\pi}{n+1}.$$

With the help of (23) we can obtain a more explicit result.

COROLLARY. *If* $a_{i,j}$ *is the* i, j*th element of* $D_n^{-1}(\beta M_m(\alpha))$ *and*

$$2\cosh\phi_r=\alpha-\frac{2}{\beta}\cos\frac{r\pi}{n+1},$$

then $a_{i,j}=a_{j,i}$ *for all* i *and* j, *and for* $i\leq j$

$$(28) \quad a_{i,j}=[\beta(n+1)]^{-1}\sum_{r=1}^{n}(-1)^{r+1}\left(1-\cos\frac{2r\pi}{n+1}\right)\frac{\sinh i\phi_r \sinh(m+1-j)\phi_r}{\sinh\phi_r \sinh(m+1)\phi_r} .$$

In the case $i=1$, this reduces to

$$(29) \qquad a_{1,j}=[\beta(n+1)]^{-1}\sum_{r=1}^{n}(-1)^{r+1}\left(1-\cos\frac{2r\pi}{n+1}\right)\frac{\sinh(m+1-j)\phi_r}{\sinh(m+1)\phi_r} .$$

4. **In the formulae of** the two previous sections, if $\rho=\Delta y/\Delta x=1$ we have $\alpha=4$, $\beta=1$ and there is some simplification in the resulting calculations. It is pointed out by Hyman [1, p. 322] that the case $\rho=1$ is the one which gives the most accurate results. Hence it is suggested that in arranging the lattice points of a rectangle, one should attempt to have ρ approximately one. We now give a method of finding a correction, when ρ is approximately one, to the solution obtained by assuming $\rho=1$. It is found that in this way we can make use of tables prepared for the case $\rho=1$.

We write $\rho^2=1+\delta$. The equations (1) then become

$$(30) \qquad (4+2\delta)z_{i,j}=(1+\delta)(z_{i,j+1}+z_{i,j-1})+(z_{i+1,j}+z_{i-1,j})+a_{i,j} .$$

Let

$$\Delta x_{i,j}=2x_{i,j}-x_{i,j+1}-x_{i,j-1}$$
$$\Box x_{i,j}=4x_{i,j}-x_{i+1,j}-x_{i-1,j}-x_{i,j+1}-x_{i,j-1} .$$

Then (30) may be written

$$(31) \qquad \Box z_{i,j}=-\delta\Delta z_{i,j}+a_{i,j} .$$

We suppress the first term on the right of (31) and find $u_{i,j}^{(1)}$ so that

$$(32) \qquad \Box u_{i,j}^{(1)}=a_{i,j}$$

and $u_{i,j}^{(1)}=z_{i,j}$ on the boundaries. Let Z denote the values of $z_{i,j}$ at the lattice points, with similar notation for $U^{(r)}$ and $V^{(r)}$ with $r\geq 1$. Let $Z=U^{(1)}+V^{(1)}$, then $U^{(1)}$ is an approximation to the values of Z with error $V^{(1)}$ for which an equation is obtained by subtracting (32) from (31). Thus

$$\Box v_{i,j}^{(1)}=-\delta\Delta z_{i,j}=-\delta\Delta v_{i,j}^{(1)}-\delta\Delta u_{i,j}^{(1)}$$

and $V^{(1)}$ is zero on the boundary.

We now find $U^{(2)}$ such that

$$\Box u_{i,j}^{(2)}=-\delta\Delta u_{i,j}^{(1)}$$

and $U^{(2)}$ is zero on the boundary. Writing $V^{(1)}=U^{(2)}+V^{(2)}$ we obtain, by subtraction,

$$\square v^{(2)}_{i,j} = -\delta \Delta v^{(1)}_{i,j} .$$

Proceeding in this manner we obtain for $r \geq 1$

$$V^{(r)} = V^{(r+1)} + U^{(r+1)}$$

$$\square u^{(r+1)}_{i,j} = -\delta \Delta u^{(r)}_{i,j} \tag{33}$$

$$\square v^{(r+1)}_{i,j} = -\delta \Delta v^{(r)}_{i,j} \tag{34}$$

where $V^{(r)}$ and $U^{(r)}$, $r \geq 2$ are zero on the boundary. A formal solution of equations (30) is thus

$$Z = \sum_{r=1}^{\infty} U^{(r)} . \tag{35}$$

We observe that equations (32) and (33) to determine $U^{(r)}$ are the equations (1) where $\rho = 1$ and where different sets of values are successively used in place of the $a_{i,j}$. The formal solution (35) will be the solution provided $V^{(r)}$ tends to 0 as r tends to ∞. This will certainly be the case if, given any arbitrary $X^{(1)}$ we can show that the iteration

$$\square x^{(r+1)}_{i,j} = -\delta \Delta x^{(r)}_{i,j}$$

leads to the result $X^{(r)} \to 0$ as $r \to \infty$. In the next two sections we obtain the condition on δ that this should be the case, and we obtain an estimate of the error if we take $Z = \sum_{r=1}^{s} U^{(r)}$.

5. **We proceed to** the solution of (33) (and (34)) when $r \geq 2$. The equation may be written

$$-u^{(r+1)}_{i,j+1} + 4u^{(r+1)}_{i,j} - u^{(r+1)}_{i,j-1} = u^{(r+1)}_{i+1,j} + u^{(r+1)}_{i-1,j} - \delta(-u^{(r)}_{i,j+1} + 2u^{(r)}_{i,j} - u^{(r)}_{i,j-1}) .$$

If $U^{(r)}_i$ is the vector $(u^{(r)}_{i,1}, u^{(r)}_{i,2}, \cdots, u^{(r)}_{i,n})$ then since all boundary values are zero, these n equations can be written

$$M_n(4) U^{(r+1)}_i = U^{(r+1)}_{i+1} + U^{(r+1)}_{i-1} - \delta M_n(2) U^{(r)}_i . \tag{36}$$

If now $\mathcal{M}$ is the $m \times m$ matrix of matrices defined by

$$\mathcal{M} = \begin{pmatrix} M_n(4), & -I, & 0, & \cdot \\ -I, & M_n(4), & -I, & \cdot \\ 0, & -I, & M_n(4), & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$\mathcal{M}^*$ is the $m \times m$ matrix of matrices defined by

$$\mathcal{M}^* = \begin{pmatrix} M_n(2), & 0, & 0, & \cdot \\ 0, & M_n(2), & 0, & \cdot \\ 0, & 0, & M_n(2), & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and $U^{(r)}$ is the vector $(U_1^{(r)},\cdots, U_m^{(r)})$, then since all boundary values are zero the m equations (36) can be written

$$\mathcal{M}U^{(r+1)}=-\delta\mathcal{M}^*U^{(r)}$$

and so

$$U^{(r+1)}=-\delta\mathcal{M}^{-1}\mathcal{M}^*U^{(r)}. \tag{37}$$

In the case $r=1$, we must take into account some boundary values. Thus let $Z'=(Z_1', Z_2',\cdots, Z_m')$ where Z_i' $(i=1,\cdots, m)$ is the vector $(z_{i,0}, 0,\cdots, 0, z_{i,n+1})$. Then the solution of (33) for $r=1$ is

$$U^{(2)}=-\delta\mathcal{M}^{-1}(\mathcal{M}^*U^{(1)}-Z'). \tag{38}$$

Returning to (37) we have

$$U^{(r)}=(-\delta\mathcal{M}^{-1}\mathcal{M}^*)^{r-2}U^{(2)}.$$

Hence $U^{(r)}$ and $V^{(r)}$ tend to zero as r tends to ∞ provided that a circle of radius $|\delta|^{-1}$ and center the origin contains the spectrum of $\mathcal{M}^{-1}\mathcal{M}^*$.

The spectrum of $\mathcal{M}^{-1}\mathcal{M}^*$ is found most easily by considering the matrix $\mathcal{M}^{*-1}\mathcal{M}$. Writing $M=M_n(2)$, we have

$$\begin{aligned}\mathcal{M}^{*-1}\mathcal{M}\\ &=\begin{pmatrix} M^{-1}, & 0, & 0, & \cdot\\ 0, & M^{-1}, & 0, & \cdot\\ 0, & 0, & M^{-1}, & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}\cdot\begin{pmatrix} M+2I, & -I, & 0, & \cdot\\ -I, & M+2I, & -I, & \cdot\\ 0, & -I, & M+2I, & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}\\ &=\begin{pmatrix} I+2M^{-1}, & -M^{-1}, & 0, & \cdot\\ -M^{-1}, & I+2M^{-1}, & -M^{-1}, & \cdot\\ 0, & -M^{-1}, & I+2M^{-1}, & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}.\end{aligned}$$

If $\{\mu_r,\ r=1,\cdots, n\}$ is the spectrum of M, we may use a theorem of Williamson [5, Theorem 1] to find that the spectrum of $\mathcal{M}^{*-1}\mathcal{M}$ consists of the spectra of the n $m\times m$ matrices

$$\begin{pmatrix} 1+2\mu_r^{-1}, & -\mu_r^{-1}, & 0, & \cdot\\ -\mu_r^{-1}, & 1+2\mu_r^{-1}, & -\mu_r^{-1}, & \cdot\\ 0, & -\mu_r^{-1}, & 1+2\mu_r^{-1}, & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}=\mu_r^{-1}M_m(\mu_r+2).$$

By (14) this is the set $\mu_r^{-1}\left(\mu_r+2+2\cos\dfrac{s\pi}{m+1}\right)$, $r=1,\cdots,n;\ s=1,\cdots,m$.

However by (14) also $\mu_r=2+2\cos\frac{r\pi}{n+1}$, $r=1,\cdots,n$. Thus the spectrum of $\mathscr{M}^{-1}\mathscr{M}^*$ is

$$\frac{2+2\cos\frac{r\pi}{n+1}}{4+2\cos\frac{r\pi}{n+1}+2\cos\frac{s\pi}{m+1}}=\frac{1}{1+\dfrac{\sin^2\frac{s\pi}{2(m+1)}}{\sin^2\frac{r\pi}{2(n+1)}}}, \qquad \begin{array}{l} r=1,\cdots,n. \\ s=1,\cdots,m. \end{array}$$

This spectrum therefore lies in the open interval (0, 1). $Z=\sum_{r=1}^{\infty}U^{(r)}$ is thus a solution of (1) if

$$|\rho^2-1|=|\delta|<1+\frac{\sin^2\frac{\pi}{2(m+1)}}{\sin^2\frac{n\pi}{2(n+1)}}$$

and certainly if $|\delta|\leq 1$.

6. **We shall now** estimate the error if we take $Z=\sum_{r=1}^{s}U^{(r)}$. We suppose that $|\delta|<1$, and consider first the case $s\geq 2$. Since the spectrum of $\mathscr{M}^{-1}\mathscr{M}^*$ lies in the open interval $(-1, 1)$, using (37) we have

$$\text{(39)} \qquad \begin{aligned} Z-\sum_{r=1}^{s}U^{(r)} &= \sum_{r=s+1}^{\infty}U^{(r)}=\sum_{r=1}^{\infty}(-\delta\mathscr{M}^{-1}\mathscr{M}^*)^rU^{(s)} \\ &= -\delta\mathscr{M}^{-1}\mathscr{M}^*(I+\delta\mathscr{M}^{-1}\mathscr{M}^*)^{-1}U^{(s)}. \end{aligned}$$

Now the spectrum of the matrix $(I+\delta\mathscr{M}^{-1}\mathscr{M}^*)^{-1}$ is

$$\left[1+\delta\left\{1+\frac{\sin^2\frac{s\pi}{2(m+1)}}{\sin^2\frac{r\pi}{2(n+1)}}\right\}^{-1}\right]^{-1}, \qquad r=1,\cdots,n;\ s=1,\cdots,m.$$

If $\delta>0$ this lies within a circle of radius one, while if $\delta<0$ it lies within a circle of radius $(1+\delta)^{-1}=(1-|\delta|)^{-1}$. Therefore we obtain from (39)

$$\text{(40)} \qquad \begin{aligned} \left\|Z-\sum_{r=1}^{s}U^{(r)}\right\| &< |\delta|(1-|\delta|)^{-1}\|U^{(s)}\|, && \text{when} \quad \delta<0, \\ &< |\delta|\,\|U^{(s)}\|, && \text{when} \quad \delta>0^1. \end{aligned}$$

[1] The norm $\|T\|$ of a matrix is $\sup\|Tx\|/\|x\|$, where $\|x\|$ is the square root of the sum of the squares of the coordinates of the vector x. If T is symmetric it is known that $\|T\|=|\lambda|$ where λ is the characteristic root of T of maximum modulus.

Consider now the case $r=1$. By (37) and (38) we have

$$Z-U^{(1)}=\sum_{r=0}^{\infty}(-\delta\mathscr{M}^{-1}\mathscr{M}^{*})^{r}(-\delta\mathscr{M}^{-1})(\mathscr{M}^{*}U^{(1)}-Z')$$

(41)
$$=\sum_{r=1}^{\infty}(-\delta\mathscr{M}^{-1}\mathscr{M}^{*})^{r}(U^{(1)}-\mathscr{M}^{*-1}Z')$$

$$=-\delta\mathscr{M}^{-1}\mathscr{M}^{*}(I+\delta\mathscr{M}^{-1}\mathscr{M}^{*})^{-1}(U^{(1)}-\mathscr{M}^{*-1}Z') .$$

We wish now to obtain a formula corresponding to (40). We observe that

$$\mathscr{M}^{*-1}=\begin{pmatrix} M_n^{-1}(2) , & 0 , & 0 , & \cdot \\ 0 , & M_n^{-1}(2) , & 0 , & \cdot \\ 0 , & 0 , & M_n^{-1}(2) , & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and the i, jth element $c_{i,j}$ of $M_n^{-1}(2)$ is given by (24) and (8) as

$$c_{i,j}=c_{j,i}=\frac{D_{i-1}(2)D_{n-j}(2)}{D_n(2)}=\frac{i(n-j+1)}{n+1} , \qquad i\le j .$$

By direct multiplication $\mathscr{M}^{*-1}Z'$ is thus a vector $P=(P_1, P_2, \cdots, P_m)$, where

$$P_i=M_n^{-1}(2)Z_i'=(n+1)^{-1}(nz_{i,0}+z_{i,n+1}, (n-1)z_{i,0}+2z_{i,n+1}, \cdots, z_{i,0}+nz_{i,n+1}) .$$

Now

$$\|P\|^2=\sum_{i=1}^{m}\|P_i\|^2$$
$$=\sum_{i=1}^{m}\sum_{j=1}^{n}(n+1)^{-2}((n-j+1)z_{i,0}+jz_{i,n+1})^2$$
$$=(n+1)^{-2}\sum_{j=1}^{n}\{j^2(\|Z_0\|^2+\|Z_{n+1}\|^2)+2j(n-j+1)(Z_0, Z_{n+1})\}\ ^{2}$$
$$=\frac{n}{6(n+1)}[(2n+1)(\|Z_0\|^2+\|Z_{n+1}\|^2)+(2n+4)(Z_0, Z_{n+1})]$$
$$\le\frac{n(2n+1)}{6(n+1)}\{\|Z_0\|^2+\|Z_{n+1}\|^2+2(Z_0, Z_{n+1})\}$$
$$\le\frac{n}{3}(\|Z_0\|+\|Z_{n+1}\|)^2 .$$

Thus

$$\|\mathscr{M}^{*}Z'\|=\|P\|\le\sqrt{\frac{n}{3}}(\|Z_0\|+\|Z_{n+1}\|) .$$

[2] (Z_0, Z_{n+1}) is the inner product $\sum_{i=1}^{m} z_{i,0}\, z_{i,n+1}$, and $|(Z_0, Z_{n+1})|\le\|Z_0\|\,\|Z_{n+1}\|$.

From (41) using the same arguments as for (40) we obtain

$$\|Z-U^{(1)}\| \leq |\delta| \left\{ \|U^{(1)}\| + \sqrt{\frac{n}{3}} (\|Z_0\| + \|Z_{n+1}\|) \right\}, \qquad \text{when } \delta > 0,$$

$$\leq |\delta| (1-|\delta|)^{-1} \left\{ \|U^{(1)}\| + \sqrt{\frac{n}{3}} (\|Z_0\| + \|Z_{n+1}\|) \right\}, \qquad \text{when } \delta < 0.$$

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UNIVERSITY OF NATAL
DURBAN, SOUTH AFRICA

Added in proof: Some of the results of §§ 1, 2, and 3 have been found also by:

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THE MAPPINGS OF THE POSITIVE INTEGERS INTO THEMSELVES WHICH PRESERVE DIVISION

MORGAN WARD

1. Introduction, First Theorem. Let L denote the lattice of the integers 0, 1, 2, $\cdots$ partially ordered by division. We study here mappings

$$\phi: \quad \phi_0, \phi_1, \phi_2, \cdots, \phi_n = \phi(n), \cdots$$

of L into itself which preserve division; that is,

(i) *If n divides m, then ϕ_n divides ϕ_m.*

Since ϕ_1 divides every ϕ_n and every ϕ_n divides ϕ_0, we lose little generality by assuming

(ii) $\phi_0 = 0, \quad \phi_1 = 1.$

Any mapping with properties (i) and (ii) will be called a divisibility sequence on L.

A mapping ϕ is said to be of "*positive character*" if

(iii) $\phi_n > 0$ for $n > 0$.

A divisibility sequence of positive character will be called a *normal sequence* or *normal mapping* of L.

In many instances, we are interested in the occurrence of multiples of some assigned modulus m among the terms of a normal sequence ϕ. If $\phi_r \equiv 0 \pmod m$ for some $r > 0$, we call m a *divisor* of ϕ and r a "*place of apparition*" of m in ϕ. If in addition $\phi_s \not\equiv 0 \pmod m$ for every proper divisor s of r, r is called a "*rank of apparition*" of m in ϕ. If m is not a divisor of ϕ, we assign to it the rank of apparition zero, which is consistent with the definitions.

It follows that every modulus m has at least one rank of apparition in ϕ. If each modulus has exactly one rank of apparition, we say that ϕ "*admits a rank function*". Indeed if the rank of m in ϕ is denoted by $\rho(m)$ then ρ is a divisibility sequence. Furthermore

(iv) $\phi_n \equiv 0 \pmod m$ *if and only if* $n \equiv 0 \pmod{\rho_m}$.

Under this condition, multiples of any integer m if they appear at all in ϕ are regularly spaced as in the identity mapping $i(n) = n$.

Normal sequences are of common occurrence in number theory; the totient function and its various generalizations [3, chap. 5] is a

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familiar example. For other examples and generalizations see [3, chap. 17], [4], [6], [9], [10].

Normal sequences with property (iv) are of considerable arithmetical interest, and special instances, notably the Lucas sequences [6] have been intensively studied [1], [5].

We study here general properties of all divisibility sequences and in particular develop necessary and sufficient conditions that a normal sequence shall admit a rank function. Our first main result is as follows.

THEOREM 1. *A necessary and sufficient condition that a normal mapping* ϕ *admit a rank function is that it have the following property*:

(v) $$\phi(pn) \smile \phi(qn) = \phi(n) \qquad p,\ q \text{ any distinct primes.}$$

Here we are using the lattice notation explained in § 3; the left side of (v) is the greatest common divisor of $\phi(pn)$ and $\phi(qn)$.

2. Further Results, Second Theorem. Our other results are formulated in terms of the notion of the "generator" of a normal sequence. Let

(2.1) $$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

be the prime factorization of any positive integer n of L. Define a new mapping ψ of L by $\psi(0)=0$, $\psi(1)=1$ and

(2.2) $$\psi(n) = \phi(n) \div \bigcap_{1 \le i \le k} \phi\left(\frac{n}{p_i}\right), \qquad n > 1.$$

Then ψ is called the generator of ϕ. It has properties (ii) and (iii), but not in general property (i). It is shown in § 5 that formula (2.2) may be inverted to express ϕ in terms of ψ thus:

(2.3) $$\phi(n) = \bigcap_{(c)} \prod_{1 \le i \le r} \psi(c_i).$$

Here (c): $1 = c_1, c_2, \cdots, c_{r-1}, c_r = n$ is a complete chain of divisors of n in the lattice L, c_i covering c_{i+1} for $i = 1, 2, \cdots, r-1$. The indicated least common multiple $\bigcap$ of the products $\prod \psi(c_i)$ is to be extended over all such chains (c) of divisors of n.

For example, if $n=12$, there are three complete chains: 1, 2, 4, 12; 1, 2, 6, 12 and 1, 3, 6, 12. Thus (2.3) becomes

$$\phi(12) = \psi(1)\psi(2)\psi(4)\psi(12) \cap \psi(1)\psi(2)\psi(6)\psi(12) \cap \psi(1)\psi(3)\psi(6)\psi(12).$$

Conversely, it turns out that if we start off with a mapping ψ of positive character with $\psi_0 = 0$, $\psi_1 = 1$ and define ϕ by (2.3), then ϕ is a normal mapping, and ψ is its generator. The relationships between arithmetical properties of ϕ and ψ are developed in §§ 6 and 7.

If ϕ is of positive character, we may define a new numerical function ζ by the Dedekind-Möbius inversion formulas [2, p. 61]

$$(2.4) \qquad \zeta(n) = \prod_{d \supseteq n} \phi\left(\frac{n}{d}\right)^{\mu(d)}; \qquad \phi(n) = \prod_{d \supseteq n} \zeta(d) .$$

Here μ as usual is the Möbius function.

ζ is uniquely determined by ϕ, but does not define a mapping of L because $\zeta(n)$ is not necessarily an integer. If $\zeta(n)$ is an integer for every n, ϕ is evidently a normal sequence; we call ζ in this case the "*Dedekind generator*" of ϕ.

THEOREM 2. *If ϕ is a normal sequence, then a necessary and sufficient condition that ϕ admit a rank function is that its Dedekind generator should exist, and be equal to its ordinary generator.*

The best known instance of this theorem is when ϕ is the Lucas sequence $\phi_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ where $p = \alpha + \beta$, $q = \alpha\beta$ are co-prime integers chosen so that $|pq| > 1$; $|p^2 - 4q| > 0$. Then ψ is the Sylvester [7] cyclotomic sequence

$$\psi_n = \prod_{\substack{1 \le r \le n \\ r \cup n = 1}} \left(\alpha - e^{2\pi i \frac{r}{n}} \beta\right).$$

3. **Notations.** We use whenever convenient the standard notations of lattice algebra for arithmetical division and its associated operations over L considered as a distributive residuated lattice [8], [11]. We thus write $a \supseteq b$, $a \not\supseteq b$ and $a \supset b$ for "a divides b", "a does not divide b" and "a properly divides b". If neither $a \supseteq b$ nor $b \supseteq a$, we say a and b are "*non-comparable*". If $a \supseteq b$ and $a \supseteq x \supseteq b$ implies either $a = x$ or $b = x$ we say "*a covers b*".

$a \cup b$, $a \cap b$ and ab stand respectively for the greatest common divisor (g.c.d.), least common multiple (l.c.m.), and product of a and b. If $a_1, a_2, \cdots, a_k$ are k given integers of L, we write $\bigcup a_i$, $\bigcap a_i$ and $\prod a_i$ for their g.c.d., l.c.m. and product suppressing the range of i where no confusion can arise.

If $x * y$ denotes any one of the three operations $x \cup y$, $x \cap y$ or xy in L, and ϕ is any mapping of L, we say that ϕ is "$*$-factorable" if $\phi(x * y) = \phi(x) * \phi(y)$ whenever $x \cup y = 1$ and "completely $*$-factorable" if $\phi(x * y) = \phi(x) * \phi(y)$ for every x, y. The star-product of two mappings ϕ and θ is defined as usual by $(\phi * \theta)_n = \phi_n * \theta_n$.

In proofs we use when convenient $\Longrightarrow$ and $\Longleftrightarrow$ for "implies", and "implies and is implied by". We use without specific mention the familiar formulas [12]

$$b\cup a_i = \cup ba_i\,, \qquad b\cap a_i = \cap ba_i\,,$$
$$b\cup 0 = b\cap 1 = b\,; \quad b\cup 1 = 1, \quad b\cap 0 = 0\,.$$

4. Divisibility Sequences, Binary Sequences. Let ϕ be any divisibility sequence; that is, a mapping of L with properties (i) and (ii) of the introduction. Define $\alpha_0 = \beta_0 = 0$ and

$$\begin{array}{llll} \alpha_n = \phi_n\,, & \beta_n = 1 & \text{if} & \phi_n \neq 0 \\ \alpha_n = \bigcap\limits_{\substack{x \supseteq n \\ \phi_x \neq 0}} \phi_x\,, & \beta_n = 0 & \text{if} & \phi_n = 0. \end{array}$$

Then α and β are divisibility sequences, and $\phi = \alpha\beta$. Furthermore α is a normal mapping of L, while β consists exclusively of zeros and ones. We call β a "*binary* (*divisibility*) *sequence*".

We may immediately obtain a binary sequence from any divisibility sequence by reducing each term modulo 2. More generally, if m is any modulus, we may obtain from the divisibility sequence ϕ a binary sequence θ which describes the distribution of multiples of m in ϕ by letting $\theta_n = 0$ or 1 according as $\phi \equiv 0$ or $\phi \not\equiv 0 \pmod m$. The sequences obtained in this manner from linear divisibility [12] or elliptic divisibility sequences [13] are usually periodic.

Again, if E is any subset of L with the properties that 0 is not in E and if x is in E, so is every divisor of x, then the characteristic function of E is evidently a binary sequence. A simple example is the set of square-free integers; the characteristic function is μ^2.

Let β be any binary sequence. If $\beta_k = 0$, k is called a zero of β. If in addition $\beta_d \neq 0$ for $d \supset k$, k is called a prime zero of β. The prime zeros of β evidently form a multiplicative basis for the set of all zeros of β. Perhaps the most interesting property of this basis is expressed by the following theorem whose proof is left to the reader.

THEOREM. *The zeros of a binary divisibility sequence have a finite basis if and only if the sequence is periodic. The period of the sequence is then the* l.c.m. *of the prime zeros of its basis.*

5. The Generator of A Normal Sequence. From now on, all mappings considered are of positive character. Let ψ be any such mapping with $\psi_0 = 0$, $\psi_1 = 1$ and define a new mapping ϕ by means of formula (2.3) and $\phi_0 = 0$, $\phi_1 = 1$. Then ϕ is evidently normal. Hold n fixed, and let (2.1) be its prime decomposition. Each complete chain

$$(c):\quad 1=c_1,\ c_2,\ \cdots,\ c_{r-1},\ c_r=n;\qquad c_i \text{ covers } c_{i+1}$$

in the sublattice of all divisors of n is of the same length $r=a_1+a_2+\cdots+a_k+1$, while c_{r-1} is one of the k elements n/p_i which cover n. We may accordingly group the chains into k mutually exclusive classes C_i by putting into class C_i all chains (c) with $c_{r-1}=n/p_i$. But any chain of class C_i consists of a complete chain of divisors of n/p_i plus the fixed element $c_r=n$. Hence formula (2.3) may be written

$$\phi_n=\bigcap_{C_i}\bigcap_{(c')}(\psi_{c'_1}\cdots\psi_{c'_{r-1}})\psi_n$$

where the inner l.c.m. is taken over all complete chains (c') of divisors of n/p_i. Thus by (2.3) again

$$\phi_n=\bigcap_i \phi(n/p_i)\psi(n)=\psi(n)[\phi_{n/p_1}\cap\cdots\cap\phi_{n/p_k}].$$

Therefore ψ is the generator of ϕ as defined in formula (2.2).

Conversely, if we define ψ by (2.2), we find by direct calculation that (2.3) holds for small n. We therefore proceed by induction and assume that (2.3) is true for all integers less than n, and hence in particular for the k integers n/p_i which cover n.

On transforming the right side of (2.3) as in the first part of this proof, we obtain by (2.2) and the hypothesis of the induction

$$\bigcap_{(c)}\prod_i \psi_{c_i}=\bigcap_{C_i}\prod_i \psi_{c'_i}\psi_n=\bigcap_i(\phi_{n/p_i}\psi_n)=\psi_n\bigcap_i\phi_{n/p_i}=\phi_n\,.$$

Thus the formulas (2.2) and (2.3) are equivalent.

6. Factorable sequences. Various factorability properties of normal sequences may be elegantly stated as properties of its generator. We postpone the consideration of g.c.d. factorability until the next section, since it is intimately connected with the existence of a rank function. We omit proofs of the results stated here, since we merely wish to show the importance of the notion of a generator.

Either of the following two conditions is necessary and sufficient for a normal sequence ϕ with generator ψ to be product-factorable:

$$\phi_n=\prod_{p^t\supseteq n}\psi_{p^t}\,. \tag{6.1}$$

Here the product is extended over all prime powers p^t dividing n.

$$\psi_{nm}=\psi_n\cup\psi_m\qquad n,\ m \text{ co-prime.} \tag{6.2}$$

A necessary and sufficient condition for ϕ to be l.c.m.-factorable is that

$$\psi_n=1\,,\qquad n \text{ not a power of a prime.} \tag{6.3}$$

Any one of the following three sets of conditions are necessary and sufficient for ϕ to be completely product factorable:

(6.4) $$\psi(mn)=\psi(m\cap n)=\psi(m)\cup\psi(n), \qquad n, m>1.$$

(6.5) $$\psi(mn)=\psi(m)\cup\psi(n) \qquad \text{if } m, n \text{ are co-prime}$$

and $\psi(p^a)=\psi(p)$ for every prime p.

(6.6) $$\psi(n)=\psi(p_1, \cdots, p_k)=\phi(p_1)\phi(p_2), \cdots, \phi(p_k) .$$

Here as in (2.1), $p_1, p_2, \cdots, p_k$ are the distinct prime factors of n.

7. G.C.D. factorable mappings. A mapping ϕ is said to be *completely* g.c.d. *factorable* if it has the property

(vi) $$\phi(n\cup m)=\phi(n)\cup\phi(m) .$$

Every such mapping evidently preserves division.

LEMMA 7.1. (Ward [14]): *Conditions* (iv) *and* (vi) *are equivalent for normal mappings of L; that is, a normal mapping admits a rank function if and only if it is completely* g.c.d. *factorable.*

Proof. Assume that ϕ is a normal mapping satisfying Condition (iv). Let $\rho=\rho(k)$ be the rank of $k=\phi_n\cup\phi_m$ in ϕ. Then ρ is positive. Also $k\supseteq\phi_n, \phi_m\Longrightarrow\rho\supseteq n, m\Longrightarrow\rho\supseteq n\cup m\Longrightarrow k\supseteq\phi(n\cup m)$. But by (i),

$$n\cup m\supseteq n, m\Longrightarrow\phi_{n\cup m}\supseteq\phi_n, \phi_m\Longrightarrow\phi_{n\cup m}\supseteq k .$$

Hence $\phi_{n\cup m}=\phi_n\cup\phi_m$ and (iv) implies (vi).

Conversely, let ϕ be a normal mapping with property (vi), and let k be any modulus. If k is not a divisor of ϕ, the rank of k is zero, and (iv) is satisfied. If k is a divisor of ϕ, let ϕ_r be the first term with positive index r which k divides. By (i),

$$n\equiv 0 \pmod r\Longrightarrow\phi_n\equiv 0 \pmod k.$$

Assume conversely that $\phi_n\equiv 0 \pmod k$. Then by (vi), $\phi_{n\cup r}\equiv 0 \pmod k$. But $0<n\cup r\leq r$. Hence $n\cup r=r$ or $n\equiv 0 \pmod r$. In other words,

$$\phi_n\equiv 0 \pmod k\Longrightarrow n\equiv 0 \pmod r.$$

Hence r is the rank of k in ϕ. Since k was arbitrary, (vi) implies (iv), which completes the proof.

The factorability condition on ϕ may be replaced by an equivalent condition on its generator ψ.

LEMMA 7.2 *A normal mapping ϕ admit a rank function if and* *only if its generator ψ satisfies the condition*

(vii) $$\psi(n)\cup\psi(m)=1 \qquad n,\ m \text{ non-comparable.}$$

Proof. Assume that ϕ is normal, and admits a rank function, but that (vii) is false. Then there exist integers n, m and a prime q such that

(7.1) $$\psi(n)\equiv\psi(m)\equiv 0 \pmod q, \text{ but } n \not\supset m,\ m \not\supset n .$$

By formula (2.2),

$$(7.1) \Longrightarrow \phi(n)\equiv\phi(m)\equiv 0 \pmod q .$$

Suppose that q^a exactly divides $\phi(n)$ and q^b exactly divides $\phi(m)$. We may evidently assume that $b\geq a$. Let r be the rank of q^a in ϕ. Then since $n\equiv m\equiv 0 \pmod r$, we have $n\cup m\equiv 0 \pmod r$. But if (2.1) gives the factorization of n so that $p_1, p_2, \cdots, p_k$ are its distinct prime factors, then

(7.2) $$\phi(n/p_i)\not\equiv 0 \bmod q^a, \qquad 1\leq i\leq h.$$

For in the contrary case, $\psi(n)\equiv 0 \pmod q$ and (2.2) together imply

$$\phi(n)\equiv\psi(n)\bigcap_i\phi\left(\frac{n}{p_i}\right)\equiv 0 \pmod{q^{a+1}}$$

which is a contradiction.

Now

$$(7.2) \Longrightarrow n/p_i\not\equiv 0 \pmod r \qquad i=1, 2, \cdots, k.$$

But $n\equiv 0 \pmod r$. Hence $n=r$ and $n\supseteq n\cup m\supseteq m$ contradicting (7.1). Therefore (iv) implies (vii).

Assume conversely that ϕ is normal with generator ψ satisfying (vii). To show that ϕ then admits a rank function, it will suffice to prove that *every prime power q^a has a unique rank of apparition in ϕ.* If q^a is not a divisor of ϕ then it has the unique rank zero. If q^a is a divisor of ϕ then there exists a positive index r such that

(7.3) $$\phi_r\equiv 0 \pmod{q^a}, \quad \phi_n\not\equiv 0 \pmod{q^a}, \qquad 0<n<r.$$

To prove that r is the rank of q^a in ϕ, it will suffice to show that if $\phi_n\equiv 0 \bmod q^a$ then $n\equiv 0 \pmod r$. This we do by contradiction. For otherwise, there exists a least positive $n>r$ such that $\phi_n\equiv 0 \bmod q^a$, but n, r noncomparable. Evidently, $\psi_r\equiv 0 \pmod q$. Hence $\psi_n\not\equiv 0 \pmod q$ by Condition (vii). But then formula (2.2) implies that $\phi(n/p_i)\equiv 0 \pmod{q^a}$ for some prime divisor p_i of n. Therefore, by the minimal choice of n, either $r \supseteq n/p_i$ or $n/p_i \supseteq r$. In the first case, $r \supset n$. In the second case $n/p_i \leq r$ so that by (7.3), $n/p_i = r$ and $r \supset n$. In either case $r \not\supset n$ is contradicted. Hence (vii) implies (iv), which completes the proof of the lemma.

8. Proof of Theorem 1. In view of Lemma 7.1, the proof of Theorem 1, requires only the demonstration that if ϕ is normal, Condition (v) implies Condition (vi); for $pn \cup qn = n$ so that the implication (vi)$\Longrightarrow$(v) is trivial. Note also that (v) is essentially a weakening of (vi), since it amounts to asserting (vi) only in the special case when $n \cup m$ covers both n and m.

Let ϕ be normal, and s a fixed positive integer. Then the normal mapping θ defined by

$$\theta(n) = \phi(sn)/\phi(s), \qquad n = 0, 1, 2, \cdots \tag{8.1}$$

is called a subsequence of ϕ. The following lemma is an easy consequence of this definition.

LEMMA 8.1. *If* ϕ is *normal, and has the property* (v), *then so has every subsequence of* ϕ.

LEMMA 8.2. *If* ϕ *is normal, and has the property* (v), *then* ϕ *is* g.c.d. *factorable; that is*

$$\phi(n) \cup \phi(m) = 1 \quad \text{if} \quad n \cup m = 1. \tag{viii}$$

Note that by (ii), (viii) is a special case of (vi); the proof is by induction on the number of prime factors of n and m. First if n and m are distinct primes p and q, then (viii) follows from (v) on taking $n = 1$.

Suppose that $n = p$ and m is the product of $l \geq 2$ primes, $m = q_1, q_2, \cdots, q_l$ where the q_i are distinct from p but not necessarily distinct from one another. Assume that (viii) has been proved for $n = p$ and m a product of $l-1$ primes. Now take $p = p$, $q = q_l$ and $n = m/q_l$ in (v). Then

$$\phi(pm/q_l) \cup \phi(m) = \phi(m/q_l).$$

Now

$$q_l \not\supseteq p \Longrightarrow p \supseteq pm/q_l \Longrightarrow \phi(p) \supseteq \phi(pm/q_l).$$

Consequently,

$$\phi(p) \cup \phi(m) = \phi(p) \cup \phi(pm/q_l) \cup \phi(m) = \phi(p) \cup \phi(m/q_l) = 1$$

by the hypothesis of the induction. Hence (viii) is true if n is a prime number.

Next assume that $n=p_1p_2\cdots p_k$ is the product of $k\geq 2$ primes p_i distinct from all the primes q_j dividing m so that $n\cup m=1$, and also assume that (viii) has been proved for n a product of $k-1$ primes. Now apply (v) with $p=p_k$, $q=q_1$ and $n=nm/p_kq_1$. Thus

$$\phi(nm/q)\cup\phi(nm/p)=\phi(nm/pq)\,.$$

Now

$$n\supseteq nm/q\Longrightarrow\phi(n)\supseteq\phi(nm/q)\Longrightarrow\phi(n)\cup\phi(nm/p)=\phi(n)\cup\phi(nm/q)\cup\phi(nm/p)$$
$$=\phi(n)\cup\phi(nm/pq)=\phi(n)\cup\phi(nm/pq_1)\,.$$

Repeat this argument replacing m successively by

$$m/q_1,\ m/q_1q_2,\ \cdots,\ m/q_1q_2\cdots q_l=1$$

and leaving n and p unchanged; we find that

$$\phi(n)\cup\phi(nm/p)=\phi(n)\cup\phi(nm/pq_1)$$
$$=\phi(n)\cup\phi(nm/pq_1q_2)=\cdots=\phi(n)\cup\phi(n/p)=\phi(n/p).$$

But

$$m\supseteq nm/p\Longrightarrow\phi(m)\supseteq\phi(nm/p)\Longrightarrow\phi(n)\cup\phi(m)=\phi(n)\cup\phi(nm/p)\cup\phi(m)$$
$$=\phi(n/p)\cup\phi(m)=1$$

by the hypothesis of the induction. Hence (viii) is true for every n prime to m, completing the proof of Lemma 8.2.

Theorem 1 may now be proved as follows: Let ϕ be a normal mapping satisfying (v) and let both n and m be positive, since (vi) is trivially satisfied if n or m is zero. Let $s=n\cup m$. Then $n=n's$, $m=m's$ with $n'\cup m'=1$. Consider the subsequence θ of ϕ defined by (8.1). By Lemma 8.1, θ has property (v). Hence Lemma 8.2 implies

$$\theta(n')\cup\theta(m')=1\Longrightarrow\phi(n)/\phi(s)\cup\phi(m)/\phi(s)=1$$
$$\Longrightarrow\phi(n)\cup\phi(m)=\phi(s)=\phi(n\cup m)\,.$$

Hence (v) implies (vi), completing the proof of Theorem 1.

9. Proof of second theorem—necessity. Assume that ϕ is normal, and admits a rank function, and let ψ be its generator. We shall show that

(ix) $$\phi_n=\prod_{d\supseteq n}\psi_d$$

so that ψ is the Dedekind generator of ϕ. The proof is based on a consequence of Dedekind's cross-classification principle [2]; namely

LEMMA 9.1. *If $a_1, a_2, \cdots, a_k$ are positive integers, then*

$$a_1 \cap a_2 \cap \cdots \cap a_k = \Pi\, a_1\, \Pi\, (a_1 \cup a_2 \cup a_3) \cdots \div \Pi\, (a_1 \cup a_2)\, \Pi\, (a_1 \cup a_2 \cup a_3 \cup a_4) \cdots.$$

This result is a generalization of the familiar formula $a_1 \cap a_2 = a_1 a_2 \div a_1 \cup a_2$ and is perhaps easiest proved by showing that the highest powers of p dividing both sides of the formula are the same.

On applying the result to formula (2.2), we obtain

$$\begin{aligned}\psi(n) &= \phi(n) \div \left[\phi\left(\frac{n}{p_1}\right) \cap \phi\left(\frac{n}{p_2}\right) \cap \cdots \cap \phi\left(\frac{n}{p_k}\right)\right] \\ &= \phi(n)\, \Pi \left(\phi\left(\frac{n}{p_1}\right) \cup \phi\left(\frac{n}{p_2}\right)\right) \Pi \left(\phi\left(\frac{n}{p_1}\right) \cup \phi\left(\frac{n}{p_2}\right) \cup \phi\left(\frac{n}{p_3}\right) \cup \phi\left(\frac{n}{p_4}\right)\right) \cdots \\ &\quad \div \Pi\, \phi\left(\frac{n}{p_1}\right) \Pi \left(\phi\left(\frac{n}{p_1}\right) \cup \phi\left(\frac{n}{p_2}\right) \cup \phi\left(\frac{n}{p_3}\right)\right) \cdots.\end{aligned}$$

Now since ϕ admits a rank function, ϕ is completely g.c.d. factorable by Lemma 7.1. Therefore the formula above may be written

$$\begin{aligned}\psi(n) &= \phi(n)\, \Pi\, \phi\left(\frac{n}{p_1 p_2}\right) \Pi\, \phi\left(\frac{n}{p_1 p_2 p_3 p_4}\right) \cdots \\ &\quad \div \Pi\, \phi\left(\frac{n}{p_1}\right) \Pi\, \phi\left(\frac{n}{p_1 p_2 p_3}\right) \cdots \\ &= \Pi_{d \supseteq n}\, \phi\left(\frac{n}{d}\right)^{\mu(d)}\end{aligned}$$

where μ is the Möbius function. Hence (ix) follows by the Dedekind inversion formula, completing the proof of the necessity.

10. Proof of second theorem—sufficiency.

Now assume that ϕ is normal, and that its Dedekind generator exists and equals its ordinary generator; that is, Condition (ix) is satisfied. We shall show that

(vi) $$\psi(n) \cup \psi(m) = 1. \qquad n, m \text{ non-comparable.}$$

Hence it will follow from Lemma 7.2 that (ix) is a sufficient condition for ϕ to admit a rank function.

Assume n, m non-comparable. Then if $l = n \cap m$, $n < l$ and $m < l$. Let $q_1, q_2, \cdots, q_s$ be the distinct prime factors of l, and let p be any prime p^{a_n}, p^{b_n} the highest powers of p dividing $\phi(n)$ and $\psi(n)$ respectively.

Now by formula (2.2),

$$\phi_l = \psi_l [\phi_{l/q_1} \cap \cdots \cap \phi_{l/q_s}].$$

Hence $a_l = b_l + a_{l/q}$, where $a_{l/q}$ is the largest of $a_{l/q_1}, \cdots, a_{l/q_s}$. But by (ix), $a_l = \sum_{d \supseteq l} b_d$. Hence $b_d = 0$ unless $d = l$ or $d \supseteq l/q$.

Not both b_n and b_m are positive. For since $n \supset l$ and $m \supset l$, in the contrary case $n \supseteq l/q$ and $m \supseteq l/q$ by the remark above. But then $l = n \cap m \supseteq l/q$ so that $q=1$, contrary to q a prime.

It follows that p does not divide both $\psi(n)$ and $\psi(m)$. Since p was an arbitrarily chosen prime, (v) follows, which completes the proof of Theorem 2.

In closing, note that it follows from Theorem 2 and Lemma 7.2 that if ϕ has the Dedekind generator ζ (that is,

$$\zeta(n) = \Pi\ \phi(n/a)^{\mu(d)}$$

is an integer for every n); then a necessary and sufficient condition that ϕ should admit a rank function is that its Dedekind generator satisfy the condition $\zeta(n) \cup \zeta(m) = 1$ if n, m are non-comparable.

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California Institute of Technology, Pasadena

WEAK LOCALLY MULTIPLICATIVELY-CONVEX ALGEBRAS[1]

SETH WARNER

Let E be an algebra over the reals or complex numbers, E' a total subspace of the algebraic dual E^* of vector space E. We first discuss the following natural questions: When is the weak topology $\sigma(E, E')$ defined on E by E' locally m-convex? When is multiplication continuous for $\sigma(E, E')$, that is, when is $\sigma(E, E')$ compatible with the algebraic structure of E? We then apply our results to certain weak topologies on the algebra of polynomials in one indeterminant without constant term.

1. Weak topologies.

Let K be either the reals or complex numbers, E a K-algebra. A topology $\mathscr{T}$ on E is *locally multiplicatively-convex* (which we abbreviate henceforth to "locally m-convex") if it is a locally convex topology and if there exists a fundamental system of idempotent neighborhoods of zero (a subset A of E is idempotent if $A^2 \subseteq A$). Multiplication is then clearly continuous at (0, 0) and hence everywhere, so $\mathscr{T}$ is compatible with the algebraic structure of E. If A is idempotent, so is its convex envelope, its equilibrated envelope (a subset V of E is called equilibrated if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$), and its closure for any topology on E compatible with the algebraic structure of E. Hence if $\mathscr{T}$ is locally m-convex, zero has a fundamental system of convex, equilibrated, idempotent, closed neighborhoods. (For proofs of these and other elementary facts about locally m-convex algebras, see §§ 1–3 of [8] or [1].) Henceforth, E' is a total subspace of the algebraic dual of E.

LEMMA 1. *Let W be a weak, equilibrated neighborhood of zero (that is, for the topology $\sigma(E, E')$), J a subspace of E, and $g \in E'$ such that $J \subseteq W \subseteq W \cup W^2 \subseteq \{g\}^0$. Then J, JE, and EJ are contained in the kernel of g.*

Proof. Let $x \in J$, $y \in E$. As W is equilibrated and absorbing, let $\lambda > 0$ be such that $\lambda y \in W$. For all positive integers m, $\lambda^{-1}mx \in J$, and therefore $mxy = (\lambda^{-1}mx)(\lambda y) \in JW \subseteq W^2 \subseteq \{g\}^0$; hence $|g(mxy)| \leq 1$ for all positive integers m, and therefore $g(xy) = 0$. Hence JE is contained in

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the kernel of g. Similarly for EJ. Also $|g(mx)| \leq 1$ for all $x \in J$ and all positive integers m, and therefore $g(x)=0$ for all $x \in J$.

LEMMA 2. *Let V be a weak neighborhood of zero. Then $L = \bigcap[u^{-1}(0) \mid u \in V^0]$ is a weakly closed subspace of finite codimension.*

Proof. L is clearly a weakly closed subspace. By definition of $\sigma(E, E')$ there exist $h_1, h_2, \cdots, h_n$ in E' such that $\{h_1, h_2, \cdots, h_n\}^0 \subseteq V$. Thus if $|h_i(z)| \leq 1$ for $1 \leq i \leq n$, then $|u(z)| \leq 1$ for all $u \in V^0$. Then if $x \in \bigcap_{i=1}^{n} h_i^{-1}(0)$, for any positive integer m $|h_i(mx)| = 0 < 1$ for $1 \leq i \leq n$ and hence $|u(mx)| \leq 1$, so $u(x)=0$ for all $u \in V^0$. Hence $\bigcap_{i=1}^{n} h_i^{-1}(0) \subseteq L$. Since the codimension of $\bigcap_{i=1}^{n} h_i^{-1}(0)$ is at most n, so also the codimension of L is at most n.

LEMMA 3. *Let $E_1, E_2, \cdots, E_n$ be finite-dimensional, Hausdorff topological K-vector spaces, F a topological K-vector space. Any multilinear transformation from $E_1 \times E_2 \times \cdots \times E_n$ into F is continuous.*

Proof. This lemma is well known, and follows from Theorem 2 of [3, p. 27] just as Corollary 2 of that theorem does.

THEOREM 1. *$\sigma(E, E')$ is a locally m-convex topology on E if and only if for all $g \in E'$, the kernel of g contains a weakly closed ideal of finite codimension.*

Proof. Necessity: Let $g \in E'$. Let V be a weakly closed, convex, equilibrated, idempotent neighborhood of zero such that $V \subseteq \{g\}^0$. Let $L = \bigcap [u^{-1}(0) \mid u \in V^0]$. Then clearly $L \subseteq V^{00}$, but since V is weakly closed, convex, and equilibrated, $V^{00} = V$ (see [4]). By Lemma 2 L is a weakly closed subspace of finite codimension. We assert L is an ideal: Let $x \in L$, $y \in E$. Choose $\lambda > 0$ such that $\lambda y \in V$. For all positive integers m, $\lambda^{-1}mx \in L$; hence $mxy = (\lambda^{-1}mx)(\lambda y) \in LV \subseteq V^2 \subseteq V$. Hence for all positive integers m and any $u \in V^0$, $|u(mxy)| \leq 1$; hence $u(xy)=0$ for all $u \in V^0$, so $xy \in L$. Similarly $yx \in L$, so L is an ideal. Now let $J = L \cap g^{-1}(0)$. Then J is a weakly closed subspace of finite codimension contained in the kernel of g. It remains to show J is an ideal. Now $J \subseteq L \subseteq V = V \cup V^2 \subseteq \{g\}^0$; hence by Lemma 1 $JE \subseteq g^{-1}(0)$ and $EJ \subseteq g^{-1}(0)$. Also $JE \subseteq LE \subseteq L$ and $EJ \subseteq EL \subseteq L$. Therefore $JE \subseteq L \cap g^{-1}(0) = J$ and $EJ \subseteq L \cap g^{-1}(0) = J$, so J is an ideal.

Sufficiency: It clearly suffices to show that for all $g \in E'$ there exists an idempotent neighborhood V of zero such that $V\subseteq\{g\}^0$. Let J be a weakly closed ideal of finite codimension contained in $g^{-1}(0)$. Then $F=E/J$ is a finite-dimensional algebra with a Hausdorff topology compatible with the vector space structure of F. Multiplication is a bilinear transformation from $F\times F$ into F, and hence by Lemma 3 multiplication is continuous. But also, any finite-dimensional, Hausdorff, K-vector space has its topology defined by a norm (this follows from Theorem 2 of [3, p. 27]); and by a familiar property of normed spaces with a continuous multiplication, the norm may be so chosen that F is a normed algebra [6, p. 50]. Let φ be the continuous canonical homomorphism from E onto F, and let $g=\bar{g}\circ\varphi$. $\bar{g}$ is continuous on F, so we may select an idempotent neighborhood U of zero in F such that $v\in U$ implies $|\bar{g}(v)|\leq 1$. Then $V=\varphi^{-1}(U)$ is a neighborhood of zero for $\sigma(E, E')$. As U is idempotent and φ a homomorphism, V is idempotent. Finally, if $x\in V$ then $\varphi(x)\in U$, and therefore $|g(x)|=|\bar{g}(\varphi(x))|\leq 1$, so $x\in\{g\}^0$; hence $V\subseteq\{g\}^0$, and the theorem is completely proved.

THEOREM 2. *Multiplication in E is continuous for $\sigma(E, E')$ if and only if for all $g\in E'$, the kernel of g contains a weakly closed subspace J of finite codimension such that JE and EJ are also contained in the kernel of g.*

Proof. Necessity: Let $g\in E'$. Then since $\{g\}^0$ is a neighborhood of zero, we may choose a weakly closed, convex, equilibrated neighborhood W of zero such that $W\cup W^2\subseteq\{g\}^0$. Let $L=\bigcap[u^{-1}(0)\mid u\in W^0]$. Then clearly $L\subseteq W^{00}=W$, since W is weakly closed, convex, and equilibrated. By Lemma 2 L is a weakly closed subspace of finite codimension. Let $J=L\cap g^{-1}(0)$. Then J is also a weakly closed subspace of finite codimension contained in the kernel of g. Also $J\subseteq L\subseteq W\subseteq W\cup W^2\subseteq\{g\}^0$, so by Lemma 1, JE and EJ are contained in the kernel of g.

Sufficiency: It suffices to show that for any $g\in E'$ and any $a\in E$, there exist neighborhoods W and V of zero in E such that $W^2\subseteq\{g\}^0$ and $Va\cup aV\subseteq\{g\}^0$ ([5, p. 49]). Let $I=g^{-1}(0)$ and let J be a weakly closed subspace of finite codimension contained in I such that $EJ\subseteq I$ and $JE\subseteq I$. Let φ and ψ respectively be the canonical maps from E onto E/J and from E onto E/I. Let $g=\bar{g}\circ\psi$. We assert the map $(\varphi(x), \varphi(y))\to\psi(xy)$ is a well-defined bilinear map from $(E/J)\times(E/J)$ into E/I: If $x-x'\in J$ and $y-y'\in J$, then $xy-x'y\in JE\subseteq I$ and $x'y-x'y'\in EJ\subseteq I$; hence $xy-x'y'=(xy-x'y)+(x'y-x'y')\in I+I=I$. The map is therefore well-defined; bilinearity is easily seen. Both (E/J) and (E/I) are finite-dimensional Hausdorff topological K-vector spaces, so by

Lemma 3 the above bilinear map is continuous. Hence there exists a neighborhood U of zero in E/J such that if $\varphi(x)$, $\varphi(y) \in U$, then $\psi(xy) \in \{\bar{g}\}^0$. If $W=\varphi^{-1}(U)$, then W is a neighborhood of zero for $\sigma(E, E')$; if $x, y \in W$, then $\varphi(x)$, $\varphi(y) \in U$ and hence $|g(xy)|=|\bar{g}(\psi(xy))| \leq 1$, so $xy \in \{g\}^0$. Thus $W^2 \subseteq \{g\}^0$. Now let $a \in E$. We assert the maps $\varphi(x) \rightarrow \psi(ax)$ and $\varphi(x) \rightarrow \psi(xa)$ are well-defined, linear maps from E/J into E/I: For if $x-x' \in J$, then $ax-ax' \in EJ \subseteq I$ and $xa-x'a \in JE \subseteq I$, so the maps are well-defined. Linearity is immediate. Since E/J and E/I are finite dimensional and Hausdorff, again by Lemma 3 these maps are continuous. Hence we may choose a neighborhood P of zero in E/J such that if $\varphi(x) \in P$ then $\psi(ax)$, $\psi(xa) \in \{\bar{g}\}^0$. Then $V=\varphi^{-1}(P)$ is a neighborhood of zero for $\sigma(E, E')$. If $x \in V$, then $\varphi(x) \in P$ and hence $|g(ax)|=|\bar{g}(\psi(ax))| \leq 1$ and similarly $|g(xa)| \leq 1$. Hence $aV \cup Va \subseteq \{g\}^0$, and the theorem is completely demonstrated.

Here is an example of a Banach algebra E with topological dual E' such that multiplication is not continuous for the associated weak topology $\sigma(E, E')$. Let E be the algebra of all continuous functions from the compact interval $[0, 1]$ into K with the uniform topology. If $\mu(f)=\int_0^1 f(t)dt$ (dt is the usual Lebesgue complex-valued measure if K is the complex numbers), then $\mu \in E'$. But μ does not satisfy the restrictions of Theorem 2: Let J be any weakly closed subspace contained in the kernel of μ such that $JE \subseteq \mu^{-1}(0)$. If $f \in J$, then $f\bar{f} \in JE \subseteq \mu^{-1}(0)$ ($\bar{f}=f$ if K is the reals); hence $\int_0^1 |f(t)|^2 dt=0$ and so, since f is continuous, $f=0$. Therefore $J=\{0\}$. But since E is infinite-dimensional, J is not of finite codimension. Hence by Theorem 2, multiplication is not continuous for $\sigma(E, E')$.

2. Algebras of polynomials. If E is any locally m-convex algebra, E' its topological dual, $\mathscr{M}(E)$ is the set of all continuous multiplicative linear forms, $\mathscr{M}^-(E)$ the set of all nonzero continuous multiplicative linear forms. $\mathscr{M}(E)$ and $\mathscr{M}^-(E)$ are topologized as subsets of E'; $\sigma(E', E)$.

In [9] Šilov proved the following theorems:

(1) If E is a normed C-algebra (C is the complex numbers) with identity e, generated by e and another element x (that is, if all elements of E are of form $\alpha_0 e+\alpha_1 x+\cdots+\alpha_n x^n$), then $\mathscr{M}^-(E)$ is homeomorphic with a compact subset of C whose complement is connected; (2) every such subset of C arises in this manner.

Here we give elementary analogues of these theorems for locally m-convex algebras.

Proposition 1. *If E is a locally m-convex Hausdorff algebra generated by a single element x, then $f \to f(x)$ is a homeomorphism from $\mathscr{M}(E)$ onto a subset of K.*

Proof. The map is surely continuous and is one-to-one since x generates E. To show $f(x) \to f$ is continuous, it suffices to show $f(x) \to f(z)$ is continuous for all $z \in E$; but as x generates E it suffices for this to show $f(x) \to f(x^n)$ is continuous for all positive integers n. But $f(x^n) = f(x)^n$, so $f(x) \to f(x^n)$ is simply a restriction of the map $\lambda \to \lambda^n$ from K into K, which is surely a continuous map. Hence $f \to f(x)$ is a homeomorphism into K.

Proposition 2. *Let E be an algebra over any field F. The set M of nonzero multiplicative linear forms is a linearly independent subset of E^*, the algebraic dual of E.*

Proof. In Theorem 12 of [2, p. 34], Artin proves that if G is a group, F a field, then the set of all nonzero homomorphisms from G into the multiplicative semi-group of F is a linearly independent subset of the vector space $\mathscr{F}(G, F)$ of all functions from G into F. The proof remains valid if "semi-group" replaces "group" in the statement of the theorem, and thus modified the theorem may be applied to the multiplicative semi-group of an algebra to yield the desired result.

Henceforth, $K[X]$ is the K-algebra of all polynomials in one indeterminant, E the subalgebra of those without constant term. $K[X]$ has a base $\{e_i\}_{i=0}^{\infty}$ with multiplication table $e_i e_j = e_{i+j}$; $\{e_i\}_{i=1}^{\infty}$ is a base for E. For $\lambda \in K$ we let f_λ be the linear form defined on E by: $f_\lambda(e_j) = \lambda^j$. Also for every positive integer i, g_i is the linear form defined on E by: $g_i(e_i) = 1$, $g_i(e_j) = 0$ for $j \neq i$.

LEMMA 4. *The set of all multiplicative linear forms on E is $[f_\lambda \mid \lambda \in K]$.*

Proof. $f_\lambda(e_j e_k) = f_\lambda(e_{j+k}) = \lambda^{j+k} = \lambda^j \lambda^k = f_\lambda(e_j) f_\lambda(e_k)$. This suffices to show f_λ is multiplicative. Conversely, if f is any multiplicative linear form, let $\lambda = f(e_1)$. Then for any positive integer i, $f(e_i) = f(e_1^i) = f(e_1)^i = \lambda^i$. Hence $f = f_\lambda$.

LEMMA 5. $\{f_\lambda\}_{\lambda \in K, \lambda \neq 0} \cup \{g_i\}_{i=1}^{\infty}$ *is a linearly independent subset of E^*.*

Proof. Suppose $\sum_{i=1}^{n} \alpha_i g_i + \sum_{j=1}^{p} \beta_j f_{\lambda_j} = 0$, where the λ_j are distinct from each other and all different from zero. Then for $m > n$, $g_i(e_m) = 0$

for $1 \leq i \leq n$, so $\sum_{j=1}^{p} \beta_j f_{\lambda_j}(e_m)=0$. The subspace of E generated by $\{e_j\}_{j=n+1}^{\infty}$ is clearly a subalgebra; the restrictions of the f_{λ_j}, $1 \leq j \leq p$, to this algebra are again clearly distinct from each other and different from zero. Hence by Proposition 2 applied to this subalgebra, all $\beta_j=0$. Hence $\sum_{i=1}^{n} \alpha_i g_i=0$; but $\alpha_i=\alpha_i g_i(e_i)=\sum_{j=1}^{n} \alpha_j g_j(e_i)=0$, so the lemma is proved.

LEMMA 6. *Let $\{\lambda_i\}_{i=1}^{\infty}$ be a denumerable family of distinct nonzero elements of K. Then $\{f_{\lambda i}\}_{i=1}^{\infty}$ separates the points of E.*

Proof. For $\lambda \neq 0$, each f_λ has a unique extension to a multiplicative linear form on $K[X]$ obtained by setting $f_\lambda(e_0)=1$. Let $x=\sum_{i=1}^{n} \alpha_i e_i \in E$. Then $x=\sum_{i=0}^{n} \alpha_i e_i$ in $K[X]$ where $\alpha_0=0$. Suppose $f_{\lambda_j}(x)=0$ for $1 \leq j \leq n+1$. Then $\sum_{i=0}^{n} \alpha_i \lambda_j^i=0$ for $1 \leq j \leq n+1$. But the determinant of the system of linear equations $\sum_{i=0}^{n} \zeta_i \lambda_j^i=0$, $1 \leq j \leq n+1$, is

$$\begin{vmatrix} \lambda_1^0 & \lambda_1^1 & \cdots & \lambda_1^n \\ \lambda_2^0 & \lambda_2^1 & \cdots & \lambda_2^n \\ \vdots & \vdots & & \vdots \\ \lambda_{n+1}^0 & \lambda_{n+1}^1 & \cdots & \lambda_{n+1}^n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n+1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j < n+1} (\lambda_i - \lambda_j) \neq 0$$

(this is the Vandermonde determinant). Hence the above system of linear equations has only the trivial solution, and therefore $\alpha_i=0$, $0 \leq i \leq n$, and hence $x=0$. Thus the proof is complete.

Proposition 3. If L is any subset of K containing zero, there is a Hausdorff, weak locally m-convex topology $\mathcal{T}$ on E such that the canonical map $f_\lambda \rightarrow \lambda$ maps $\mathcal{M}(E)$ homeomorphically onto L. Further if L is an infinite set, $\mathcal{T}$ may be so chosen that the completion of E : $\mathcal{T}$ is semi-simple; and if L is denumerable, $\mathcal{T}$ is metrizable.

Proof. Case 1: L is finite. Let $M=[f_\lambda \mid \lambda \in L]$, and let E' be the subspace of E^* generated by $\{g_i\}_{i=1}^{\infty} \cup M$. Clearly E' is a total subspace of E^*, and so, as E' has a denumerable linear base, $\sigma(E, E')$ is a metrizable weak topology on E. To show $\sigma(E, E')$ is locally m-convex, it clearly suffices to show that the condition of Theorem 1 holds for all members of a base of E'. The condition holds trivially for all $u \in M$, since the kernel of $u \in M$ is already a weakly closed ideal. Consider any g_i: The linear subspace generated by $\{e_j\}_{j=i+1}^{\infty}$ is clearly of finite codimension, and the multiplication table shows that it is actually an ideal. Further, it is identical with $\bigcap_{k=1}^{i} g_k^{-1}(0)$ and thus is weakly closed and contained in the kernel of g_i. Hence by Theorem 1, $\sigma(E, E')$ is locally m-convex. By Lemma 5 the set of all multiplicative linear forms in E' is M. As the topological dual of E; $\sigma(E, E')$ is E' (see [7]), M is the set of all continuous multiplicative linear forms on E; $\sigma(E, E')$, and by Proposition 1 applied to $x=e_1$, M is homeomorphic with L.

Case 2: L is infinite. Again let $M=[f_\lambda \mid \lambda \in L]$, and let E' be the subspace of E^* generated by M. By Lemma 6, E' is total. The condition of Theorem 1 is trivially satisfied by E', so $\sigma(E, E')$ is a Hausdorff, weak locally m-convex topology on E. If L is denumerable, E' has a countable base and so $\sigma(E, E')$ is metrizable. M is again the set of all continuous multiplicative linear forms on E; $\sigma(E, E')$ and is homeomorphic with L. The completion of E for this topology is E'^* ([7]), and as M generates E', M separates the points of E'^*; thus the completion of E for this topology is semi-simple by Corollary 5.5 of [8].

It is easy to see that E has no divisors of zero and that zero is the only element having an adverse; thus the Jacobson radical is $\{0\}$ and E is semi-simple. If, in Proposition 3, $L=\{0\}$ and the scalar field is the complex numbers, E is a commutative, metrizable locally m-convex algebra with no continuous nonzero multiplicative linear forms; the completion $\hat{E}$ of E then has no continuous nonzero multiplicative linear forms and hence by Corollary 5.5 of [8] is a radical algebra. Thus we have an example of a semi-simple metrizable algebra whose completion is a radical algebra. This phenomenon is also known even for normed algebras. For example, an elementary calculation shows the following is a norm on E:

$$\left\| \sum_{n=1}^{m} \alpha_n e_n \right\| = \sum_{n=1}^{m} \frac{|\alpha_n|}{n!}.$$

$\|(m-1)!e_m\|=1/m\to 0$, so $(m-1)!\,e_m\to 0$ for this norm topology. But for any $\lambda\neq 0$, $|f_\lambda((m-1)!\,e_m)|=(m-1)!\,|\lambda|^m\to\infty$, so f_λ is not continuous. Hence E has no continuous nonzero multiplicative linear forms and so, assuming the scalar field is the complex numbers, the completion of E for this norm is a radical algebra.

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HARVARD UNIVERSITY

GROUP-THEORETIC ORIGIN OF CERTAIN GENERATING FUNCTIONS

LOUIS WEISNER

1. Introduction. A linear ordinary differential equation containing a parameter n may be written in the form

$$L(x, d/(dx), n)v=0\,. \tag{1.1}$$

Substituting $A=y\partial/(\partial y)$ for n, supposing the left member a polynomial in n, we construct the partial differential operator $L=L(x, d/(dx), A)$ on functions of two independent variables. This operator is independent of n and is commutative with A. A solution of the simultaneous equation $Lu=0$, $Au=nu$, where n is a constant, has the form $u=v_n(x)y^n$, where $v=v_n(x)$ is a solution of (1.1). Conversely, if $v=v_n(x)$ is a solution of (1.1), then $u=v_n(x)y^n$ is a solution of the equations $Lu=0$, $Au=nu$.

Now suppose that, independently of the preceding considerations, we have *obtained an explicit solution* $u=g(x, y)$ *of* $Lu=0$, and that from the properties of this function we know that it has an expansion in powers of y of the form

$$g(x, y)=\sum_n g_n(x)y^n\,, \tag{1.2}$$

where n is not necessarily an integer. If termwise operation with L on this series is permissible, then L annuls each term of the series, and $v=g_n(x)$ is a solution of (1.1). Thus $g(x, y)$ is a generating function for certain solutions of (1.1). The main problem is to find $g(x, y)$; its expansion is a detail of calculation.

It is difficult, in general, to find an explicit solution of $Lu=0$, other than an artificial superposition of the functions $v_n(x)y^n$, for which the generating function reduces to a tautology. However, if the equation admits a group of transformations besides $x'=x$, $y'=ty$ ($t\neq 0$), it is possible, in many cases, to find a solution which leads to a significant generating function of the form (1.2). In this paper it will be shown in detail how generating functions for the hypergeometric functions $F(-n, \beta; \gamma; x)$ may be obtained by this method. The Kummer functions ${}_1F_1(-n; \gamma; x)$ and ${}_1F_1(\alpha; n+1, x)$, the Bessel functions $J_n(x)$ and the Hermite functions $H_n(x)$ admit similar treatment. The point to be emphasized is that the generating functions so obtained owe their existence to the fact that the partial differential equation derived from the ordinary differential equation in the manner described above is invariant with respect to a nontrivial continuous group of transformations.

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2. The hypergeometric functions $F(-n, \beta; \gamma; x)$. Suppose that γ is not an integer; then the hypergeometric equation

$$x(1-x)\frac{d^2v}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{dv}{dx}-\alpha\beta v=0 \tag{2.1}$$

has the linearly independent solutions

$$v_1=F(\alpha, \beta; \gamma; x), \text{ and } v_2=x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; x).$$

A solution which is regular at $x=0$ is a constant multiple of v_1. Substituting $-y\partial/(\partial y)$ for α, so that $-\alpha$ plays the rôle of the parameter n of § 1, we construct the operator

$$L=x(1-x)\frac{\partial^2}{\partial x^2}+xy\frac{\partial^2}{\partial x\partial y}+\{\gamma-(\beta+1)x\}\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}.$$

Setting

$$A=y\frac{\partial}{\partial y}, \qquad B=y^{-1}\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right), \tag{2.2}$$
$$C=y\left\{x(1-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\gamma-\beta x\right\},$$

it may be verified that

$$xL=CB+A^2+(\gamma-1)A,$$

$$[A, B]=-B, \quad [A, C]=C, \quad [C, B]=2A+\gamma, \tag{2.3}$$

where $[A, B]=AB-BA$.

It follows from these relations, and may be verified by direct calculations, that xL is commutative with A, B and C and hence with $R=r_1A+r_2B+r_3C+r_4$, where the r's are arbitrary constants.

The commutator relations (2.3) show that the operators 1, A, B, C generate a Lie group Γ. The elements of this group may be represented in the form e^R, but are more conveniently expressed as products of a finite number of the operators e^{aA}, e^{bB}, e^{cC}, e^d, where a, b, c, d are constants. The operator 1 generates the multiplicative group of complex numbers, while A generates the group $x'=x$, $y'=ty$ $(t\neq 0)$. We shall use these two trivial groups for purposes of normalization. We find that

$$e^{cC}e^{bB}f(x, y)=(1-cy)^{\beta-\gamma}\{1+c(x-1)y\}^{-\beta}f(\xi, \eta), \tag{2.4}$$
$$\xi=\frac{xy}{\{1+c(x-1)y\}\{(1+bc)y-b\}}, \quad \eta=\frac{(1+bc)y-b}{1-cy},$$

where $f(x, y)$ is an arbitrary function. Since xL is commutative with

B and C, it follows that *if* $f(x, y)$ *is annulled by* L, *so is the right member of* (2.4).

3. Conjugates sets of generators of the first order. The main problem with which we shall be concerned is that of solving the simultaneous partial differential equations $Lu=0$, $Ru=0$, where $R=r_2A+r_2B+r_3C+r_4$, for all choices of the ratios of the coefficients except $r_1=r_2=r_3=0$. A great deal of labor is saved by the following observation: If S is an element of the group Γ of § 2, then SRS^{-1} has the same form as R; and if u is annulled by L and R, then Su is annulled by L and SRS^{-1}. It is therefore sufficient for our purpose to find the functions annulled by L and one operator from each of the conjugate classes into which the operators R fall with respect to Γ; and to apply Γ to these functions.

For any two linear operators X and Y with a common domain of operands, we have the formal expansion

$$e^{tX}Ye^{-tX}=\sum_{k=0}^{\infty}\frac{t^k}{k!}[X, Y]_k ,$$

where

$$[X, Y]_0=Y,\ [X, Y]_1=XY-YX, \text{ and } [X, Y]_k=[X, [X, Y]_{k-1}], \quad (k=2, 3,\cdots).$$

Hence, utilizing (2.3), we have

$$(3.1)\qquad e^{aA}Be^{-aA}=e^{-a}B , \qquad e^{aA}Ce^{-aA}=e^{a}C ,$$

$$(3.2)\qquad e^{bB}Ae^{-bB}=A+bB , \qquad e^{bB}Ce^{-bB}=-2bA-b^2B+C-b\gamma ,$$

$$(3.3)\qquad e^{cC}Ae^{-cC}=A-cC , \qquad e^{cC}Be^{-cC}=2cA+B-c^2C+c\gamma .$$

Despite the use of infinite series in the derivation of these operator identities, no questions of convergence arise in their application to an operand. An arbitrary function $f(x, y)$, whose partial derivatives of the first order exist, is converted by the left member of each of these identities into a function expressed in closed form with the aid of (2.4), while the right member involves only a finite number of terms.

From the preceding identities we have

$$(3.4)\qquad (e^{cC}e^{bB})A(e^{cC}e^{bB})^{-1}=(1+2bc)A+bB-c(1+bc)C+bc\gamma .$$

It follows that R is a conjugate of $\lambda A+\alpha$, for suitable choices of the constants λ and α, except when $r_1^2+4r_2r_3=0$. In that case it may be inferred from (3.3) and

$$(3.5)\qquad (e^{B}e^{-C})B(e^{B}e^{-C})^{-1}=-C$$

that R is a conjugate of $\lambda B+\alpha$. Since only the ratios of the coefficients of R are essential, we shall choose $\lambda=1$.

4. Generating functions. The general solution of the simultaneous partial differential equations $Lu=0$, $(A+\alpha)u=0$, is a linear combination, with constant coefficients, of

$$u_1=y^{-\alpha}F(\alpha, \beta ; \gamma ; x), \quad u_2=y^{-\alpha}x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1 ; 2-\gamma ; x).$$

It follows from (3.4) that the general solution of the equations

$$Lu=0, \quad \{(1+2bc)A+bB-c(1+bc)C+\alpha+bc\gamma\}u=0$$

is a linear combination, with constant coefficients, of

$$G_1(x, y)=(1-cy)^{\alpha+\beta-\gamma}\{(1+bc)y-b\}^{-\alpha}\{1+c(x-1)y\}^{-\beta}F(\alpha, \beta ; \gamma ; \xi),$$

$$G_2(x, y)=(xy)^{1-\gamma}(1-cy)^{\alpha+\beta-\gamma}\{(1+bc)y-b\}^{\gamma-\alpha+1}\{1+c(x-1)y\}^{\gamma-\beta+1} \cdot F(\alpha-\gamma+1, \beta-\gamma+1 ; 2-\gamma ; \xi),$$

where ξ is given by (2.4). It is sufficient to consider only G_1, as the expansion of G_2 in powers of y may be obtained from that of G_1 by simple substitutions.

If $b=0$, we normalize by choosing $c=1$, so that

$$G_1=y^{-\alpha}(1-y)^{\alpha+\beta-\gamma}\{1+(x-1)y\}^{-\beta}F\left(\alpha, \beta ; \gamma ; \frac{x}{1+(x-1)y}\right).$$

This function has an expansion of the form $\sum_{n=0}^{\infty} g_n(x)y^{n-\alpha}$. As noted in § 1, $g_n(x)$ must be a solution of (2.1) with α replaced by $\alpha-n$. Since $g_n(x)$ is regular at the origin it must be a constant multiple of $F(\alpha-n, \beta ; \gamma ; x)$. The constant is determined by setting $x=0$. Thus

$$(4.1) \qquad (1-y)^{\alpha+\beta-\gamma}\{1+(x-1)y\}^{-\beta}F\left(\alpha, \beta ; \gamma ; \frac{x}{1+(x-1)y}\right) = \sum_{n=0}^{\infty}\binom{\gamma-\alpha+n-1}{n}F(\alpha-n, \beta ; \gamma ; x)y^n,$$

where $x\neq 1$, $|y|<\min(1, |1-x|^{-1})$. The region of convergence is determined by examining the singularities of the left member.

Similarly, if $b=1$ and $c=0$, we obtain

$$(4.2) \qquad (1-y)^{-\alpha}F\left(\alpha, \beta ; \gamma ; \frac{-xy}{1-y}\right)=\sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}F(-n, \beta ; \gamma ; x)y^n,$$

$|y|<\min(1, |1-x|^{-1})$, (Feldheim [1. p. 120]).

If $bc\neq 0$, we set $b=-w^{-1}$, $c=1$, and obtain

$$(4.3)\qquad (1-y)^{\alpha+\beta-\gamma}\{1+(w-1)y\}^{-\alpha}\{1+(x-1)y\}^{-\beta}F(\alpha,\beta;\gamma;\zeta) = \sum_{n=0}^{\infty}\binom{\gamma+n-1}{n}F(-n,\alpha;\gamma;w)F(-n,\beta;\gamma;x)y^n,$$

$$\zeta=\frac{wxy}{\{1+(w-1)y\}\{1+(x-1)y\}},\quad 1-\zeta=\frac{(1-y)\{1-(w-1)(x-1)y\}}{\{1+(w-1)y\}\{1+(x-1)y\}},$$

$$|y|<\min(1,\ |1-x|^{-1},\ |1-w|^{-1},\ |1-x|^{-1}|1-w|^{-1}).$$

The required coefficient in the right member is readily obtained since the left member is unaltered by the permutation $(\alpha\beta)(wx)$. The special case $w=0$ is due to Feldheim [1, p. 120].

In accordance with the analysis of § 3, we next examine the simultaneous equations $Lu=0$, $Bu=-u$. The general solution of the latter is $u=e^y f(xy)$ and is annulled by L if $f(X)$ satisfies the ordinary differential equation

$$X\frac{d^2f}{dX^2}+(\gamma+X)\frac{df}{dX}+\beta f=0.$$

Comparing with Kummer's equation

$$x\frac{d^2v}{dx^2}+(\gamma-x)\frac{dv}{dx}-\alpha v=0,$$

it follows that u is a linear combination, with constant coefficients, of

$$(4.4)\qquad u_1=e^yF(\beta;\gamma;-xy),\quad u_2=e^y(-xy)^{1-\gamma}F(\beta-\gamma+1;2-\gamma;-xy),$$

where the customary indices in Kummer's function have been omitted. The first of these functions is regular at $x=0$, and we obtain

$$(4.5)\qquad e^yF(\beta;\gamma;-xy)=\sum_{n=0}^{\infty}\frac{1}{n!}F(-n,\beta;\gamma;x)y^n$$

(Humbert [2, p. 64]).

Substituting $-wy$ for y in the first of the functions (4.4), we obtain $e^{-wy}F(\beta;\gamma;wxy)$ which is annulled by L and $B-w$. Operating on this function with e^C, as suggested by (3.3) with $c=1$, we obtain the left member of

$$(4.6)\qquad (1-y)^{\beta-\gamma}\{1+(x-1)y\}^{-\beta}\exp\left\{\frac{-wy}{1-y}\right\}F\left(\beta;\gamma;\frac{wxy}{(1-y)\{1+(x-1)y\}}\right) = \sum_{n=0}^{\infty}L_n^{(\gamma-1)}(w)F(-n,\beta;\gamma;x)y^n,\qquad |y|<\min(1,\ |1-x|^{-1}).$$

The required coefficient in the right member is obtained by setting $x=0$ and comparing with a known generating function for the Laguerre polynomials (Szegö [4, p. 97]).

By (3.5) and (4.4)

$$e^{B}e^{-C}\{e^{y}F(\beta;\gamma;-xy)\}=\exp\{1-y^{-1}\}y^{-\gamma}(1-x)^{-\beta}F\left(\beta;\gamma;\frac{-x}{y(1-x)}\right)$$

is the only linearly independent solution of $Lu=0$, $Cu=u$ that is regular at $x=0$. Its expansion, when simplified, reads

$$e^{y}(1-x)^{-\beta}F\left(\beta;\gamma;\frac{xy}{1-x}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}F(\gamma+n,\beta;\gamma;x)y^{n}, \qquad (x\neq 1). \tag{4.7}$$

We have now obtained, in normalized form, a solution of the simultaneous equations $Lu=0$, $(r_1A+r_2B+r_3C+r_4)u=0$ for each admissible choice of the r's, and its expansion in powers of y. Most of the solutions are expansible in other regions of the y-plane than those noted ; hence other generating functions may be obtained. For example, the left member of (4.2) may be written

$$y^{-\alpha}(1-y^{-1})^{-\alpha}F\left(\alpha,\beta;\lambda;\frac{x}{1-y^{-1}}\right),$$

except for a numerical factor. This function has an expansion valid for $|y|>1$. The result, when simplified by cancelling $y^{-\alpha}$ and replacing y by y^{-1}, reads

$$(1-y)^{-\alpha}F\left(\alpha,\beta;\gamma;\frac{x}{1-y}\right)=\sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}F(\alpha+n,\beta;\gamma;x)y^{n},$$
$$|y|<\min(1,|1-x|).$$

5. **Application to ultraspherical polynomials.** Special cases of some of the preceding formulas, involving ultraspherical polynomials, may be obtained by means of the representation

$$P_n^{(\lambda)}(\cos\theta)=\binom{2\lambda+n-1}{n}e^{-ni\theta}F(-n,\lambda;2\lambda;1-e^{2i\theta}),$$

which may be established by comparing the differential equations satisfied by the two members of this equation. Thus we obtain from (4.3)

$$\{1-2y\cos(\theta-\varphi)+y^2\}^{-\lambda}F\left(\lambda,\lambda;2\lambda;\frac{-4y\sin\theta\sin\varphi}{1-2y\cos(\theta-\varphi)+y^2}\right)$$
$$=\sum_{n=0}^{\infty}\binom{2\lambda+n-1}{n}P_n^{(\lambda)}(\cos\varphi)P_n^{(\lambda)}(\cos\theta)y^{n}, \qquad |y|<|e^{i(\pm\varphi\pm\theta)}|.$$

Ossicini [3] expresses the left member in terms of Legendre functions of the second kind, while Watson [5] expresses the result, for $\lambda=1/2$, in terms of elliptic integrals.

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