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**REMARK ON THE PRECEDING PAPER. ALGEBRAIC
EQUATIONS SATISFIED BY ROOTS OF NATURAL NUMBERS**

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REMARK ON THE PRECEDING PAPER
ALGEBRAIC EQUATIONS SATISFIED BY ROOTS OF
NATURAL NUMBERS

E. G. STRAUS AND O. TAUSKY

In the preceding paper [1] it was shown that the polynomials in question are factors of $\Phi_n(x^k/n)$ where Φ_n is the cyclotomic polynomial of order n and k, n are positive integers. The case $k=2$ was settled in [1, Lemma 2]. It will now be shown that this is essentially the only nontrivial case. For a different treatment of a somewhat related question see K. T. Vahlen [2].

First let us remark that we can exclude the case $n=m^d$ where $d/k, d > 1$; since we may then set $y=x^{k/d}/m$ so that $\Phi_n(y^d)$ is either reducible with cyclotomic factors or equal to $\Phi_{na}(y)$. We shall refer to n and $\Phi_n(x^k/n)$ which satisfy the above exclusion as *simplified*.

THEOREM. *The simplified polynomial $\Phi_n(x^k/n)$ is irreducible for all odd k . For $k=2l$ the polynomial is reducible if and only if $\Phi_n(x^2/n)$ is reducible. In that case we have*

$$(1) \quad \Phi_n(x^k/n) = g(x^l)g(-x^l),$$

where the polynomials on the right are irreducible.

The proof is based on the following lemma.

LEMMA. *If $k > 2$ and $n^{1/k}$ is simplified then $n^{1/k}$ is not contained in a cyclotomic field.*

Proof. The Galois group of a cyclotomic field $R(\zeta)$ is Abelian and hence all subfields of $R(\zeta)$ are normal. The field $R(n^{1/k})$ is, however, not a normal field for $k > 2$.

We can now prove the Theorem. Let ζ_n be a primitive n th root of unity. A zero ω of a simplified $\Phi_n(x^k/n)$ is a zero of

$$(2) \quad x^k - n\zeta_n$$

and hence $R(\omega)$ is an algebraic extension of $R(\zeta_n)$. If the degree of $R(\omega)$ over $R(\zeta_n)$ were k then its degree over R would be kn . Hence $\Phi_n(x^k/n)$ is reducible if and only if (2) is reducible over $R(\zeta_n)$. Say

$$(3) \quad x^k - n\zeta_n = F(x)G(x) \quad F, G \in R(\zeta_n)[x].$$

Since all the roots of (2) are of the form $n^{1/k}\zeta_{kn}^s$ we have

$$F(0) = n^{l/k} \zeta \in R(\zeta_n) \qquad l = \deg F$$

where ζ is a root of unity. In other words

$$(4) \qquad n^{l/k} \in R(\zeta_n, \zeta) = R(\zeta')$$

where ζ' is a root of unity.

According to the lemma (4) is impossible if the reduced fraction $1/k$ has denominator > 2 . For k odd this means $l=0$ or k and $\Phi_n(x^k/n)$ irreducible. For k even and $0 < l < k$ we can have only $l=k/2$. In this case

$$F(0) = \pm n^{1/2} \zeta_{nk}^s, \quad G(0) = \pm n^{1/2} \zeta_{nk}^t;$$

and since both $F(0)G(0)$ and $F(0)/G(0)$ are in $R(\zeta_n)$ we obtain

$$s + t \equiv s - t \equiv 0 \pmod{k}.$$

Hence $s \equiv t \equiv 0 \pmod{l}$ so that

$$(5) \qquad F(0) = \sqrt{n \zeta_n^u} \in R(\zeta_n).$$

But we noted in [1, Lemma 1] that (5) is necessary and sufficient for the reducibility of $\Phi_n(x^2/n)$. Thus we have

$$\Phi_n(x^2/n) = g(x)g(-x) \text{ and therefore}$$

$$\Phi_n(x^k/n) = g(x^l)g(-x^l)$$

as the complete factorization of $\Phi_n(x^k/n)$ over $R[x]$.

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