THE SYMMETRY FUNCTION IN A CONVEX BODY

SHERMAN K. STEIN
THE SYMMETRY FUNCTION IN A CONVEX BODY

S. Stein

Let $K_n$ be an $n$-dimensional convex body in $n$-dimensional Euclidean space $E_n$. At each point $P$ in $K_n$ consider the largest subset $S(P)$ of $K_n$ radially symmetric with respect to the point $P$. This set is well-defined and convex for it is simply the intersection of $K_n$ with its radial reflection through the point $P$. Let $m(P)$ equal the measure of $S(P)$ and let $f(P)$ equal $m(P) V_n^{-1}$ where $V_n$ is the measure of $K_n$. Clearly $0 \leq f(P) \leq 1$ for all $P$ in $K_n$ and $f(P)=0$ only if $P$ is on the boundary of $K_n$; also $f$ is continuous. Moreover $f$ attains the value 1 only if $K_n$ is radially symmetric. The object of this note is to present various properties of this function $f$.

THEOREM 1. (Besicovitch [1], $n=2$). There is a point $P$ in $K_2$ such that $f(P)=2/3$. (In [3, p. 46] this theorem is ascribed to S. S. Konvyer.)

THEOREM 2. (Besicovitch [2], $n=2$). If $K_2$ is of constant width then there is a point $P$ in $K_2$ such that $f(P)=.840\cdots$.

H. G. Eggleston [4] studied further the symmetric function in a body of constant width.

Using a result of P. C. Hammer [5] on the ratio which the centroid of a convex body divides the chords passing through it, F. W. Levi [6] obtained the following.

THEOREM 3. If $P$ is the centroid of $K_n$ then

$$f(P) \geq 2(1+n^{-1})^{-1}.$$ 

The following properties of $f$ will be obtained.

THEOREM 4. $\int_{K_n} f = 2^{-n} V_n$.

COROLLARY. There is a point $P$ in $K_n$ such that $f(P) > 2^{-n}$.

THEOREM 5. If $a$ is a real number then the set of points $P$ in $K_n$ at which $f(P) \geq a$ is convex. Furthermore $f$ attains its maximum value at precisely one point.

Received Nov. 17, 1953, and in revised forms March 19, 1954, and Oct. 4, 1954.
COROLLARY (to proof of Theorem 5, suggested by referee). If 0 ≤ λ ≤ 1 and P and Q are in Kn then
\[ f(\lambda P + (1 - \lambda)Q) \geq \lambda f(P) + (1 - \lambda)f(Q). \]

THEOREM 6. If Kn is an n-dimensional simplex and P is its centroid, then f attains its maximum at P and f(P) = 2(n + 1)^{-1}.

Proof of Theorem 4. Consider the set of points
\[ K_{2n} = \{(P, Q) | P \in K_n, Q \in S(P)\}. \]
In a straightforward manner this set can be shown to be convex and hence measurable. By Fubini's theorem on the relation between iterated and multiple integrals, the volume \( V_{2n} \) of \( K_{2n} \) is seen to equal \( \int_{K_n} m \) and also \( \int_{K_n} h \) where \( h(Q) \) denotes the measure of the cross section of \( K_{2n} \) defined by
\[ \{(P, Q) | (Q \in \mathcal{A}), S(P) \not\subset Q\}. \]
Now \( S(P) \not\subset Q \) only if P is less than half way from Q to the boundary of \( K_n \) along the line determined by P and Q. Thus \( h(Q) = 2^{-n}V_n \) independently of Q [7, p. 38]. Thus
\[ \int_{K_n} f = V_n^{-1}\int_{K_n} h = V_n^{-1}2^{-n}(V_n)^2 = 2^{-n}V_n. \]

Proof of Corollary to Th. 4. Since the average value of f on \( K_n \) is 2^{-n} and since f(P) < 2^{-n} on (and near) the boundary of \( K_n \) there must be a point at which f exceeds 2^{-n}.

Proof of Theorem 5. Let P and Q be distinct points of \( K_n \) such that f(P) = f(Q). We shall show\(^1\) that f((P+Q)/2) > f(P). This fact, combined with the fact that \{P | f(P) ≥ a\} is closed, would prove the theorem. Consider the convex body \((S(P)+S(Q))/2\). This body is symmetric, and, if so translated that \((P+Q)/2\) is its center, lies within \( K_n \). By the Brunn-Minkowski theorem [7, p. 88] the measure of this set is strictly larger than m(P) if S(P) is not congruent to S(Q) by a translation. If S(P) is congruent to S(Q) by a translation, consider the convex hull of the set union of S(P) and S(Q). This set is clearly symmetric with respect to the point \((P+Q)/2\), lies in \( K_n \), and has a measure greater than m(P). Thus f((P+Q)/2) > f(P) = f(Q).

Proof of Corollary to Th. 5. A continuous function which satisfies

\(^1\) If P and Q are on the boundary of \( K_n \) it may happen that f((P+Q)/2) = f(P)\).
for \( \lambda = 1/2 \) and all \( P, Q \) in a line segment satisfies the inequality for all \( \lambda, 0 \leq \lambda \leq 1 \), and \( P, Q \), in the line segment.

**Proof of Theorem 6.** Since affine transformations preserve symmetry, centroids, and ratio of volumes it will be sufficient to consider the case where \( K_n \) is regular.

Let \( Q \) be the point in \( K_n \) maximizing \( f \). If \( T \) is an orthogonal transformation interchanging two of the vertices of \( K_n \), and leaving the remaining vertices fixed then \( f(Q) = f(T(Q)) \). Thus, by Theorem 5, \( T(Q) = Q \). Since this is true for each pair of vertices of \( K_n \), \( Q \) must be equidistant from all the vertices of \( K_n \). Thus \( Q = P \).

Now to compute \( f(P) \).

Let \( K'_n \) be the reflection of \( K_n \) through \( P \) of altitude \( h \) and volume \( V \). The boundary of \( K_n \cap K'_n \) is readily seen to be composed of \( 2(n+1) \) congruent \( n-1 \) dimensional sets \( B_i \), \( 1 \leq i \leq 2(n+1) \) each of volume \( V^* \). Let \( S \) denote the volume of \( K_n \cap K'_n \).

Considering \( K_n \cap K'_n \) as being composed of \( 2(n+1) \) congruent joins with the common vertex \( P \), bases \( B_i \), and altitude \( h(n+1)^{-1} \) one obtains

\[
S = 2(n+1)h(n+1)^{-1}V^*n^{-1}.
\]

On the other hand, considering \( K_n \cap K'_n \) as being obtained from \( K_n \) by the removal of \( n+1 \) congruent sets, each of which is a join of a vertex of \( K_n \) with a \( B_i \) and has an altitude \( (n-1)(n+1)^{-1}h \), one obtains

\[
S = V - (n+1)(n-1)(n+1)^{-1}hV^*n^{-1}.
\]

Elimination of the product \( hV^* \) from (1) and (2) yields

\[
S = 2(n+1)^{-1}V
\]

and thus

\[
f(P) = 2(n+1)^{-1}.
\]

**References**

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>David Blackwell</td>
<td>An analog of the minimax theorem for vector payoffs</td>
<td>1</td>
</tr>
<tr>
<td>L. W. Cohen</td>
<td>A non-archimedian measure in the space of real sequences</td>
<td>9</td>
</tr>
<tr>
<td>George Bernard Dantzig</td>
<td>Constructive proof of the Min-Max theorem</td>
<td>25</td>
</tr>
<tr>
<td>Jim Douglas</td>
<td>On the numerical integration of quasilinear parabolic differential equations</td>
<td>35</td>
</tr>
<tr>
<td>James Michael Gardner Fell</td>
<td>A note on abstract measure</td>
<td>43</td>
</tr>
<tr>
<td>Isidore Isaac Hirschman, Jr.</td>
<td>A note on orthogonal systems</td>
<td>47</td>
</tr>
<tr>
<td>Frank Harary</td>
<td>On the number of dissimilar line-subgraphs of a given graph</td>
<td>57</td>
</tr>
<tr>
<td>Newton Seymour Hawley</td>
<td>Complex bundles with Abelian group</td>
<td>65</td>
</tr>
<tr>
<td>Alan Jerome Hoffman, Morris Newman, Ernst Gabor Straus and Olga Taussky</td>
<td>On the number of absolute points of a correlation</td>
<td>83</td>
</tr>
<tr>
<td>Ernst Gabor Straus and Olga Taussky</td>
<td>Remark on the preceding paper. Algebraic equations satisfied by roots of natural numbers</td>
<td>97</td>
</tr>
<tr>
<td>Ralph D. James</td>
<td>Summable trigonometric series</td>
<td>99</td>
</tr>
<tr>
<td>Gerald R. Mac Lane</td>
<td>Limits of rational functions</td>
<td>111</td>
</tr>
<tr>
<td>F. Oberhettinger</td>
<td>Note on the Lerch zeta function</td>
<td>117</td>
</tr>
<tr>
<td>Gerald C. Preston</td>
<td>On locally compact totally disconnected Abelian groups and their character groups</td>
<td>121</td>
</tr>
<tr>
<td>Vikramaditya Singh and W. J. Thron</td>
<td>On the number of singular points, located on the unit circle, of certain functions represented by C-fractions</td>
<td>135</td>
</tr>
<tr>
<td>Sherman K. Stein</td>
<td>The symmetry function in a convex body</td>
<td>145</td>
</tr>
<tr>
<td>Edwin Weiss</td>
<td>Boundedness in topological rings</td>
<td>149</td>
</tr>
<tr>
<td>Albert Leon Whiteman</td>
<td>A sum connected with the series for the partition function</td>
<td>159</td>
</tr>
<tr>
<td>Alfred B. Willcox</td>
<td>Some structure theorems for a class of Banach algebras</td>
<td>177</td>
</tr>
<tr>
<td>Joseph Lawrence Zemmer</td>
<td>Some remarks on p-rings and their Boolean geometry</td>
<td>193</td>
</tr>
</tbody>
</table>