CORRESPONDING RESIDUE SYSTEMS IN ALGEBRAIC
NUMBER FIELDS

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In this paper we shall consider integral ideals in finite algebraic
extensions of the field \( R \) of rational numbers. Algebraic number fields
will be denoted by \( \mathcal{F} \) with subscripts or superscripts, ideals by German
letters, algebraic numbers by lower case Greek letters, and numbers
of the rational field \( R \) by lower case Latin letters.

Two ideals in the same field are equal if and only if they contain
the same numbers.

If \( \alpha_i \) is an ideal in a field \( \mathcal{F}_i \) and \( \alpha_j \) is an ideal in a field \( \mathcal{F}_j \), then
we shall write \( \alpha_i \equiv \alpha_j \) provided \( \alpha_i \) and \( \alpha_j \) generate the same ideal in some
field containing all the numbers of \( \mathcal{F}_i \) and of \( \mathcal{F}_j \) (see [1, § 37]). Two such
ideals may therefore be denoted by the same symbol and we shall speak
of an ideal \( \alpha \) without regard to a particular field. An ideal \( \alpha \) is said to
be contained in a field \( \mathcal{F} \) if it may be generated by numbers in \( \mathcal{F} \),
that is to say, if it has a basis in \( \mathcal{F} \).

Let \( \alpha \) be an ideal contained in the fields \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). We say that
\( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems modulo \( \alpha \) if for every in-
teger \( \alpha_i \) of \( \mathcal{F}_1 \), there exists an integer \( \alpha_j \) of \( \mathcal{F}_2 \) such that
\( \alpha_i \equiv \alpha_j \pmod{\alpha} \), and for every integer \( \alpha_j \) of \( \mathcal{F}_2 \), there exists an integer \( \alpha_i \) of \( \mathcal{F}_1 \) such
that \( \alpha_i \equiv \alpha_j \pmod{\alpha} \).

The problem considered in this paper is the following one: if \( \mathcal{F}_1 \)
and \( \mathcal{F}_2 \) are two fields containing an ideal \( \alpha \), under what conditions will
\( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( \alpha \). We shall show
that this problem reduces to that in which the ideal \( \alpha \) is a power of a
prime ideal and a necessary and sufficient condition for \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) to
have corresponding residue systems mod \( \alpha \) is derived in case that \( \alpha \) is
a prime ideal. A necessary (but not sufficient) condition is derived in
case \( \alpha \) is a power of a prime ideal and \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are normal over \( \mathcal{F}_1 \cap
\mathcal{F}_2 \). A special case in which the fields are of the type \( \mathcal{F}(\sqrt{\mu}) \) is con-
sidered. These fields are of interest in themselves and in view of
Corollary 7.1 seem to have a direct connection with the general
problem.

**Theorem 1.** Let \( \alpha \) be an ideal in the number fields \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) and
suppose \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( \alpha \). Then \( \alpha \)
has the same prime ideal decomposition in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \).

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Proof. Let
\[ \alpha = \psi_1 \cdots \psi_r \text{ in } \mathcal{F}_1 \]
\[ \alpha = \phi_1 \cdots \phi_s \text{ in } \mathcal{F}_2 \]
where the \( \psi_i \) are prime ideals in \( \mathcal{F}_1 \) and the \( \phi_i \) are prime ideals in \( \mathcal{F}_2 \).

Let \( \alpha \) be an integer in \( \mathcal{F}_1 \) such that \( \alpha \) is exactly divisible by \( p_\lambda \) and \( (\alpha, \alpha_i) = (l) \) for \( i = 2, \ldots, r \).

There exists an integer \( \beta \) in \( \mathcal{F}_2 \) such that \( \alpha \equiv \beta \pmod{\alpha} \) and thus in \( \mathcal{F}_1 \cup \mathcal{F}_2 \), we have \( (\beta, \alpha) = p_i \). Since \( \beta \) is in \( \mathcal{F}_2 \) and \( \alpha \subset \mathcal{F}_2 \), it follows that \( p_i \subset \mathcal{F}_2 \). In the same manner it follows that \( p_i \subset \mathcal{F}_2 \) for \( i = 1, \ldots, r \) and \( q_i \subset \mathcal{F}_1 \) for \( i = 1, \ldots, s \). Therefore in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \) we have \( \psi_1 \cdots \psi_r = q_1 \cdots q_s \).

In \( \mathcal{F}_1 \) the \( q_i \) are prime ideals and hence \( q_i | p_i \) in \( \mathcal{F}_1 \) for some \( j \). In \( \mathcal{F}_2 \) the \( p_i \) are prime ideals and therefore \( p_k | q_i \) in \( \mathcal{F}_1 \) for some \( k \). Thus in \( \mathcal{F}_1 \cup \mathcal{F}_2 \) we have \( p_k | p_i \) which implies that \( p_k = p_j = q_i \) in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \).

By renumbering and repeated application of the above argument we obtain \( r = s \) and \( p_i = q_i \) for \( i = 1, \ldots, r = s \) in \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \).

**Theorem 2.** Let \( \alpha \) be an ideal in the number fields \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). In order that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( \alpha \) it is necessary and sufficient that \( \alpha = \psi_1 \cdots \psi_r \) where \( \psi_i \) is a prime ideal in \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), and \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( \psi_i \) for \( i = 1, \ldots, r \).

**Proof.** The necessity follows from Theorem 1. Suppose \( \alpha = \psi_1 \cdots \psi_r \) in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \), where \( \psi_i \) is a prime ideal in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \), and that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( \psi_i \) for \( i = 1, \ldots, r \). Let \( \alpha \) be any integer of \( \mathcal{F}_1 \). There exist integers \( \beta_i \) in \( \mathcal{F}_2 \) such that \( \alpha \equiv \beta_i \pmod{\psi_i} \) for \( i = 1, \ldots, r \). By the Chinese remainder theorem there exists an integer \( \beta \) in \( \mathcal{F}_2 \) such that \( \beta \equiv \beta_i \pmod{\psi_i} \) for \( i = 1, \ldots, r \) and hence \( \alpha \equiv \beta \pmod{\alpha} \). It follows that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( \alpha \).

**Theorem 3.** Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two number fields, \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \), and let \( \mathcal{F} \) be a prime ideal in both \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). Suppose \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( \mathcal{F} \) and let \( \mathcal{F}_n \) be the smallest normal extension over \( \mathcal{F} \) containing \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). Then for every automorphism \( A \) in the Galois group \( \text{Gal}(\mathcal{F}_n/\mathcal{F}) \) of \( \mathcal{F}_n \) over \( \mathcal{F} \) we have \( \alpha_1 = \alpha_2 \pmod{\mathcal{F}} \) and \( \alpha_2 = \alpha_2 \pmod{\mathcal{F}_n} \) for every integer \( \alpha_1 \) in \( \mathcal{F}_1 \) and \( \alpha_2 \) in \( \mathcal{F}_2 \).

**Proof.** Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be the subgroups of \( \text{Gal}(\mathcal{F}_n/\mathcal{F}) \) which leave \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) fixed respectively. Since \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \), we have by Galois theory that \( \mathcal{G}_1 \cup \mathcal{G}_2 \) corresponds to \( \mathcal{F} \) under the Galois correspondence between subgroups and subfields. Hence \( \mathcal{G}_1 \cup \mathcal{G}_2 = \mathcal{G}(\mathcal{F}_n/\mathcal{F}) \).
Denote by \( \mathfrak{S}_i \) \((i=1, 2)\) the set of automorphisms \( A \) in \( \mathfrak{S}(\mathfrak{F}_n|\mathfrak{F}) \) such that \( \alpha_i^2 \equiv \alpha_i \pmod{\mathfrak{p}_j} \) for all integers \( \alpha_i \) in \( \mathfrak{F}_i \) for \( i=1, 2 \). The sets \( \mathfrak{S}_i \) are subgroups of \( \mathfrak{S}(\mathfrak{F}_n|\mathfrak{F}) \). Furthermore the sets \( \mathfrak{S}_i \) contain \( \mathfrak{S}_i \) for \( i=1, 2 \).

Let \( A \) be an automorphism of \( \mathfrak{S}_2 \). For every integer \( \alpha_i \) in \( \mathfrak{F}_i \) there exists an integer \( \alpha_i \) in \( \mathfrak{F}_i \) such that \( \alpha_i \equiv \alpha_i \pmod{\mathfrak{p}_j} \). Therefore \( (\alpha_i - \alpha_i)^2 \equiv 0 \pmod{\mathfrak{p}_j} \), \( \alpha_i^2 \equiv \alpha_i \pmod{\mathfrak{p}_j} \), and thus \( \alpha_i^2 \equiv \alpha_i \pmod{\mathfrak{p}_j} \) for \( i=1 \) and \( \mathfrak{S}_i \subset \mathfrak{S}_2 \). Hence the automorphism \( A \) is also in \( \mathfrak{S}_i \) and it follows that \( \mathfrak{S}_i \subset \mathfrak{S}_i \). Similarly \( \mathfrak{S}_i \subset \mathfrak{S}_2 \) and therefore \( \mathfrak{S}_i = \mathfrak{S}_2 \). Hence \( \mathfrak{S}_i \) contains \( \mathfrak{S}_i \) for \( i=1, 2 \) and \( \mathfrak{S}_2 \cup \mathfrak{S}_i = \mathfrak{S}(\mathfrak{F}_n|\mathfrak{F}) \).

**Corollary 3.1.** Under the conditions of Theorem 3 it follows that \( d_i \equiv 0 \pmod{\mathfrak{p}_j} \) and \( d_i \equiv 0 \pmod{\mathfrak{p}_j} \), where \( n_i + 1 = (\mathfrak{F}_i|\mathfrak{F}) \), \( n_2 + 1 = (\mathfrak{F}_2|\mathfrak{F}) \), and \( d_i \) denotes the relative differente of \( \mathfrak{F}_i \) over \( \mathfrak{F} \) for \( i=1, 2 \).

**Theorem 4.** Let \( \mathfrak{F}_i \subset \mathfrak{F}_i \) be two number fields and let \( \mathfrak{p} \) be a prime ideal in \( \mathfrak{F}_i \). Suppose that for every integer \( \alpha \) in \( \mathfrak{F}_i \) we have \( \alpha \equiv \alpha_i \pmod{\mathfrak{p}_j} \) for \( i = 1, \ldots, k = (\mathfrak{F}_i|\mathfrak{F}) \), where \( \alpha_i \) is the \( i \)th conjugate of \( \alpha \) in \( \mathfrak{F}_i \) over \( \mathfrak{F} \). Then \( \mathfrak{p} \) is of order \( k = (\mathfrak{F}_i|\mathfrak{F}) \) with respect to \( \mathfrak{F}_i \).

**Proof.** It is clear that \( \mathfrak{p} \) coincides with its conjugates. Moreover if \( \alpha \) is any integer in \( \mathfrak{F}_i \) and \( \alpha_1, \ldots, \alpha_k \) its conjugates over \( \mathfrak{F}_i \) then

\[
f(x) = (x - \alpha)(x - \alpha_1) \cdots (x - \alpha_k) \equiv (x - \alpha)^k \pmod{\mathfrak{p}_j}.
\]

The polynomial \( f(x) \) has its coefficients in \( \mathfrak{F}_i \) and since the field of residue classes mod \( \mathfrak{p}_j \) is separable over the field or residue classes mod \( \mathfrak{p}_j \), it must be of degree one.

**Theorem 5.** Let \( \mathfrak{F}_i \) and \( \mathfrak{F}_i \) be two number fields and \( \mathfrak{p} \) a prime ideal in both fields. Then \( \mathfrak{F}_i \) and \( \mathfrak{F}_i \) have corresponding residue systems mod \( \mathfrak{p}_j \) if and only if \( \mathfrak{p}_j \) is of order \( (\mathfrak{F}_i|\mathfrak{F}_i \cap \mathfrak{F}_i) \) in \( \mathfrak{F}_i \) over \( \mathfrak{F}_i \cap \mathfrak{F}_i \) and of order \( (\mathfrak{F}_i|\mathfrak{F}_i \cap \mathfrak{F}_i) \) in \( \mathfrak{F}_i \) over \( \mathfrak{F}_i \cap \mathfrak{F}_i \).

**Proof.** If \( \mathfrak{F}_i \) and \( \mathfrak{F}_i \) have corresponding residue systems mod \( \mathfrak{p}_j \), it follows immediately from Theorems 3 and 4 that the order of \( \mathfrak{p}_j \) satisfies the conditions of the theorem.

The converse is clear since \( \mathfrak{p}_j \) is of degree one over \( \mathfrak{F}_i \cap \mathfrak{F}_i \) and therefore every residue class mod \( \mathfrak{p}_j \) contains an integer of \( \mathfrak{F}_i \cap \mathfrak{F}_i \).

**Corollary 5.1.** Let \( \alpha \) be an ideal in the number fields \( \mathfrak{F}_i \) and \( \mathfrak{F}_i \). If \( \mathfrak{F}_i \) and \( \mathfrak{F}_i \) have corresponding residue systems mod \( \mathfrak{p}_j \), then \( (\mathfrak{F}_i|\mathfrak{F}_i \cap \mathfrak{F}_i) = (\mathfrak{F}_i|\mathfrak{F}_i \cap \mathfrak{F}_i) \).
THEOREM 6. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two number fields each normal over \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \) and let \( p \) be a prime ideal in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \). In order that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( p \) it is necessary and sufficient that the inertial group of \( p \) in \( \mathcal{F}_1 \) over \( \mathcal{F} \) be equal to the Galois group of \( \mathcal{F}_1 \) over \( \mathcal{F} \) for \( j=1,2 \).

Proof. The condition is sufficient since \( p \) is of degree one in \( \mathcal{F}_j \) over \( \mathcal{F} \) if the inertial group of \( p \) in \( \mathcal{F}_j \) over \( \mathcal{F} \) is equal to the Galois group of \( \mathcal{F}_j \) over \( \mathcal{F} \) for \( j=1,2 \).

Suppose \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( p \) and let \( \mathcal{F}_i \) denote the inertial field of \( p \) in \( \mathcal{F}_i \) over \( \mathcal{F} \). The order of \( p \) in \( \mathcal{F}_i \) over \( \mathcal{F} \) is equal to \( (\mathcal{F}_i|\mathcal{F}) \) and hence by Theorem 5 we have \( (\mathcal{F}_i|\mathcal{F}) = (\mathcal{F}_i|\mathcal{F}_i) \). It follows that \( \mathcal{F}_i = \mathcal{F} \) and hence the Galois group of \( \mathcal{F}_i \) over \( \mathcal{F} \) is equal to the inertial group of \( p \) in \( \mathcal{F}_i \) over \( \mathcal{F} \).

THEOREM 7. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two number fields each normal over \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \) and let \( p \) be a prime ideal in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \). If \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( p^j \) then the \( j^{th} \) ramification group of \( p \) in \( \mathcal{F}_k \) over \( \mathcal{F} \) is equal to the Galois group of \( \mathcal{F}_k \) over \( \mathcal{F} \) for \( k=1,2 \).

Proof. Let \( A \) be any automorphism of \( \mathcal{G}(\mathcal{F}_1 \cup \mathcal{F}_2|\mathcal{F}) \). It follows from Theorem 3 that \( \alpha_i^j = \alpha_i \pmod{p^j} \) for every integer \( \alpha_i \) in \( \mathcal{F}_i \) for \( i=1,2 \). Hence if \( A_i \) is an automorphism of \( \mathcal{G}(\mathcal{F}_i|\mathcal{F}) \), \( (i=1,2) \), it follows that \( \alpha_i^j = \alpha_i \pmod{p^j} \) since every automorphism \( A_i \) of \( \mathcal{G}(\mathcal{F}_i|\mathcal{F}) \) can be continued to an automorphism of \( \mathcal{G}(\mathcal{F}_1 \cup \mathcal{F}_2|\mathcal{F}) \). Thus the \( j^{th} \) ramification group of \( p \) in \( \mathcal{F}_i \) over \( \mathcal{F} \) is equal to the Galois group of \( \mathcal{F}_i \) over \( \mathcal{F} \) for \( i=1,2 \).

COROLLARY 7.1. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two number fields normal over \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \) and let \( p \) be a prime ideal in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \). If \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have corresponding residue systems mod \( p^j \) for \( j>1 \), then \( (\mathcal{F}_1|\mathcal{F}) = (\mathcal{F}_2|\mathcal{F}) = p^e \) where \( p \) is the rational prime belonging to \( p \).

Proof. By Theorem 7 we have \( \mathcal{G}(\mathcal{F}_1|\mathcal{F}) = \mathcal{G}_1 = \cdots = \mathcal{G}_r \) where \( \mathcal{G}_r \) is the \( j^{th} \) ramification group of \( p \) in \( \mathcal{F}_1 \) over \( \mathcal{F} \). By Theorem 5 the order \( e \) of \( p \) in \( \mathcal{F}_1 \) over \( \mathcal{F} \) is equal to \( (\mathcal{F}_1|\mathcal{F}) \). But \( \mathcal{G}_1/\mathcal{G}_r \) is cyclic of order \( e \) where \( e = p^e p^e \), \( e, p = 1, p \) the rational prime belonging to the ideal \( p \). Therefore \( (\mathcal{F}_1|\mathcal{F}) = e, p^e \). Since \( \mathcal{F}_1 = \mathcal{F}_2 \) we have \( e = 1 \) and \( (\mathcal{F}_1|\mathcal{F}) = p^e \). Therefore \( (\mathcal{F}_1|\mathcal{F}) = (\mathcal{F}_2|\mathcal{F}) = p^e \).

COROLLARY 7.2. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two number fields normal over \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \) and let \( p \) be a prime ideal in \( \mathcal{F}_1 \) and in \( \mathcal{F}_2 \). Let \( v_i \) denote
the order of ramification of \( p \) in \( \mathfrak{F}_i \) over \( \mathfrak{F} \) for \( i=1, 2 \) and suppose \( v_1 \geq v_2 \geq 2 \). If \( \mathfrak{F}_1 \) and \( \mathfrak{F}_2 \) have corresponding residue systems mod \( p^{v_2} \), then \( \Theta(\mathfrak{F}_1, \mathfrak{F}) \) is Abelian of type \((p, \cdots, p)\) where \( p \) is the rational prime belonging to \( p \).

**Proof.** If \( \mathfrak{F}_1 \) and \( \mathfrak{F}_2 \) have corresponding residue systems mod \( p^{v_2} \), it follows from Theorem 7 that \( \Theta(\mathfrak{F}_1, \mathfrak{F}) = \Theta_1 = \cdots = \Theta_{v_2} \) where \( \Theta_i \) is the \( i^{th} \) ramification group of \( p \) in \( \mathfrak{F}_1 \) over \( \mathfrak{F} \). By the definition of \( v_2 \), \( \Theta_{v_2+1} \) is the group identity. But \( \Theta_v / \Theta_{v+1} \) is Abelian of type \((p, \cdots, p)\) where \( p \) is the rational prime belonging to \( p \). It follows that \( \Theta(\mathfrak{F}_1, \mathfrak{F}) \) is Abelian of type \((p, \cdots, p)\).

The condition of Theorem 7 is not sufficient as the following example shows. Denote by \( R \) the field of rational numbers and let \( \mathfrak{F}_1 = R(\sqrt{2}), \mathfrak{F}_2 = R(\sqrt{3}), p = (\sqrt{2}) \). It is clear that the second ramification group of the ideal \((\sqrt{2})\) in \( \mathfrak{F}_1 \) over \( \mathfrak{F} \) is equal to the Galois group of \( \mathfrak{F}_1 \) over \( R \), and likewise for \( \mathfrak{F}_2 \). However \( \mathfrak{F}_1 \) and \( \mathfrak{F}_2 \) do not have corresponding residue systems mod \((\sqrt{2})^2\).

In the remainder of this paper we consider fields of the type \( \mathfrak{F}(\sqrt{\mu}) \) where \( \mathfrak{F} \) is a number field containing a \( q^{th} \) root of unity \( \zeta_{q} \neq 1 \), \( q \) is a rational prime, and \( \mu \) is an integer of \( \mathfrak{F} \) and not the \( q^{th} \) power of an integer in \( \mathfrak{F} \).

Let \( \mathfrak{P} \) be a prime ideal in \( \mathfrak{F}(\sqrt{\mu}) \) and in \( \mathfrak{F}(\sqrt{\mu_i}) \). We may suppose that \( \mathfrak{F}(\sqrt{\mu_i}) \neq \mathfrak{F}(\sqrt{\mu}) \) since the problem of corresponding residue systems is trivial in case equality holds. By Theorem 5, in order that \( \mathfrak{F}(\sqrt{\mu_i}) \) and \( \mathfrak{F}(\sqrt{\mu}) \) have corresponding residue systems mod \( \mathfrak{P} \) it is necessary and sufficient that \( \mathfrak{P} \) be of order \( q \) in \( \mathfrak{F}(\sqrt{\mu_i}) \) over \( \mathfrak{F} \) and in \( \mathfrak{F}(\sqrt{\mu}) \) over \( \mathfrak{F} \). Therefore it is necessary and sufficient that \( \mathfrak{P} \) divide the relative differente \( d_i \) of \( \mathfrak{F}(\sqrt{\mu_i}) \) over \( \mathfrak{F} \) for \( i=1, 2 \). If \( c_i \) denotes the relative conductor of \( \sqrt{\mu_i} \) for \( i=1, 2 \) then

\[(\sqrt{\mu_i})^{q-1}q = c_i d_i\]

for \( i=1, 2 \) since \((\sqrt{\mu_i})^{q-1}q\) is the relative number differente of \( \sqrt{\mu_i} \) over \( \mathfrak{F} \). It follows that \( \mathfrak{P} \) must divide \((\sqrt{\mu_i})^{q-1}q\) for \( i=1, 2 \) if \( \mathfrak{F}(\sqrt{\mu_i}) \) and \( \mathfrak{F}(\sqrt{\mu}) \) have corresponding residue systems mod \( \mathfrak{P} \).

Denote by \( \mathfrak{p} \) the prime ideal corresponding to \( \mathfrak{P} \) in \( \mathfrak{F} \). If \( \mathfrak{p} \) divides \( \mu_i \) but not \( q \) then \( \mathfrak{p} = \mathfrak{P}^a \) in \( F(\sqrt{\mu_i}) \) if and only if \( (\mu_i) = \mathfrak{p}^a \mathfrak{c} \) for \( i=1, 2 \) where \( (a_i, q) = 1 \) and \( (a_i, \mathfrak{p}) = (1) \). (See [1, p. 150]). Thus we have the following theorem.
THEOREM 8. If \( (\mathfrak{P}, q) = (1) \), then \( \mathfrak{F}(\sqrt[m]{\mu}) \) and \( \mathfrak{F}(\sqrt[n]{\mu}) \) have corresponding residue systems mod \( \mathfrak{P} \) if and only if \( (\mu_i, q) = 1 \) and \( (a_i, \nu) = (1) \) for \( i = 1, 2 \).

From Corollary 7.1 it follows that \( \mathfrak{F}(\sqrt[m]{\mu}) \) and \( \mathfrak{F}(\sqrt[n]{\mu}) \) do not have corresponding residue systems mod \( \mathfrak{P}^j \) for \( j > 1 \) in case \( (\mathfrak{P}, q) = (1) \).

We now consider prime ideals in fields \( \mathfrak{F}(\sqrt[m]{\mu}) \) which divide \( q \), that is, prime ideals which divide the ideal \( (1 - \zeta) \) where \( \zeta \neq 1 \) is a \( q \)-th root of unity. Let \( (1 - \zeta) = \mathfrak{P} \alpha \) in \( \mathfrak{F} \) where \( (\mathfrak{P}, \alpha) = (1) \) and \( \mathfrak{P} \) is a prime ideal in \( \mathfrak{F} \), and let \( q \) be a prime ideal of \( \mathfrak{F}(\sqrt[m]{\mu}) \) which divides \( \mathfrak{P} \). By Theorem 5 we are concerned only with the case in which \( q \) is of order \( q \) in \( \mathfrak{F}(\sqrt[m]{\mu}) \) over \( \mathfrak{F} \), that is \( \mathfrak{P} = q^\alpha \) in \( \mathfrak{F}(\sqrt[m]{\mu}) \). We may suppose without loss of generality that either \( (\mu, \mathfrak{P}) = (1) \) or \( (\mu, \mathfrak{P}) = (1) \). The ideal \( \mathfrak{P} \) becomes the \( q^\alpha \) power of a prime ideal in \( \mathfrak{F}(\sqrt[m]{\mu}) \) if the congruence \( \mu = \zeta \mod q^\alpha \) is not solvable for \( \xi \) in \( \mathfrak{F} \).

The main result of this paper for fields of the type \( \mathfrak{F}(\sqrt[m]{\mu}) \) is the following one: if \( \mu_1, \mu_2 \) are two integers of \( \mathfrak{F} \) such that \( \mathfrak{P} = q^{\alpha} \) in \( \mathfrak{F}(\sqrt[m]{\mu_1}) \) and in \( \mathfrak{F}(\sqrt[m]{\mu_2}) \), and \( q \) has ramification orders \( \geq v > a \) in \( \mathfrak{F}(\sqrt[m]{\mu_1}) \) and in \( \mathfrak{F}(\sqrt[m]{\mu_2}) \) over \( \mathfrak{F} \) then \( \mathfrak{F}(\sqrt[m]{\mu_1}) \) and \( \mathfrak{F}(\sqrt[m]{\mu_2}) \) have corresponding residue systems mod \( q^{\alpha - a} \).

We first consider the case in which \( (\mu, \mathfrak{P}) = (1) \).

THEOREM 9. If \( (\mu, \mathfrak{P}) = (1) \) and \( n \) is a positive integer, then \( \mathfrak{P} = q^\alpha \) in \( \mathfrak{F}(\sqrt[m]{\mu}) \) and every integer \( \alpha \) in \( \mathfrak{F}(\sqrt[m]{\mu}) \) satisfies a congruence

\[
\alpha \equiv \alpha_0 + \alpha_1 \sqrt[m]{\mu} + \cdots + \alpha_{n-1} \sqrt[m]{\mu}^{n-1} \pmod{q^n}
\]

where the \( \alpha_i \) are integers in \( \mathfrak{F} \). Furthermore the order of ramification \( v \) of \( q \) in \( \mathfrak{F}(\sqrt[m]{\mu}) \) over \( \mathfrak{F} \) is equal to \( v = \alpha q + 1 \).

Proof. Since \( (\mu, \mathfrak{P}) = (1) \), we have \( \mathfrak{P} = q^\alpha \) in \( \mathfrak{F}(\sqrt[m]{\mu}) \) where \( q \) is a prime ideal. It follows that \( \sqrt[m]{\mu} \) is exactly divisible by \( q \). Let \( n \) be any positive integer. If \( \alpha \) is any integer of \( \mathfrak{F} \) we have

\[
\alpha \equiv \alpha_0 + \alpha_1 \sqrt[m]{\mu} + \cdots + \alpha_{n-1} \sqrt[m]{\mu}^{n-1} \pmod{q^n}
\]

where the \( \alpha_i \) are residues mod \( q \) and may be chosen in \( \mathfrak{F} \) since \( q \) is of degree 1 with respect to \( \mathfrak{F} \).

The order of ramification of \( q \) is equal to \( v \) if and only if

\[
\sqrt[m]{\mu} \equiv \zeta \sqrt[m]{\mu} \mod q^v \quad \text{and} \quad \sqrt[m]{\mu} \neq \zeta \sqrt[m]{\mu} \mod q^{v+1}.
\]

Hence \( v = \alpha q + 1 \) since \( (1 - \zeta) = \mathfrak{P}^\alpha \), \( \mathfrak{P} = q^\alpha \), and \( (\mathfrak{P}, a) = (1) \).
THEOREM 10. If \( \mu_1, \mu_2 \) are two integers of \( \mathfrak{R} \) each exactly divisible by \( \mathfrak{D} \), then \( \mathfrak{R}(\sqrt[\mu_1]) \) and \( \mathfrak{R}(\sqrt[\mu_2]) \) have corresponding residue systems mod \( q^{aq+1-a} \).

Proof. Choose a fixed residue system mod \( \mathfrak{D} \) in \( \mathfrak{R} \) consisting of \( q^{th} \) powers, which is possible since \( \mathfrak{D} \) is a prime ideal in \( \mathfrak{R} \). Represent the residue class 0 by 0 and let \( n = a(q - 1) \). Since \( \mu_1 \) is exactly divisible by \( \mathfrak{D} \) we have

\[
\mu_1 = \alpha_1^1 \mu_1 + \cdots + \alpha_i^i \mu_1^n \quad (mod \mathfrak{D}^{n+1})
\]

where the \( \alpha_i^1 \) belong to the fixed residue system mod \( \mathfrak{D} \) chosen above. Hence

\[
(\sqrt[\mu_1] - \alpha_1 \sqrt[\mu_1] - \cdots - \alpha_i^i \sqrt[\mu_1]^n) = \mu_1 - \alpha_1 \mu_1 - \cdots - \alpha_i^i \mu_1^n \quad (mod \mathfrak{D}^{n+1})
\]

It follows that

\[
\sqrt[\mu_1] = \alpha_1 \sqrt[\mu_1] + \cdots + \alpha_i^i \sqrt[\mu_1]^n \quad (mod \mathfrak{D}^{n+1})
\]

and by Theorem 9, \( \mathfrak{R}(\sqrt[\mu_1]) \) and \( \mathfrak{R}(\sqrt[\mu_2]) \) have corresponding residue systems mod \( q^{aq+1-a} \).

By Theorem 7 the fields \( \mathfrak{R}(\sqrt[\mu_1]) \) and \( \mathfrak{R}(\sqrt[\mu_2]) \) do not have corresponding residue systems mod \( q^{aq+1} \) where \( v \) is the order of ramification of \( q \). The following theorem gives a sufficient condition for \( \mathfrak{R}(\sqrt[\mu_1]) \) and \( \mathfrak{R}(\sqrt[\mu_2]) \) to have corresponding residue systems mod \( q^v \).

THEOREM 11. Let \( \mu_1, \mu_2 \) be two integers of \( \mathfrak{R} \) each exactly divisible by \( \mathfrak{D} \). If \( \mu_1 = \mu_2 \) (mod \( \mathfrak{D}^{aq+1} \)) then \( \mathfrak{R}(\sqrt[\mu_1]) \) and \( \mathfrak{R}(\sqrt[\mu_2]) \) have corresponding residue systems mod \( q^{aq+1} \), that is, mod \( q^v \) where \( v \) is the order of ramification of \( q \).

Proof. Since \( \mu_1 = \mu_2 \) (mod \( \mathfrak{D}^{aq+1} \)) and \( (\sqrt[\mu_1] - \sqrt[\mu_2]) = \mu_1 - \mu_2 \) (mod \( q \)) it follows that \( \sqrt[\mu_1] = \sqrt[\mu_2] \) (mod \( \mathfrak{D}^{aq+1} \)). Suppose

1.) \( \sqrt[\mu_1] = \sqrt[\mu_2] \) (mod \( q^m \)) and \( \sqrt[\mu_1] \neq \sqrt[\mu_2] \) (mod \( q^{m+1} \)).

For any polynomial \( p(x, y) \) with integral coefficients such that \( y \) occurs in every term we have

\[
qp(\sqrt[\mu_1], \sqrt[\mu_2]) = qp(\sqrt[\mu_2], \sqrt[\mu_2]) \quad (mod q^{m+1}q).
\]

Thus

2.) \( (\sqrt[\mu_1] - \sqrt[\mu_2]) = \mu_1 - \mu_2 \) (mod \( \mathfrak{D}^{aq+1}q^m \)).

If \( \mu_1 - \mu_2 \neq 0 \) (mod \( \mathfrak{D}^{aq+1}q^m \)) then
\[ q(aq+1) < aq(q-1)+m+1 \text{ since } \mu_1 \equiv \mu_2 \pmod{\Sigma^{aq+1}}. \]

Therefore \( q < -aq+m+1 \) and \( m \geq aq+1 \). On the other hand if \( \mu_1 - \mu_2 \equiv 0 \pmod{\Sigma^{aq+1}} \) then
\[
(\sqrt{\mu_1} - \sqrt{\mu_2})^a \equiv 0 \pmod{\Sigma^{aq+1}}
\]
from 2.). Thus by 1.) we have \( mq \geq aq(q-1)+m+1, m > aq \), and hence \( m \geq aq+1 \). Therefore in either case \( m \geq aq+1 \) and we have by 1.)
\[
\sqrt{\mu_1} - \sqrt{\mu_2} \equiv 0 \pmod{q^{aq+1}}.
\]

Let \( \alpha \) be any integer of \( \overline{\mathbb{F}}(\sqrt[\alpha]{\mu_1}) \) and \( v \) the order of ramification of \( q \), that is, \( v = aq + 1 \). By Theorem 9
\[
\alpha = \alpha_0 + \alpha_1 \sqrt{\mu_1} + \cdots + \alpha_{q-1} \sqrt{\mu_1^{q-1}} \pmod{q^a}
\]
where the \( \alpha_i \) are integers in \( \overline{\mathbb{F}} \). Let
\[
\beta = \alpha_0 + \alpha_1 \sqrt{\mu_2} + \cdots + \alpha_{q-1} \sqrt{\mu_2^{q-1}}.
\]
Then \( \alpha \equiv \beta \pmod{q^a} \) and \( \overline{\mathbb{F}}(\sqrt[\alpha]{\mu_1}) \) and \( \overline{\mathbb{F}}(\sqrt[\beta]{\mu_2}) \) have corresponding residue systems mod \( q^a \).

The condition \( \mu_1 \equiv \mu_2 \pmod{\Sigma^{aq+1}} \) in Theorem 11 may be replaced by \( \mu_1 \equiv \mu \sigma \pmod{\Sigma^{aq+1}} \) where \( \sigma \) is in \( \overline{\mathbb{F}} \).

We now consider the case in which \( (\mu, \Sigma) = (1) \) and the congruence \( \mu \equiv \xi^a \pmod{\Sigma^a} \) is not solvable for \( \xi \) in \( \overline{\mathbb{F}} \), that is, \( (\mu, \Sigma) = (1) \) and \( \Sigma = q^a \) in \( \overline{\mathbb{F}}(\sqrt[\mu]{\mu}) \). Let \( k \) be the largest integer such that the congruence \( \mu \equiv \xi^a \pmod{\Sigma^k} \) is solvable for \( \xi \) in \( \overline{\mathbb{F}} \). Clearly \( 0 < k < aq \) and \( k \) is the largest integer such that the congruence \( \sqrt[k]{\mu} \equiv \xi \pmod{q^k} \) is solvable for \( \xi \) in \( \overline{\mathbb{F}} \).

**Theorem 12.** Let \( \mu \) be an integer of \( \overline{\mathbb{F}} \) such that \( (\mu, \Sigma) = (1) \) and \( \Sigma = q^a \) in \( \overline{\mathbb{F}}(\sqrt[\mu]{\mu}) \). Let \( k \) be the largest integer such that \( \mu \equiv \xi^a \pmod{\Sigma^k} \) is solvable for \( \xi \) in \( \overline{\mathbb{F}} \). Then the order of ramification \( v \) of \( q \) with respect to \( \Sigma \) is equal to \( aq+1-k \).

**Proof.** Let \( \alpha \) in \( \overline{\mathbb{F}} \) be a solution of the congruence \( \mu \equiv \xi^a \pmod{\Sigma^k} \) with \( k \) maximal. Since \( \mu - \alpha^a \) is exactly divisible by \( \Sigma^k \), it follows that \( \sqrt[k]{\mu} - \alpha \) is exactly divisible by \( q^k \). Furthermore we have \( (k, q) = 1 \) (see [1, p. 153]). Thus there exist positive integers \( x \) and \( y \) such that \( kx = 1 + qy \).

Let \( \pi \) be an integer of \( \overline{\mathbb{F}} \) such that \( (\pi) = \alpha \Sigma \) where \( (\alpha, \Sigma) = (1) \) and \( \alpha \) is an ideal of \( \overline{\mathbb{F}} \). There exists an ideal \( \mathfrak{c} \) in \( \overline{\mathbb{F}} \) such that \( \alpha \mathfrak{c} = (\omega) \) is principal and \( \mathfrak{c} \) is prime to \( \Sigma \).
Now, let
\[ \rho = \left( \sqrt[\varphi]{\mu - \alpha} \right)^{\varphi} \]

Then
\[ (\rho) = \left( \sqrt[\varphi]{\mu - \alpha} \right)^{\varphi} \zeta_{\varphi} = \left( \varphi \mu - \alpha \right)^{\varphi} = \left( \sqrt[\varphi]{\mu - \alpha} \right)^{\varphi} \]

and
\[ (\omega^q \rho) = \left( \sqrt[\varphi]{\mu - \alpha} \right)^{\varphi} \zeta_{\varphi} \]

The ideal fraction on the right in the last equation is an integral ideal exactly divisible by \( q \), and therefore \( \omega^q \rho \) is an integer of \( \mathbb{Z} \) exactly divisible by \( q \). It follows that the order of ramification of \( q \) is equal to \( v \) if and only if \( \omega^q \rho - (\omega^q \rho)^q \) is exactly divisible by \( \varphi^q = q^{x-1}q^y \). Now
\[ (\sqrt[\varphi]{\mu - \alpha})^2 = [(\sqrt[\varphi]{\mu - \alpha} - \sqrt[\varphi]{\mu - \alpha})^2] \]
\[ = (\sqrt[\varphi]{\mu - \alpha})^2 + x(\sqrt[\varphi]{\mu - \alpha})^2 - (\sqrt[\varphi]{\mu - \alpha} + \sqrt[\varphi]{\mu - \alpha}) + \cdots \]

Therefore
\[ (\sqrt[\varphi]{\mu - \alpha})^2 \equiv (\sqrt[\varphi]{\mu - \alpha})^2 \quad (\text{mod } q^{k(x-1)}(1 - \varphi)) \]
\[ \equiv (\sqrt[\varphi]{\mu - \alpha})^2 \quad (\text{mod } q^{k(x-1)}q^{y\varphi}) \]

since \( 0 < k < aq \) and \( (1 - \varphi) = \varphi_a \) with \( (\varphi, a) = (1) \). Furthermore this congruence holds exactly mod \( q^{k(x-1)}q^{xy} \). It follows that \( kx - 1 + v = k(x - 1) + aq \) and \( v = aq + 1 - k \).

**Theorem 13.** Let \( \mu_1, \mu_2 \) be two integers of \( \mathbb{F} \) each prime to \( \varphi \) and such that \( \varphi = q^a \) in \( \mathbb{F}(\mu_1) \) (and \( \mathbb{F}(\mu_2) \)). Let \( k_i \) be the largest integer such that the congruence \( \mu_i \equiv a_i \ (\text{mod } \varphi^{k_i}) \) is solvable for \( a_i \), an integer of \( \mathbb{F} \) \((i = 1, 2)\). Let \( v_i = aq + 1 - k_i \) for \( i = 1, 2 \), and suppose \( v_1 \geq v_2 > a \). Then \( \mathbb{F}(\mu_1) \) and \( \mathbb{F}(\mu_2) \) have corresponding residue systems mod \( q^{v_1 - v} \).
Proof. Since $\mu_i - \alpha_i$ is exactly divisible by $S^{k_i}$ it follows that $\sqrt{\mu_i} - \alpha_i$ is exactly divisible by $q^{k_i}$ for $i=1, 2$. Since $(k_i, q)=1$ we have positive integers $x_i$ and $y_i$ such that $k_i x_i = 1 + q y_i$ for $i=1, 2$. Let $\pi$ be an integer of $\mathfrak{R}$ exactly divisible by $S$. Using the method of Theorem 12 we obtain an integer

$$\theta_i = \frac{\omega^{y_i}(\sqrt{\mu_i} - \alpha_i)^{x_i}}{\pi^{y_i}}$$

of $\mathfrak{R}(\sqrt{\mu_i})$ which is exactly divisible by $q$ for $i=1, 2$.

We now show that $\theta_i$ is congruent to an integer of $\mathfrak{R}$ mod $\mathfrak{S}^{v_i-a}$ for $i=1, 2$. We have

$$\theta_i = \frac{\omega^{y_i}(\sqrt{\mu_i} - \alpha_i)^{x_i}}{\pi^{y_i}} = \frac{\omega^{y_i}(\lambda_i - \rho_i q)^{x_i}}{\pi^{y_i}}$$

where $\lambda_i$ is an integer of $\mathfrak{R}$ and $\lambda_i \equiv 0 \pmod{\mathfrak{S}^{k_i}}$. Hence since $\rho_i$ is divisible by $q^{k_i}$

$$\theta_i = \frac{\omega^{y_i}(\lambda_i^{x_i} - x_i \lambda_i^{x_i-1} \rho_i q + \cdots)}{\pi^{y_i}}$$

$$= \frac{\omega^{y_i} \lambda_i^{x_i}}{\pi^{y_i}} \quad (\text{mod } \mathfrak{S}^{a q + 1 - k_i - a})$$

$$= \frac{\omega^{y_i} \lambda_i^{x_i}}{\pi^{y_i}} \quad (\text{mod } \mathfrak{S}^{v_i-a})$$

But the expression on the right of the last congruence is an integer of $\mathfrak{R}$, so that $\theta_i$ is congruent to an integer of $\mathfrak{R}$ mod $\mathfrak{S}^{v_i-a}$.

We now show that the $q$th power of every integer of $\mathfrak{R}(\sqrt{\mu_i})$ is congruent to an integer of $\mathfrak{R}$ mod $\mathfrak{S}^{v_i-a}$ for $i=1, 2$.

Let $\beta$ be any integer of $\mathfrak{R}(\sqrt{\mu_i})$ and let $n=v_i-a$. Since $\theta_i$ is exactly divisible by $q$ we have $\beta \equiv \beta_0 + \beta_1 \theta_1 + \cdots + \beta_{n-1} \theta_1^{n-1} \pmod{q^n}$, where the $\beta_i$ are residues mod $q$ and may be chosen in $\mathfrak{R}$ since $q$ is of degree 1 over $\mathfrak{R}$. Hence

$$[\beta - (\beta_0 + \cdots + \beta_{n-1} \theta_1^{n-1})]^q$$

$$\equiv \beta^q - (\beta_0 + \cdots + \beta_{n-1} \theta_1^{n-1})^q \pmod{q}$$

$$\equiv \beta^q - (\beta_0^q + \cdots + \beta_{n-1}^q \theta_1^{n-1}) \pmod{q}$$

$$\equiv \beta^q - \sigma \pmod{\mathfrak{S}^{v_i-a}},$$

where $\sigma$ is an integer of $\mathfrak{R}$. It follows that $\beta^q \equiv \sigma \pmod{\mathfrak{S}^{v_i-a}}$. 
If \( \beta \) and \( \beta' \) are two integers of \( \mathbb{H}(\sqrt{\mu_1}) \) such that \( \beta^q = \sigma \) (mod \( \mathbb{O}^{a_1} \)) and \( \beta'^q = \sigma \) (mod \( \mathbb{O}^{a_1} \)), then \( \beta = \beta' \) (mod \( q^{a_1-a} \)). Also if \( \beta^q = \sigma \) (mod \( \mathbb{O}^{a_1} \)) and \( \beta'^q = \sigma' \) (mod \( \mathbb{O}^{a_1} \)) where \( \sigma, \sigma' \) are integers of \( \mathbb{H} \), then \( \sigma = \sigma' \) (mod \( \mathbb{O}^{a_1} \)). The number of residue classes mod \( q^{a_1-a} \) in \( \mathbb{H}(\sqrt{\mu_1}) \) is equal to the number of residue classes mod \( \mathbb{O}^{a_1} \) in \( \mathbb{H} \). It follows that if \( \sigma \) is any integer of \( \mathbb{H} \) there exists an integer \( \beta \) of \( \mathbb{H}(\sqrt{\mu_1}) \) such that \( q^\beta = \sigma \) (mod \( \mathbb{O}^{a_1} \)).

Similarly, if \( \gamma \) is any integer of \( \mathbb{H}(\sqrt{\mu_2}) \) there exists an integer \( \tau \) of \( \mathbb{H} \) such that \( \gamma^q = \tau \) (mod \( \mathbb{O}^{a_2} \)). There exists an integer \( \beta \) of \( \mathbb{H}(\sqrt{\mu_1}) \) such that \( \beta^q = \tau \) (mod \( q^{a_2} \)). Since \( v_1 \geq v_2 \) we have \( \beta^q = \tau^q \) (mod \( \mathbb{O}^{a_2} \)) and therefore \( \beta = \gamma \) (mod \( q^{a_2} \)).

**Theorem 14.** If \( \mu_1, \mu_2 \) are two integers of \( \mathbb{H} \) such that \( \mathfrak{d} = q^\mathfrak{a} \) in \( \mathbb{H}(\sqrt{\mu_1}) \) and in \( \mathbb{H}(\sqrt{\mu_2}) \), and \( q \) has ramification orders \( \geq v > a \) in \( \mathbb{H}(\sqrt{\mu_1}) \), \( \mathbb{H}(\sqrt{\mu_2}) \) over \( \mathbb{F} \), then \( \mathbb{H}(\sqrt{\mu_1}) \) and \( \mathbb{H}(\sqrt{\mu_2}) \) have corresponding residue systems mod \( q^{a_2} \).

**Proof.** We need only to consider the case in which \( \mu_1 \) is exactly divisible by \( \mathfrak{d} \) and \( \mu_2 \) is prime to \( \mathfrak{d} \), the other two cases following from Theorems 10 and 13.

Let \( v_1 = aq + 1 \) be the order of ramification of \( q \) in \( \mathbb{H}(\sqrt{\mu_1}) \) over \( \mathbb{F} \), and let \( v_2 \) be the order of ramification of \( q \) in \( F(\sqrt{\mu_2}) \) over \( \mathbb{F} \). From Theorem 12 it follows that \( v_1 - 1 = aq \geq v_2 \).

Let \( \alpha \) be any integer of \( \mathbb{H}(\sqrt{\mu_1}) \) and let \( n = aq - a \). Since \( \sqrt{\mu_1} \) is exactly divisible by \( q \), it follows that

\[
\alpha = \alpha_0 + \alpha_1 \sqrt{\mu_1} + \cdots + \alpha_{a-1} \sqrt{\mu_1}^{a-1} \quad \text{(mod } q^n)\]

where the \( \alpha_i \) are integers in \( \mathbb{H} \). Hence

\[
\alpha^q = \alpha_0^q + \alpha_1^q \mu_1 + \cdots + \alpha_{a-1}^q \mu_1^{a-1} \quad \text{(mod } \mathbb{O}_n) \\
\equiv \sigma \quad \text{(mod } \mathbb{O}^{aq-a})
\]

where \( \sigma \) is an integer of \( \mathbb{H} \). Using the method of Theorem 13, there exists an integer \( \beta \) of \( \mathbb{H}(\sqrt{\mu_1}) \) such that \( \beta^q = \sigma \) (mod \( \mathbb{O}^{aq-a} \)). Therefore \( \alpha^q = \beta^q \) (mod \( \mathbb{O}^{a_2} \)) and \( \alpha = \beta \) (mod \( q^{a_2} \)). Thus \( \mathbb{H}(\sqrt{\mu_1}) \) and \( \mathbb{H}(\sqrt{\mu_2}) \) have corresponding residue systems mod \( q^{a_2} \) where \( v_2 \geq v > a \).

**Theorem 15.** Let \( \mu_1, \mu_2 \) be two integers of \( \mathbb{H} \), each prime to \( \mathfrak{d} \), such that \( \mathfrak{d} = q^\mathfrak{a} \) in \( \mathbb{H}(\sqrt{\mu_1}) \) and in \( \mathbb{H}(\sqrt{\mu_2}) \). Suppose \( \mu_1 \equiv \mu_2 \) (mod \( \mathbb{O}^a \)) and let \( k \) be the largest integer such that the congruences \( \mu_1 \equiv \alpha^k \) (mod \( \mathbb{O}^k \)) and \( \mu_2 \equiv \alpha^k \) (mod \( \mathbb{O}^k \)) are solvable for \( \alpha \) an integer of \( \mathbb{H} \).
Then \( \mathfrak{F}(\sqrt[\nu]{\mu_1}) \) and \( \mathfrak{F}(\sqrt[\nu]{\mu_2}) \) have corresponding residue systems mod \( q^u \) where \( v = aq + 1 - k \).

**Proof.** Since \( \mu_1 \equiv \mu_2 \pmod{\mathfrak{S}^a} \) it follows that \( \sqrt[\nu]{\mu_1} \equiv \sqrt[\nu]{\mu_2} \pmod{\mathfrak{S}^a} \) using the method of Theorem 11. We have \( kx = 1 + qy \) and following Theorem 12 it is sufficient to show that

\[
(\sqrt[\nu]{\mu_1} - \alpha)^x \equiv (\sqrt[\nu]{\mu_2} - \alpha)^x \pmod{q^{u+qy}}.
\]

We have

\[
(\sqrt[\nu]{\mu_2} - \alpha)^x = [(\sqrt[\nu]{\mu_1} - \alpha) + (\sqrt[\nu]{\mu_2} - \sqrt[\nu]{\mu_1})]^x
\]

\[
= (\sqrt[\nu]{\mu_1} - \alpha)^x + x(\sqrt[\nu]{\mu_1} - \alpha)^{x-1}(\sqrt[\nu]{\mu_2} - \sqrt[\nu]{\mu_1}) + \cdots
\]

\[
= (\sqrt[\nu]{\mu_1} - \alpha)^x \pmod{q^{k(x-1)q^a}}
\]

\[
\equiv (\sqrt[\nu]{\mu_1} - \alpha)^x \pmod{q^{u+qy}}.
\]

Thus \( \mathfrak{F}(\sqrt[\nu]{\mu_1}) \) and \( \mathfrak{F}(\sqrt[\nu]{\mu_2}) \) have corresponding residue systems mod \( q^u \) where \( v = aq + 1 - k \) is the order of ramification of \( q \) in \( \mathfrak{F}(\sqrt[\nu]{\mu_1}) \) and \( \mathfrak{F}(\sqrt[\nu]{\mu_2}) \) over \( \mathfrak{S} \).

We remark that if \( \mathfrak{F}(\sqrt[\nu]{\mu_1}) \neq \mathfrak{F}(\sqrt[\nu]{\mu_2}) \) then \( \sqrt[\nu]{\mu_1} \neq \sqrt[\nu]{\mu_2} \pmod{q^{a(q+1)}} \) for otherwise we would have corresponding residue systems mod \( q^{a+1} \) contrary to Theorem 7.

In Theorem 15 we may replace the condition \( \mu_1 \equiv \mu_2 \pmod{\mathfrak{S}^a} \) by \( \mu_1 \equiv \beta \mu_2 \pmod{\mathfrak{S}^a} \) with \( \beta \) in \( \mathfrak{S} \).

**Theorem 16.** Let \( \mu_1, \mu_2 \) be two integers of \( \mathfrak{S} \) such that \( \Sigma = q^a \) in \( \mathfrak{F}(\sqrt[\nu]{\mu_1}) \) and in \( \mathfrak{F}(\sqrt[\nu]{\mu_2}) \) and the orders of ramification of \( q \) in \( \mathfrak{F}(\sqrt[\nu]{\mu_1}) \) and \( \mathfrak{F}(\sqrt[\nu]{\mu_2}) \) over \( \mathfrak{S} \) are \( \geq aq \). In order that \( \mathfrak{F}(\sqrt[\nu]{\mu_1}) \) and \( \mathfrak{F}(\sqrt[\nu]{\mu_2}) \) have corresponding residue systems mod \( q^{aq} = \Sigma^a \) it is necessary and sufficient that the following congruences be solvable in \( \mathfrak{S} \):

\[
\sum_{e_0+e_1+\cdots+e_{q-1}=q} \frac{q!}{e_0!e_1!\cdots e_{q-1}!} \alpha_{e_0}^{\alpha_{e_1}} \cdots \alpha_{e_{q-1}}^{\alpha_{e_{q-1}}} \equiv 0 \pmod{\mathfrak{S}^a}
\]

\[
\sum_{e_0+e_1+\cdots+e_{q-1}=q} \frac{q!}{e_0!e_1!\cdots e_{q-1}!} \alpha_{e_0}^{\alpha_{e_1}} \cdots \alpha_{e_{q-1}}^{\alpha_{e_{q-1}}} \equiv \mu_i \pmod{\mathfrak{S}^a},
\]

where \( \alpha_0, \cdots, \alpha_{q-1} \) are integers of \( \mathfrak{S} \) and \( e_0, e_1, \cdots, e_{q-1}, m \) are nonnegative integers.
Proof. Since the orders of ramification of $q$ in $\mathfrak{g}(\sqrt[\mu_j])$ over $\mathfrak{g}$ are $\geq aq$ for $j=1, 2$, then either $\sqrt[\mu_j]$ is exactly divisible by $q$ or $\sqrt[\mu_j]$ is prime to $q$ and there exists an integer $\xi_j$ of $\mathfrak{g}$ such that $\sqrt[\mu_j]-\xi_j$ is exactly divisible by $q$. In either case 1, $\sqrt[\mu_j], \cdots, \sqrt[\mu_j]^{-1}$ form a basis for the residue system mod $q^n$, $n$ a given positive integer.

If $\mathfrak{g}(\sqrt[\mu_1])$ and $\mathfrak{g}(\sqrt[\mu_2])$ have corresponding residue systems mod $q^aq$, we have

1.) $\sqrt[\mu_1]=\alpha_1+\alpha_2\sqrt[\mu_2]+\cdots+\alpha_{q-1}\sqrt[\mu_2]^{-1}$ (mod $\Sigma^a$)

2.) $\mu_1=\alpha_1+\alpha_2\sqrt[\mu_2]+\cdots+\alpha_{q-1}\sqrt[\mu_2]^{-1}$ (mod $\Sigma^a$)

and the congruences of the theorem follow.

Conversely if the congruences of the theorem are valid then 2.) is valid and 1.) follows. Interchanging the roles of $\mu_1$ and $\mu_2$, the converse follows.

Theorem 17. If $\mathfrak{g}=R(\zeta)$, $q=3$, and $\mathfrak{g}(\sqrt[\mu_1])$ and $\mathfrak{g}(\sqrt[\mu_2])$ have corresponding residue systems mod $(1-\zeta)$, then either $\mu_1=\alpha^3\mu_2^e$ (mod $3(1-\zeta)$) where $\alpha$ is in $R(\zeta)$ and $e=1$ or 2, or $\mu_1=\mu_2=0$ (mod $(1-\zeta)$).

Proof. In $R(\zeta)$ the ideal $(1-\zeta)$ is a prime ideal, that is, $(1-\zeta)=\Sigma$. Since $\mathfrak{g}(\sqrt[\mu_1])$ and $\mathfrak{g}(\sqrt[\mu_2])$ have corresponding residue systems mod $(1-\zeta)$ we have $(1-\zeta)=q^a$, and the orders of ramification of $q$ in $\mathfrak{g}(\sqrt[\mu_1]), \mathfrak{g}(\sqrt[\mu_2])$ over $\mathfrak{g}$ are $\geq 3$, and hence either 3 or 4. In either case 1, $\sqrt[\mu_1], \sqrt[\mu_2]$ form a basis for the residue system mod $(1-\zeta)$ in $\mathfrak{g}(\sqrt[\mu_1])$ for $j=1, 2$.

Since $\mathfrak{g}(\sqrt[\mu_1])$ and $\mathfrak{g}(\sqrt[\mu_2])$ have corresponding residue systems mod $(1-\zeta)$, we have

$\sqrt[\mu_1]=\alpha_1+\alpha_2\sqrt[\mu_2]+\alpha_3\sqrt[\mu_2]^2$ (mod $(1-\zeta)$)

$\mu_1=\alpha_1^3+\alpha_2^2\mu_2+\alpha_3^2\mu_2^2+3P(\sqrt[\mu_2])$ (mod $3(1-\zeta)$)

where $P(x)$ is a polynomial with coefficients in $R(\zeta)$. It follows that $P(\sqrt[\mu_1])$ is congruent to a number in $R(\zeta)$ mod $(1-\zeta)$, and the coefficients of $\sqrt[\mu_2]$ and $\sqrt[\mu_2]^2$ in $P(\sqrt[\mu_2])$ must vanish mod $(1-\zeta)$. Thus

$$2\alpha_1^3\alpha_1+\alpha_2^2\alpha_1^3+\alpha_2^2\alpha_2^2=0$$ (mod $(1-\zeta)$)
\[ \alpha_0 \alpha_1^2 + \alpha_1 \alpha_2^2 \mu + \alpha_2^3 \alpha \equiv 0 \pmod{(1 - \zeta)}. \]

By considering two cases, \( \mu \equiv 0 \pmod{(1 - \zeta)} \) and \( \mu \not\equiv 0 \pmod{(1 - \zeta)} \), the conclusion of the theorem follows from the last two congruences.

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