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DISTRIBUTION OF MATRICES IN A FINITE FIELD

L. CARLITZ AND JOHN HERBERT HODGES

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1. Introduction and notation. This paper is mainly concerned with the distribution with respect to characteristic polynomial and factors of the characteristic polynomial, of square matrices with elements in a finite field $GF(q)$. The method employed is to investigate the properties of the polynomials in question, that is, the matrix problems are reduced to problems concerning polynomials. In this connection see a recent paper by Walker [5] on Fermat's theorem for algebras; incidentally Walker's Theorem 3 had been proved earlier in [1; § 7].

The properties of matrices assumed here may be found in [4]. German capitals $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ will denote square matrices with elements in $GF(q)$. Polynomials in an indeterminate x with coefficients in $GF(q)$ will be denoted by $F(x), M(x), \dots$ in § 2 and simply by F, M, \dots elsewhere.

The number of partitions of the positive integer m into at most r parts will be denoted by $\pi_r(m)$, with $\pi_m(m) = \pi(m)$, the number of unrestricted partitions of m . The symbol $\pi'_r(m)$ will denote the *weighted* partition into at most r parts:

$$(1.1) \quad \pi'_r(m) = \sum_{k_1 + 2k_2 + \dots + rk_r = m} q^{k_1 + k_2 + \dots + k_r},$$

with $\pi'_m(m) = \pi'(m)$, the unrestricted weighted partition.

In Theorem 1 below the number of non-derogatory matrices of order m is given in terms of the Euler ϕ -function for $GF[q, x]$.

If $F = F(x)$ is a polynomial of degree m and $F = P_1^{r_1} \dots P_s^{r_s}$, where the P_i are distinct irreducible polynomials, we find (Theorem 2) that the number of *classes* of similar matrices of order m with characteristic polynomial $F(x)$ is

$$(1.2) \quad C_m(F) = \pi(r_1) \dots \pi(r_s).$$

Theorem 3 determines the total number $N(m)$ of distinct classes of similar matrices of order m as

$$(1.3) \quad N(m) = \pi'(m),$$

where $\pi'(m)$ is defined in (1.1) with $r = m$.

We also find (Theorem 4) the number of distinct classes of similar matrices of order m with minimum polynomial of degree r , where r is a fixed integer $\leq m$. Finally in § 4 we consider a polynomial problem

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which is suggested by the problem of determining the number of admissible minimum polynomials of fixed degree r for matrices of order m .

2. Non-derogatory matrices over $GF(q)$. Let A be a non-derogatory matrix of order m with elements in $GF(q)$, that is a matrix for which the characteristic and minimum polynomials are identical. Then it is well known that $\mathfrak{A}\mathfrak{S}=\mathfrak{S}\mathfrak{A}$ if and only if $\mathfrak{S}=F(\mathfrak{A})$, where $F(x)$ is a scalar polynomial of degree $\leq m-1$. Moreover if $M(x)$ denotes the characteristic polynomial of \mathfrak{A} , then \mathfrak{S} is non-singular if and only if $(F(x), M(x))=1$. Clearly, corresponding to every such polynomial $F(x)$, there is a unique primary polynomial $G(x)$ of degree m such that $(F(x), M(x))=1$, if and only if $(G(x), M(x))=1$. Thus, the number of distinct non-singular matrices \mathfrak{S} which commute with \mathfrak{A} is the number of primary (sometimes called *monic*) polynomials $G(x)$ of degree m such that $(G(x), M(x))=1$. This number, which is the Euler function for $GF[q, x]$, the polynomial domain in x over $GF(q)$, is given in [2; 21] by the formula

$$(2.1) \quad \phi(M(x))=q^m \prod_{P(x)|M(x)} \left(1 - \frac{1}{|P(x)|}\right),$$

where $P(x)$ runs through all primary prime divisors of $M(x)$ and $|P(x)|=q^e$, where $\deg P(x)=e$.

We recall that similar matrices have the same characteristic polynomial and that if two non-derogatory matrices have the same characteristic polynomial they are similar. Thus, as \mathfrak{S} runs through all the non-singular matrices of order m , the form $\mathfrak{S}^{-1}\mathfrak{A}\mathfrak{S}$ runs through the set of all non-derogatory matrices of order m having characteristic polynomial $M(x)$, each one appearing as many times as \mathfrak{A} appears, namely $\phi(M(x))$. If we let g_m denote the number of non-singular matrices of order m , we have

THEOREM 1. *The number of non-derogatory matrices of order m in $GF(q)$ is*

$$(2.2) \quad Y(m)=g_m \sum_{\deg M(x)=m} \frac{1}{\phi(M(x))},$$

where the sum is over primary $M(x)$ only, $\phi(M(x))$ is the Euler function and $g_m = \prod_{r=0}^{m-1} (q^m - q^r)$ is the number of non-singular matrices of order m .

3. Distribution of classes of similar matrices in $GF(q)$. If \mathfrak{A} and \mathfrak{B} are matrices of order m with the elements in $GF(q)$, we will say that \mathfrak{A} and \mathfrak{B} are in the same *class* if and only if they are similar. If $F(x)$ is the characteristic polynomial of a matrix \mathfrak{A} of order m , then

$$(3.1) \quad F = H_1 H_2 \cdots H_m,$$

where $H_{i+1} | H_i$, and the H_i are the invariant factors of $x^m - \mathfrak{A}$. In particular we call H_1 the first invariant factor. (In the remainder of this paper a polynomial $F(x)$ will simply be denoted by the letter F .) If we put

$$(3.2) \quad H_i = E_i H_{i+1} \quad \text{and} \quad H_m = E_m,$$

then we also have

$$(3.3) \quad F = E_1 E_2^2 E_3^3 \cdots E_m^m.$$

Let $C_m(F)$ denote the number of distinct classes of order m having characteristic polynomial F . Then it is clear that $C_m(F)$ is the number of distinct representations of F in the form of (3.3). If we also have

$$(3.4) \quad F = P_1^{r_1} P_2^{r_2} \cdots P_s^{r_s},$$

where the P_i are distinct prime polynomials, it follows that

$$(3.5) \quad C_m(F) = C_m(P_1^{r_1}) C_m(P_2^{r_2}) \cdots C_m(P_s^{r_s}).$$

For any prime polynomial P and positive integer r , $C_m(P^r)$ is seen to be the number of unrestricted partitions of r , or $\pi(r)$. Thus in view of (3.5) we have proved the following.

THEOREM 2. *If F is a polynomial of order m with coefficients in $GF(q)$ and F has the factorization (3.4), then the number of distinct classes of order m having characteristic polynomial F is*

$$(3.6) \quad C_m(F) = \pi(r_1) \pi(r_2) \cdots \pi(r_s).$$

Let $N(m)$ denote the number of distinct classes of matrices of order m . Then it is clear that

$$N(m) = \sum_{\deg F = m} C_m(F),$$

where the sum is over primary F only. In view of the definition of $C_m(F)$ and the factorization (3.3) we may write

$$(3.7) \quad \sum_F \frac{C_m(F)}{|F|^s} = \sum_{E_1} \frac{1}{|E_1|^s} \sum_{E_2} \frac{1}{|E_2|^{2s}} \cdots \sum_{E_m} \frac{1}{|E_m|^{ms}},$$

where the sums are over all primary F, E_1, \dots, E_m . Since we have

$$(3.8) \quad \zeta(s) = \sum_F \frac{1}{|F|^s} = \prod_P \left(1 - \frac{1}{|P|^s} \right)^{-1} = \sum_{k=0}^{\infty} \frac{q^k}{q^{ks}} = (1 - q^{1-s})^{-1},$$

which converges absolutely for real $s > 1$, (3.7) becomes

$$(3.9) \quad \sum_F \frac{C_m(F)}{|F|^s} = \sum_{k_1=0}^{\infty} \frac{q^{k_1}}{q^{k_1 s}} \sum_{k_2=0}^{\infty} \frac{q^{k_2}}{q^{2k_2 s}} \cdots \sum_{k_m=0}^{\infty} \frac{q^{k_m}}{q^{m k_m s}} .$$

It therefore follows that $N(m)$ is the coefficient of q^{-ms} in the right member of (3.9). Calculating this coefficient we get the following theorem.

THEOREM 3. *The number of distinct classes of similar matrices of order m in $GF(q)$ is*

$$(3.10) \quad N(m) = \sum_{k_1+2k_2+\cdots+mk_m=m} q^{k_1+k_2+\cdots+k_m} = \pi'(m) .$$

Let $N(m, r)$ denote the number of distinct classes of matrices of order m for which $\text{deg } H_1 = r$, where H_1 is the first invariant factor as defined in (3.1). Then $N(m, r)$ will be the coefficient of $q^{-rs}q^{-mt}$ in the series

$$(3.11) \quad \sum_{\substack{H_i \\ |H_i|^{r_i}}} \frac{1}{|H_1|^s |H_1 H_2 \cdots H_m|^t} ,$$

where the H_i are all primary.

In view of the definition of the polynomials E_i , the series in (3.11) is equal to

$$\sum_{\substack{E_i \\ |E_i|^{r_i}}} \frac{1}{|E_1 E_2 \cdots E_m|^s |E_1 E_2^2 \cdots E_m^m|^t} \sum_{x_1} \frac{1}{|E_1|^{s+t}} \cdots \sum_{E_m} \frac{1}{|E_m|^{s+mt}} .$$

Then with $s+t > 1$, this product may be written as

$$(3.12) \quad \zeta(s+t)\zeta(s+2t)\cdots\zeta(s+mt) = \frac{1}{(1-q^{1-s-t})(1-q^{1-s-2t})\cdots(1-q^{1-s-mt})} .$$

In view of (3.8) it is clear that the coefficient of $q^{-rs}q^{-mt}$ in the right member of (3.12) is the same as in the product

$$\frac{1}{(1-q^{1-s-t})(1-q^{1-s-2t})\cdots(1-q^{1-s-mt})\cdots}$$

By means of a well-known identity [3; 278], this second product is equal to the series

$$(3.13) \quad \sum_{k=0}^{\infty} (q^{1-s-t})^k \frac{1}{(1-q^{-t})(1-q^{-2t})\cdots(1-q^{-kt})} .$$

By choosing $k=r$ in (3.13) the coefficient of q^{-rs-mt} may be easily obtained. We get the following theorem.

THEOREM 4. *The number of distinct classes of similar matrices of order m in $GF(q)$ for which $\text{deg } H_1=r$, where H_1 is the first invariant factor as defined in (3.1), is given by*

$$N(m, r) = q^r \sum_{i_1+2i_2+\dots+r i_r = m-r} 1 = q^r \pi_r(m-r).$$

4. Another problem. Let us consider the product

$$(4.1) \quad \prod \left\{ 1 + \frac{1}{|P|^s} \left(\frac{1}{|P|^t} + \frac{1}{|P|^{2t}} + \dots \right) + \frac{1}{|P|^{2s}} \left(\frac{1}{|P|^{2t}} + \frac{1}{|P|^{3t}} + \dots \right) + \dots \right\},$$

taken over all primary prime polynomials P in $GF(q)$. In order to determine an interval of convergence for this product, we consider the associated series

$$(4.2) \quad \sum_P \left\{ \frac{1}{|P|^s} \left(\frac{1}{|P|^t} + \dots \right) + \frac{1}{|P|^{2s}} \left(\frac{1}{|P|^{2t}} + \dots \right) + \dots \right\}.$$

The series may be written more simply as

$$(4.3) \quad \sum_P \frac{|P|^{-s-t}}{(1-|P|^{-t})(1-|P|^{-s-t})}.$$

For t real and positive, the denominators of the terms in (4.3) approach 1 as $\text{deg } P$ grows large, so that we need only consider the numerators. Comparing with (3.8) we see that the series and consequently the product (4.1) converge absolutely for real s, t such that $t > 0$ and $s+t > 1$.

It is clear that the product (4.1) is equal to the series

$$(4.4) \quad \sum_{H|F} \frac{1}{|H|^s |F|^t},$$

where the sum is over all pairs H, F of primary polynomials over $GF(q)$ such that $H|F$ and every distinct prime factor of F is a factor of H . Thus F and H may be thought of as characteristic and minimum polynomial, respectively, of some matrix. Letting $T(m, r)$ denote the number of such pairs for which $\text{deg } F=m$ and $\text{deg } H=r$, it is clear that $T(m, r)$ is the coefficient of q^{-rs-mt} in the series (4.4). Unfortunately, however, it does not seem possible to get a simple explicit formula for $T(m, r)$.¹

If we take $s=0$, then (4.1) and (4.4) converge for real $t > 1$, and denoting by $T(m)$ the coefficient of q^{-mt} in the series (4.4), we have

$$(4.5) \quad T(m) = \sum_{r=0}^{\infty} T(m, r).$$

¹ We note that, were it not for possible repetitions of H in (4.4), the number $T(m, r)$ would be the number of admissible minimum polynomials of degree $r=m$ for matrices of order m .

With $s=0$, (4.1) simplifies to

$$(4.6) \quad \prod_P \left\{ 1 + \frac{|P|^{-t}}{1-|P|^{-t}} + \frac{|P|^{-2t}}{1-|P|^{-t}} + \dots \right\} = \prod \frac{1-|P|^{-t}+|P|^{-2t}}{(1-|P|^{-t})^2}$$

$$= \prod \frac{(1+|P|^{-3t})(1-|P|^{-3t})}{(1-|P|^{-t})(1-|P|^{-2t})(1-|P|^{-3t})}.$$

Using (3.8) this is seen to be equal to

$$(4.7) \quad \frac{\zeta(t)\zeta(2t)\zeta(3t)}{\zeta(6t)} = (1-q^{-6t}) \sum_{k_1=0}^{\infty} \frac{q^{k_1}}{q^{k_1 t}} \cdot \sum_{k_2=0}^{\infty} \frac{q^{k_2}}{q^{2k_2 t}} \cdot \sum_{k_3=0}^{\infty} \frac{q^{k_3}}{q^{3k_3 t}}.$$

Computing the coefficient of q^{-m} in the product series on the right side of (4.7) gives the following theorem.

THEOREM 5. *If $T(m, r)$ is the number of pairs H, F of polynomials over $GF(q)$ such that $H|F$, every distinct prime factor of F is a factor of H , $\deg F=m$ and $\deg H=r$, then*

$$(4.8) \quad T(m) = \sum_{r=0}^m T(m, r) = \pi'_3(m) - q\pi'_3(m-6),$$

where $\pi'_3(m)$ is the weighted partition defined by (1.1).

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Louis Auslander, <i>Remark on the use of forms in variational calculations</i>	209
Hubert Spence Butts, Jr. and Henry B. Mann, <i>Corresponding residue systems in algebraic number fields</i>	211
L. Carlitz and John Herbert Hodges, <i>Distribution of matrices in a finite field</i>	225
Paul Civin and Bertram Yood, <i>Invariant functionals</i>	231
David James Dickinson, Henry Pollak and G. H. Wannier, <i>On a class of polynomials orthogonal over a denumerable set</i>	239
Bernard Friedman and Luna Mishoe, <i>Eigenfunction expansions associated with a non-self-adjoint differential equation</i>	249
Luna Mishoe and G. C. Ford, <i>On the uniform convergence of a certain eigenfunction series</i>	271
John W. Green, <i>Mean values of harmonic functions on homothetic curves</i>	279
Charles John August Halberg, Jr. and Angus E. Taylor, <i>On the spectra of linked operators</i>	283
Chuan Chih Hsiung, <i>Some integral formulas for closed hypersurfaces in Riemannian space</i>	291
Norman D. Lane, <i>Differentiable points of arcs in conformal n-space</i>	301
Louis F. McAuley, <i>A relation between perfect separability, completeness, and normality in semi-metric spaces</i>	315
G. Power and D. L. Scott-Hutton, <i>The slow shearing motion of a liquid past a semi-infinite plane</i>	327
A. C. Schaeffer, <i>Entire functions</i>	351
Edward Silverman, <i>An intrinsic inequality for Lebesgue area</i>	363
Choy-Tak Taam, <i>Asymptotic relations between systems of differential equations</i>	373
Ti Yen, <i>Quotient algebra of a finite AW^*-algebra</i>	389