Pacific Journal of Mathematics

INVARIANT FUNCTIONALS

PAUL CIVIN AND BERTRAM YOOD

Vol. 6, No. 2 December 1956

INVARIANT FUNCTIONALS

PAUL CIVIN AND BERTRAM YOOD

1. Introduction. Let E be a normed linear space and G a solvable group of bounded linear operators on E. If there exists a non-trivial bounded linear functional invariant under G then there exists $x_0 \in E$ such that $\inf \|T(x_0)\| > 0$, $T \in G_1$, the convex envelope of G. Assume that such an x_0 exists. If G is bounded then there exists an invariant functional [7]. If G is unbounded, however, such a functional may or may not exist.

For simplicity we discuss here the abelian case. In a previous work [7] it was shown that the invariant functional exists if there is a constant $K > 0$ such that to each $U \in G_1$ there corresponds $V \in G_1$ where $\|V\| \leq K$ and $\|VU\| \leq K$. A consequence of this condition is that for each $x \in E$

$$(1) \qquad \inf_{\substack{\|T\| \leq K \\ T \in G_1}} \|T(x)\| \leq K \inf_{T \in G_1} \|T(x)\| .$$

Now call an element y *stable* if (1) holds for some $K = K(y)$ for all x of the form $U(y)$, $U \in G_1$. We show here that the invariant functional exists if E is complete and if there exists an open set S in E such that for all $x \in S$, $T \in G$, x and $T(x) - x$ are stable. An analogous result is shown to hold if G is solvable.

The problem of the existence and extension of functionals invariant under solvable groups of operators has been considered by Agnew and Morse and by Klee (see [3] for references). These authors use for E any real linear space while we take E to be a Banach space in order to utilize category arguments.

2. Notations. Let E be a Banach space and $\mathfrak{E}(E)$ be the set of all bounded operators on E. Let H be a (multiplicative) semi-group in $\mathfrak{E}(E)$. By H_1 we mean the convex envelope of H (the smallest convex subset of $\mathfrak{E}(E)$ which contains H). As in [7] we adopt the following notation. By $B(H)$ we mean the linear manifold generated by elements of the form $T(x) - x$, $x \in E$, $T \in H$. By $Z(H)$ we mean $\{x \in E \mid \inf \|T(x)\| = 0, T \in H\}$.

We introduce the following notation. An element $x \in E$ is *stable* with respect to H if there exist positive numbers K, L such that

$$\inf_{\substack{\|T\| \leq K \\ T \in H}} \|T(y)\| \leq L \inf_{T \in H} \|T(y)\|$$

for all y of the form $U(x)$, $U \in H$.

Received April 8, 1955.

We use the following symbolism,

$$\delta(y, H) = \inf_{T \in H} \|T(y)\|$$

$$\delta(y, H, r) = \inf_{\substack{\|T\| \leq r \\ T \in H}} \|T(y)\| .$$

It is readily seen that x is stable with respect to H if and only if there exists a constant $r > 0$ such that

$$(2) \qquad \delta(y, H, r) \leq r\delta(y, H)$$

for all y of the form $U(x)$, $U \in H$. Such an r is called a constant connected with the stability of x with respect to H. If x is stable with respect to H and if the right-hand side of (2) is zero for all y of the form $U(x)$, $U \in H$, we say that x is *null-stable* with respect to H.

If G is a solvable group, then $G^{(i)}$ will represent the ith derived subgroup.

3. Invariant functionals for solvable groups.

3.1 LEMMA. *If $y_1, \cdots, y_n$ are null-stable with respect to H then so is $y_1 + \cdots + y_n$.*

Proof. It is enough to show this for $y_1 + y_2$. Let M be the maximum of the constants in the definition of the null-stability of y_1 and y_2. Take $U \in H$, $\varepsilon > 0$. There exist $V_i \in H$, $\|V_i\| \leq M$, $i = 1, 2$ such that $\|V_1 U(y_1)\| < \varepsilon/(2M)$ and $\|V_2 V_1 U(y_2)\| < \varepsilon/2$. Then $\|V_2 V_1 U(y_1 + y_2)\| < \varepsilon$ with $\|V_2 V_1\| \leq M^2$. Similarly we see that $y_1 + \cdots + y_n$ is null-stable with constant M^n if M is the maximum of the constants connected with the y_i.

3.2 LEMMA. *Let E be a Banach space and G a solvable group of bounded linear operators on E. Then either* (a) *every element of E is null-stable with respect to G_1 or* (b) *there exists a non-void open set of E containing only elements not null-stable with respect to G_1 or* (c) *the set of elements not stable with respect to every $G_1^{(i)}$ is dense.*

Proof. Let $Q_n = \{x \in E \mid x$ is stable with respect to each $G_1^{(i)}$ with constant $n\}$, $n = 1, 2, \cdots$. We show that Q_n is closed. Let $x_m \in Q_n$, $x_m \rightarrow y$. Then for each i and each x_m we have

$$(1) \qquad \delta(x_m, G_1^{(i)}, n) \leq n\delta(x_m, G_1^{(i)}) .$$

We show that (1) also holds for y. If $\delta(y, G_1^{(i)}, n) = 0$ this is clear. Otherwise set $\delta = \delta(y, G_1^{(i)}, n)$ and take $0 < 2\varepsilon < \delta$. Select $T \in G_1^{(i)}$. Choose m

so large that

$$(2)\qquad \|T(y-x_m)\|<\varepsilon/n\ , \qquad \|y-x_m\|<\varepsilon/n\ .$$

Then from (1) and (2) we obtain

$$(3)\qquad n\|T(y)\|\geqq n\|T(x_m)\|-\varepsilon\geqq\delta(x_m, G_1^{(i)}, n)-\varepsilon\geqq\delta(y, G_1^{(i)}, n)-2\varepsilon\ .$$

Since $\varepsilon>0$ is arbitrary in (3),

$$(4)\qquad n\|T(y)\|\geqq\delta(y, G_1^{(i)}, n)\ .$$

Since the T of (4) is arbitrary in $G_1^{(i)}$, (1) holds for y. Since the same argument is applicable to every $V(y)$, $V\in G_1^{(i)}$ as well as for y and for each i, $y\in Q_n$.

Suppose that some Q_n contains an open sphere S. Let Σ be the collection of elements of S which are null-stable with respect to G_1. If Σ is dense in S we show that $\Sigma=S$. For let $y_m\in\Sigma$, $m=1,\cdots$, $y_m\to z\in S$. For each m, $U\in G_1$, we have $\delta(U(y_m), G_1, n)=0$. This implies that $\delta(U(z), G_1, n)=0$ which in turn shows that $z\in\Sigma$. In this case by Lemma 3.1, (a) holds since the set of elements which are null-stable with respect to G_1 forms a linear manifold with interior. If Σ is not dense in S then erthe is an open subset S_1 of S on which (b) holds.

Suppose next that no Q_n contains a sphere. By a theorem of Baire, the intersection P of the sets $E-Q_n$ is dense. If $x\in P$, then x fails to be stable with respect to at least one of the semi-groups $G_1^{(i)}$, for otherwise $x\in Q_n$ for all sufficiently large n.

3.3 LEMMA. *Let G be a solvable group in $\mathfrak{E}(E)$. If $S\in G_1^{(i)}$, $T\in G^{(i)}$, $x\in E$ then $S[T(x)-x]$ can be expressed in the form $z+TS(x)-S(x)$ where $z\in B(G^{(i+1)})$, $i=0,\cdots,n-1$.*

Proof. Let

$$S=\sum_{j=1}^{m}\alpha_jS_j,\quad \alpha_j\geqq 0\ ,\qquad \sum_{j=1}^{m}\alpha_j=1,\quad S_j\in G^{(i)}\ .$$

For each $j=1,\cdots,m$ there exists $U_j\in G^{(i+1)}$ such that $S_jT=U_jTS_j$. Then

$$(5)\qquad \begin{aligned} S[T(x)-x]&=\sum_{j=1}^{m}\alpha_j[U_jTS_j(x)-S_j(x)]\\ &=\sum_{j=1}^{m}[U_jTS_j(\alpha_jx)-TS_j(\alpha_jx)]+TS(x)-S(x)\end{aligned}$$

which is in the required form.

3.4 LEMMA. *If $S\in G_1^{(i)}$, $T\in G^{(i+1)}$, $x\in E$ then $S[T(x)-x]\in B(G^{(i+1)})$.*

This follows from Lemma 3.3.

3.5 LEMMA. *Let H be a semi-group in $\mathfrak{C}(E)$. Suppose that x is stable with respect to H and that $U(x) \in Z(H)$ for all $U \in H$. Then x is null-stable with respect to H.*

This follows directly from the definitions.

3.6 LEMMA. *Let G be a group in $\mathfrak{C}(E)$, $x \in E$ where x is stable with respect to G_1. Then $TW(x) - W(x) \in Z(G_1)$ for all $T \in G$, $W \in G_1$.*

Proof. Set $V = (I + T + \cdots + T^{s-1})/s$. Then $V[TW(x) - W(x)] = [T^s W(x) - W(x)]/s$. Let r be the constant connected with the stability of x. Then since $T^{-s} \in G$,

$$\delta(T^s W(x), G_1, r) \leqq r\delta(T^s W(x), G_1) \leqq r \| W(x) \| .$$

Pick $U \in G_1$, $\|U\| \leqq r$ where $\|UT^s W(x)\| < r\|W(x)\| + 1$. Then

$$\|UV[TW(x) - W(x)]\| < (2r\|W(x)\| + 1)/s .$$

This shows that $TW(x) - W(x) \in Z(G_1)$.

3.7 LEMMA. *Let G be a group in $\mathfrak{C}(E)$. Let $x \in E$ where $(T-I)(x)$ is stable with respect to G_1 for all $T \in G$. Then $(T-I)U(x)$ is also stable for all $T \in G$, $U \in G$.*

Proof. Observe that $(T-I)U(x) = U(U^{-1}TU - I)(x)$. Since $(U^{-1}TU - I)(x)$ is stable with respect to G_1 it follows readily that so is $(T-I)U(x)$.

3.8 THEOREM. *Let E be a Banach space and G a solvable group of bounded linear operators on E. Let Q be the set of elements of E stable with respect to each $G_1^{(i)}$. If there exists a non-void open subset $\mathfrak{S}$ of Q such that $(T-I)\mathfrak{S} \subset Q$ for each $T \in G$ then every element of $B(G)$ is null-stable with respect to G_1. If also there is at least one element of E not null-stable with respect to G_1 then there exists a non-trivial invariant functional.*

Proof. Assume the condition on the set $\mathfrak{S}$. We show by induction starting with n, where $G^{(n)} = \{I\}$, that $B(G^{(j)})$ consists entirely of elements null-stable with respect to $G_1^{(j)}$, $j = 0, \cdots, n$. This is automatic for $j = n$; suppose that it holds for $j = i+1, \cdots, n$. Let $S, T \in G^{(i)}$, $x \in \mathfrak{S}$. In the notation of Lemma 3.3, we can write $S[T(x) - x] = z + TS(x) - S(x)$ where z is a linear combination of elements of the form $U_j TS_j(x) - TS_j(x)$, $U_j \in G^{(i+1)}$, $S_j \in G^{(i)}$. By hypothesis and Lemma 3.7, $U_j TS_j(x) - TS_j(x) \in Q$.

For any $V \in G_1^{(i)}$, $V[U_jTS_j(x)-TS_j(x)] \in B(G^{(i+1)}) \subset Z(G_1^{(i+1)}) \subset Z(G_1^{(i)})$ by Lemma 3.4 and the induction hypothesis. Hence by Lemma 3.5, $U_jTS_j(x)-TS_j(x)$ is null-stable with respect to $G_1^{(i)}$ and thus, by Lemma 3.1 so is z.

Consider the constant r connected with the null-stability of z with respect to $G_1^{(i)}$. Take $\varepsilon > 0$. Since $x \in \mathfrak{S}$, by Lemma 3.6 there exists $W \in G_1^{(i)}$ such that $\| W[TS(x)-S(x)]\| < \varepsilon/(2r)$. Furthermore there exists $R \in G_1^{(i)}$, $\|R\| \leqq r$ such that $\|RW(z)\| < \varepsilon/2$. Therefore $\|RW[ST(x)-S(x)]\| < \varepsilon$ which shows that $S[T(x)-x] \in Z(G_1^{(i)})$ for all $S \in G_1^{(i)}$. Since $T(x)-x \in Q$ it follows from Lemma 3.5 that $T(x)-x$ is null-stable with respect to $G_1^{(i)}$. Let $P=\{x \in E \mid T(x)-x$ is null-stable with respect to $G_1^{(i)}\}$. By Lemma 3.1, P is a linear manifold. But $\mathfrak{S} \subset P$. Therefore $P=E$. In view of Lemma 3.1, every element of $B(G^{(i)})$ is null-stable with respect to $G_1^{(i)}$. This completes the induction.

Suppose also that some element of E is not null-stable with respect to G_1. Then (a) and (c) of Lemma 3.2 are ruled out. Thus there exists a sphere in E given by Lemma 3.2 which by the above is disjoint with $B(G)$. Hence, by the Hahn-Banach theorem there exists a bounded linear functional $\neq 0$ which vanishes on $B(G)$. This is an invariant functional.

4. Positive invariant functionals. We point out next that the arguments used above and in [7] for $B(G) \subset Z(G_1)$ have wider applicability than is apparent on the surface and in particular contain implicitly results obtained by Krein and Rutman [5].

In the terminology of [5] by a *linear semi-group* $\mathfrak{K}$ in a real normed linear space E is meant a (proper) subset of E where $\alpha x+\beta y \in \mathfrak{K}$ if x, $y \in \mathfrak{K}$ and $\alpha \geqq 0$, $\beta \geqq 0$ are scalars. We say that $x \leqq y$ $(y \geqq x)$ if $y-x \in \mathfrak{K}$, $x, y \in E$. Suppose that $\mathfrak{K}$ is given with $\operatorname{Int}(\mathfrak{K})$ non-void.

Let G be a multiplicative semi-group of linear operators on E. Following [6] we call G left-solvable if there exists a finite sequence of sub-semi-groups $G=G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(n)}=\{I\}$ such that given $T, U \in G^{(i)}$, $i=0, \cdots, n-1$ there exists $V \in G^{(i+1)}$ with $TU=VUT$.

The following is an extension of [5, Theorem 3.1].

4.1 THEOREM. *Let G be a left solvable semi-group of linear operators on E such that $A(\mathfrak{K}) \subset \mathfrak{K}$, $A \in G$. Suppose that $v \in \operatorname{Int}(\mathfrak{K})$ and*

(a) *for some $\sigma > 0$, $A(v) \geqq \sigma v$, $A \in G$, and*

(b) *for some $r > 0$, given $U \in G_1^{(i)}$ there exists $T \in G_1^{(i)}$ such that*

$$T(v) \leqq rv, \; TU(v) \leqq rv \tag{1}$$

$i=0, \cdots, n-1$. Then there exists a bounded linear functional x^* on E, invariant with respect to G and $x^*(x) > 0$, $x \in \operatorname{Int}(\mathfrak{K})$.

Let $v \in \mathrm{Int}(\mathfrak{K})$. As in [5] we define for each $x \in E$, $|x|_v = \inf t$, where $t > 0$ and satisfies $-tv \leqq x \leqq tv$. $|x|_v$ is a semi-norm[1] for E. Let A be a linear operator on E, $A(\mathfrak{K}) \subset \mathfrak{K}$. Since $v \in \mathrm{Int}(\mathfrak{K})$, if $\alpha > 0$ is sufficiently large, then

$$(2) \qquad -\alpha v \leqq 0 \leqq A(v) \leqq \alpha v$$

It is easy to see that $|A(v)|_v = \inf \alpha$, $\alpha > 0$ satisfying (1). If $-tv \leqq x \leqq tv$ then for α satisfying (1),

$$-t\alpha v \leqq -tA(v) \leqq A(x) \leqq tA(v) \leqq t\alpha v$$

from which we see that $|A(x)|_v \leqq |A(v)|_v |x|_v$. Since $|v|_v = 1$ we see that A is bounded with respect to the semi-norm and

$$(3) \qquad |A|_v = |A(v)|_v \, .$$

We define $Z(G_1^{(i)})$ in terms of the semi-norm $|x|_v$. By the formulas (1), (2) and (3) it is seen that for $T \in G_1^{(i)}$ there exists $V \in G_1^{(i)}$, $|V|_v \leqq r$ where $|VT|_v \leqq r$. The arguments of [6, Theorem 3] are unaffected by the use of the semi-norm rather than a true norm. As noted by Robison [6, Theorem 6.8] in this situation we then obtain $B(G) \subset Z(G_1)$.

Let $x \in \mathrm{Int}(\mathfrak{K})$. There exists $\alpha > 0$ such $x \geqq \alpha v$. For each $A \in G_1$, by (a), $A(x) \geqq \alpha\sigma v$. Moreover if $A(x) \leqq \beta\sigma v$, $0 < \beta < \alpha$, then $\beta\sigma v \leqq \alpha\sigma v$ which is impossible by [5, p. 11]. Hence $|A(x)|_v \geqq \alpha\sigma$. This shows that $\mathrm{Int}(\mathfrak{K}) \cap Z(G_1) = \phi$. By the above, $B(G) \cap \mathrm{Int}(\mathfrak{K}) = \phi$. An application of [4, Corollary 1.2] gives the existence of the desired functional.

As a consequence of Theorem 4.1 we obtain the following.

4.2. COROLLARY. *Let G be a left solvable semi-group of operators on E satisfying the requirements of Theorem* 4.1, *and let* $v \in \mathrm{Int}(\mathfrak{K})$. *Then for any* $w \in \mathrm{Int}(\mathfrak{K})$, $T_j \in G$, $j = 1, 2, \cdots, n$,

$$(4) \qquad \sum_{j=1}^{n} p_j T_j(w) \in \mathfrak{K} \text{ implies that } \sum_{j=1}^{n} p_j \geqq 0 \, .$$

When $\mathfrak{K}$ is the positive cone in a space E of bounded functions on a set S, and G is a semi-group of linear operations on E induced by a semi-group Γ of one-to-one transformations of S onto S, Hadwiger and Nef [2] have shown that the statement (4) is fundamental in the theory of integration systems.

[1] We mean $|ax| = |a||x|$, $x \in E$, a real, and $|x+y| \leqq |x| + |y|$, $x, y \in E$. (See [1], p. 93). In particular $|x| \geqq 0$ for all x.

REFERENCES

1. N. Bourbaki, *Les structures fondamentales de l'analyse* XV (*Espaces vectoriels topologiques*), Actualités Scientifiques et Industrielles, No. 1189, Paris, 1953.
2. H. Hadwiger and W. Nef, *Zur axiomatischen Theorie der invarianten Integration in abstrakten Räumen*, Math. Z., **60** (1954), 305–319.
3. V. L. Klee, Jr., *Invariant extension of linear functionals*, Pacific J. Math., **4** (1954) 37–46.
4. M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Uspehi Mat. Nauk (n. s.), **3** (1948), 3–95. (In Russian)
5. ———, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl. no. 26, New York, 1950.
6. G. B. Robison, *Invariant integrals over a class of Banach spaces*, Pacific J. Math., **4** (1954), 123–150.
7. B. Yood, *On fixed points for semi-groups of linear operators*, Proc. Amer. Math. Soc., **2** (1951), 225–233.
8. M. M. Day, *Means for bounded functions and ergodicity of the bounded representations of semi-groups*, Trans. Amer. Math. Soc., **69** (1950), 276–291.

UNIVERSITY OF OREGON

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, Alfred Horn at the University of California, Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, c/o University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 6, No. 2 December, 1956